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The Magic of Summation Sequences: Applied Combinatorics and Graph Theory

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Abstract

Mathematics is a useful basis for motivating self-working card tricks. In this paper, we expand the realm of self-working magic tricks through the examination of a novel type of universal cycle called a Summation Sequence. An analysis of the graph theoretical and combinatorial structure of Summation Sequences reveals, firstly, the existence of operations that preserve the properties of Summation Sequences, and, secondly, a special type of Summation Sequence called a Symmetric Sequence. We apply our most important results to the realm of card magic, where we exploit the properties of Summation Sequences to motivate various card effects.

1 Introduction

For centuries, mathematics has been used to motivate "self-working" card tricks, meaning that instead of sleight of hand, the effect of the trick relies on some sort of mathematical principle.

Combinatorial mathematics has been applied to self-working magic with great success (Diaconis, Graham, and Gardner, 2016). In particular, a family of combinatorial structures called Universal Cycles are employed in these magic tricks.

A Universal Cycle can be described somewhat informally as follows: Given a collection of mathematical objects F, a Universal Cycle for F is a sequence of numbers S such that each member of the collection F is somehow "coded" up at least once by some subsequence of S (Chung, Diaconis, & Graham, 1992).

The most widely studied type of Universal Cycle, both within the realm of mathematical card magic and within mathematics more generally, are De Bruijn sequences. Given an alphabet a and a window w, a De Bruijn sequence is a cyclic sequence of numbers between 0 and a-1 such that each possible string of symbols of length w appears exactly once in the De Bruijn sequence as a subsequence. The following is a De Bruijn sequence for alphabet $a = 2$ and window $w = 3$:

S: 00010111

Binary De Bruijn sequences (i.e., those with alphabet $a = 2$) have particular usage in self-working magic tricks, where the 0s and 1s of the sequences can be understood as representing the color of the cards (black and red) in a standard deck (Diaconis, Graham, and Gardner, 2016).

We seek to expand the application of

Universal Cycles to self-working magic tricks through the investigation of a new type of Universal Sequence which has not been investigated in either the mathematical literature or the literature on magic until now: Summation Sequences. In this paper, we will analyze Summation Sequences combinatorially and graph theoretically, applying some of our results to the realm of self-working card tricks.

A Summation Sequence is *a* cyclic sequence of numbers S with alphabet a and window w where every possible sum between 0 and $w(a - 1)$ occurs once as a sum of w-many adjacent numbers in the sequence. Note that the values comprising the alphabet of a Summation Sequence are the numbers between 0 and $(a - 1)$ inclusive – that is, the first a-many numbers begin with 0.

Given a particular alphabet a and window w, we will talk of the $<a, w>$ Summation Sequences. The following is a simple <3,2> Summation Sequence:

S_1: 00122

Since the window here is 2, we verify that this is a summation sequence by verifying that each number from 0 to 4 (i.e., $w(a - 1) = 2(3 - 1) = 4$) occurs as the sum of two consecutive numbers in the sequence:

$$0 + 0 = 0$$
$$0 + 1 = 1$$
$$1 + 2 = 3$$
$$2 + 2 = 4$$
$$2 + 0 = 2$$

Note that Summation Sequences are cyclic: we include the sum obtained from the last number in the sequence and the first number, "wrapping around" the sequence. Also, every Summation Sequence must contain w many consecutive elements with value of 0 (in order to obtain 0 as a sum) and w-many consecutive elements with value a - 1 (in order to obtain $w(a - 1)$ as a sum).

Given a Summation Sequence, we obtain the corresponding Derived Sequence by listing the sums obtained in the Summation sequence in the order in which they occur. As an example, we present the following Derived Sequence corresponding to the <3,2> Summation Sequence above:

D_1: 01342

Due to their cyclical nature, the following two cycles are identical (and treated as such when counting the number of $<a, w>$ Summation Sequences below):

S_1: 00122

S_2: 01220

We also can see that the length l of any Summation Sequence S is $w(a - 1) + 1$. This fact is illustrated by S_1 and S_2 above, but can be proven more generally by noting that the Derived Sequence must contain all sums between 0 and $w(a - 1)$ inclusive. Consequently, the length of the Derived Sequence—which is identical to the length of the original Summation Sequence—will always be one greater than the value of $w(a - 1)$.

Now that we have introduced Summation Sequences, the remainder of the paper will proceed as follows: In section 2 we will provide a graph theoretical and combinatorial investigation into the properties of Summation Sequences. We will begin by discussing sequence existence in 2.1. We will then analyze Summation Sequence graphs, and how their structure allows us to perform property-preserving operations on these sequences, in 2.2 and 2.3 respectively. Finally, in 2.3, we will discuss a special type of Summation Sequence, called a Symmetric Sequence, which has unique properties of its own. In section 3, we will apply what we discovered in section 2 to the realm of self-working card magic. We first note how we can translate the mathematical operations of a sequence to a deck of cards, and then provide two magic effects that can be formulated through the exploitation of Summation Sequence properties. Lastly, in section 4, we will note some directions for future research.

2 Summation Sequences

2.1 Sequence Existence

Now that we know the length of any sequence S, we can find conditions that must hold for a sequence with alphabet a and window w to exist. For example, from Section 1, we know that a sequence must contain w-many consecutive elements with value of 0, and w-many consecutive elements with value (a - 1). However, a sequence S must contain at least one other element, otherwise, due to the cyclical nature of S, it would obtain the sum $0 + n(a - 1)$ twice for any n such that $0 < n < w$. So, we know the following is true, where l is length:

$$l \geq 2w + 1$$

$$w(a - 1) + 1 \geq 2w + 1$$

$$w(a - 1) \geq 2w$$

$$wa - w \geq 2w$$

$$wa \geq 3w$$

$$a \geq 3$$

Thus, no Summation Sequences exist with an alphabet less than 3. Hence, unlike De Bruijn sequences, there are no Summation Sequences that contain only 0's and 1's.

We can continue to investigate the constraints that w and a impose upon Summation Sequences by exploring Summation Sequences and Derived Sequences concurrently. Given an alphabet a and window w, consider a Summation Sequence S:

$$S = (S_1, S_2\text{é} \quad S_{w(a-1)+1})$$

and its Derived Sequence D:

$$D = (D_1, D_2\text{é} \quad D_{w(a-1)+1})$$

We know that the sum of the elements of the Derived Sequence is equal to the sum of the elements of the corresponding Summation Sequence times the window w, as each element of the Summation Sequence will be counted w many times in the Derived Sequence:

$$\sum_{i=1}^{w(a-1)+1} D_i = w\left(\sum_{i=1}^{w(a-1)+1} S_i\right)$$

Since the Derived Sequence contains each number from 0 to $w(a - 1)$ exactly once (and applying a classic combinatorial formula), we know that the sum of the Derived Sequence is equal to the maximum value of the Summation Sequence, times the maximum value of the Summation sequence plus one, divided by two:

$$\sum_{i=1}^{w(a-1)+1} D_i = \sum_{i=1}^{w(a-1)} i = \frac{(w(a-1))(w(a-1)+1)}{2} = \frac{w^2a^2 - 2w^2a + w^2 + wa - w}{2}$$

Substituting:

$$\frac{w^2a^2 - 2w^2a + w^2 + wa - w}{2} = w\left(\sum_{i=1}^{w(a-1)+1} S_i\right)$$

And simplifying:

$$wa^2 - 2wa + w + a - 1 = 2\left(\sum_{i=1}^{w(a-1)+1} S_i\right)$$

In the final equation, we see that if both a and w are even, the left side of the equation is odd (and this formula is even otherwise). However, as the right side of the equation is multiplied by 2, it's always even.

Thus, there are no Summation Sequences with even a and w. This squares up with the data

w a	2	3	4	5	6	7
2	0	2	0	66	0	16988
3	0	2	6	78	?	?
4	0	2	0	?	0	?
5	0	2	?	?	?	?
6	0	2	0	?	0	?
7	0	2	?	?	?	?

Fig. 1. The number of Summation Sequences which exist for different alphabet and window, as generated through various self-built computer programs and manual graph-theoretic analysis.

given in Figure 1, which shows the number of distinct Summation Sequences for those alphabet and window pairs <*a, w*> for which we have been able to determine this number of distinct <*a, w*> Summation Sequences (for example, there are 78 distinct Summation Sequences for alphabet 5 and window 3).

Figure 1 also illustrates a fact that we will note but not prove here: There exist Summation Sequences for all alphabets greater than 2 (in particular, there exist <*a*, *w*> Summation Sequences for any alphabet *a* and window *w* such that *a* > 2, and *w* odd). Thus, the result with which we began this section is the strongest such result, constraining Summation Sequence existence solely in terms of the size of the alphabet.

2.2 Graphs

In this section, we provide an outline of some basic graph-theoretical results regarding Summation Sequences. We will restrict our attention to the simplest case, where $w = 2$. These techniques can be extended to Summation Sequences with larger windows, however.

We begin with the total (labelled) graph corresponding to <5,2> Summation Sequences, and then investigate <5, 2> Summation Sequences via an examination of subgraphs of this graph. In the total graph, each node represents a number in the sequence alphabet, and each edge represents the sum of the two nodes they connect.

A Summation Sequence corresponds to a cyclic path through the total graph such that each number from 0 to $w(a - 1)$ occurs as the label of an edge in the path exactly once. Note that, in the <5, 2> case, this means any such path must pass through the reflexive edges from 0 to 0 and from 4 to 4, and any such path must also pass through the edge from 0 to 1 and the edges from 3 to 4, since these are the only edges with 0, 8, 1, and 7 as labels, respectively.

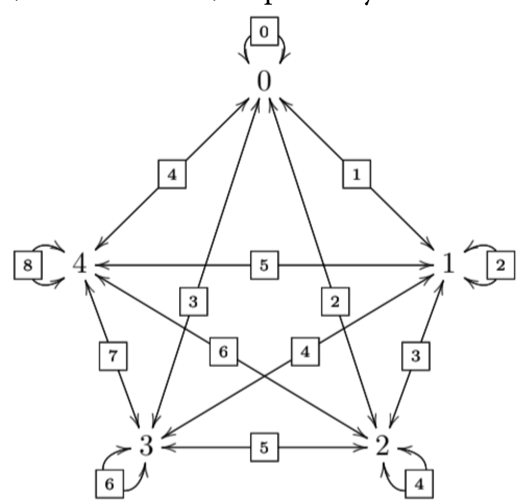

Fig. 2. The Total Graph for <5,2> Sequences

In Figure 2, we have made all possible connections between the five numbers in the alphabet, including the reflexive loops from a node to itself. As noted in Figure 1, there are 66 paths through this graph which correspond to a <5,2> Summation Sequence.

Below, we have drawn five digraphs, each omitting six edges and zero nodes from the total graph.

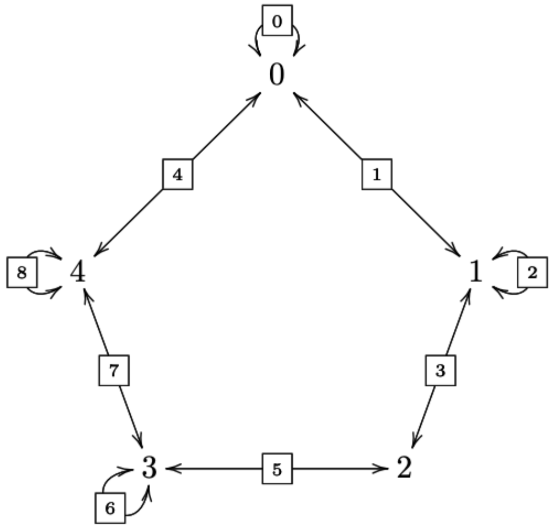

Fig. 3. Digraph G1 for <5,2> Sequences

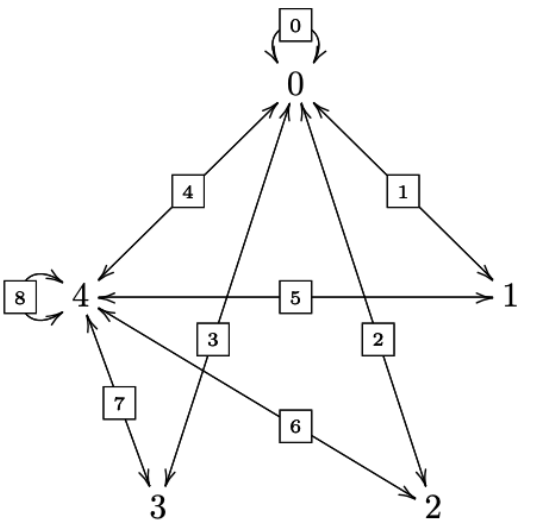

Fig. 4. Digraph G2 for <5,2> Sequences

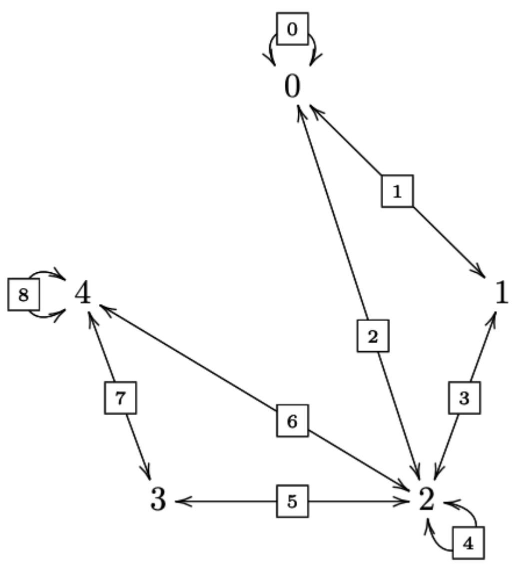

Fig. 5. Digraph G3 for <5,2> Sequences

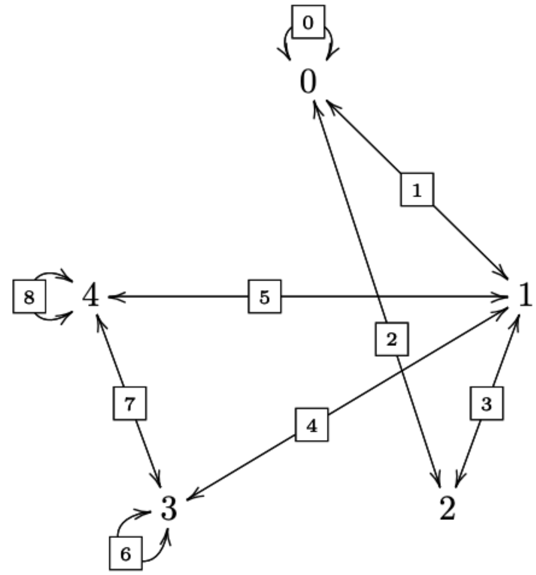

Fig. 6. Digraph G4 for <5,2> Sequences

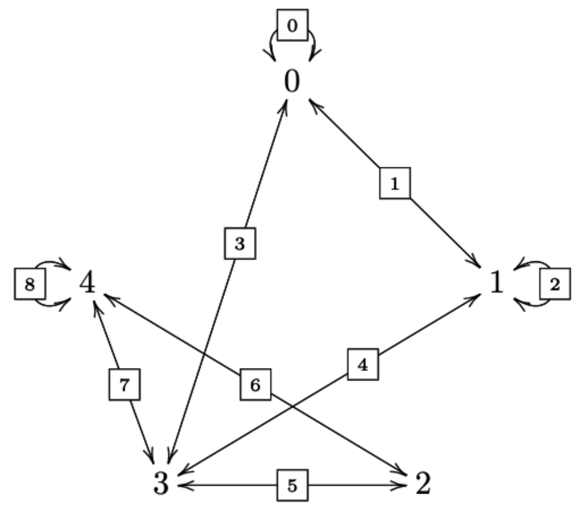

Fig. 7. Digraph G5 for <5,2> Sequences

Each distinct <5,2> Summation Sequence corresponds to an Eulerian path (i.e., a path that passes through each edge exactly once) through one of the digraphs given in Figures 3 through 7.[1] As stated before, there exist 66 distinct <5,2> Summation Sequences. Here, we note how many of these sequences are obtained from each graph:

G1: 2

G2: 48

G3: 8

G4: 4

G5: 4

[1]For a good introduction to the basics of graph theory and combinatorics, including the concept of Eulerian paths, see (Harris, Hirst, & Mossinghoff, 2010).

For example, the four distinct Eulerian paths through G5, corresponding to four distinct Summation Sequences, are:

S_3: 001134423

S_4: 001132443

S_5: 003244311

S_6: 003442311

The interconnectedness of the digraph G2 results in it containing many Eulerian circuits (and hence coding up many Summation Sequences), while the opposite can be said for G1. Further analysis of Summation Sequence graphs will follow in the next two sections.

2.3 Operations

There are a number of operations we can perform on Summation Sequences, and thus also their graphs, that preserve certain properties. In particular, many of these operations take a Summation Sequence as input and give a distinct Summation Sequence as output.

2.3.1 Inverse

Consider a Summation Sequence S:

$$S = (s_1, s_2 \ldots s_k)$$

The inverse S^{-1} of S is:

$$S^{-1} = (s_k, s_{k-1} \ldots s_1)$$

Returning to our simple example from earlier, the inverse of:

$$S_1 = 00122$$

is:

$$S_1^{-1} = 22100 = 00221$$

In other words, to obtain the inverse S^{-1} of S, we read S backwards. It is evident that the inverse of an $<a, w>$ Summation Sequence is also an $<a, w>$ Summation Sequence, since each sum that occurs in the original sequence occurs in the inverted sequence, but in the opposite order.

2.3.2 Codifference

Consider an $<a, w>$ Summation Sequence S:

$$S = (s_1, s_2 \ldots s_k)$$

The codifference S^{-d} of S:

$$S^{-d} = ((a - 1) - s_1, (a - 1) - s_2 \ldots (a - 1) - s_k)$$

For example, given the <5, 2> Summation Sequence from the previous subsection:

$$S_4 = 001132443$$

The corresponding codifference sequence is:

$$S_4^{-d} = 443312001 = 001443312$$

In this example, $(a - 1) = 4$, so each element is subtracted from 4 to obtain the values in the codifference sequence. The reader is invited to verify that the codifference sequence of an $<a, w>$ Summation Sequence is also an $<a, w>$ Summation Sequence.

2.3.3 Loop Rotation

Consider a Summation Sequence S:

$$S = (s_1 \ldots R, T, Q \ldots s_k)$$

where R and Q are subsequences of length l where $l \geq w - 1$, $R = Q^{-1}$, and T is a subsequence of any length. Then the loop rotation sequence corresponding to S is:

$$S^* = (s_1 \ldots R, T^{-1}, Q \ldots s_k)$$

From a graph theoretical lens, consider G5 from Figure 6, and notice its loop containing nodes 3, 4, and 2. Viewed in terms of Eulerian paths on G5, the loop rotation sequence corresponding to S is the Summation Sequence corresponding to the path that traverses this loop in the opposite direction than that taken in the Eulerian path on G5 corresponding to S. To illustrate this, consider the following Summation Sequence obtained from the digraph G5:

$$S_3 = 001134423$$

Note that 442 is between two equivalent elements (i.e., two subsequences of length 1 that are trivially inverses of one another). So, we can perform our loop rotation operation to obtain a second sequence S_4^*:

$S_3^* = 001132443$

Consideration of the corresponding Eulerian paths through G5 should make it clear that the loop rotation of an <*a, w*> Summation Sequence is also an <*a, w*> Summation Sequence (the sequence merely passes through the "loop" 34423 in the opposite direction).

2.4 Symmetric Sequences

Symmetric Sequences are a special type of Summation Sequence with their own unique properties. A Symmetric Sequence is such that any two elements which are *d* distance away from the center point in the sequence sum to the same value, which is (a - 1). Something that is important to note is that if the Summation Sequence has an odd length (which is the case unless *w* is odd and *a* is even), that the central element's value must be ½(a - 1), or be the middlemost value in the alphabet.

To illustrate this, consider the following <5, 2> Symmetric Sequence (which corresponds to a Eulerian path in G2):

S_7:	001420344
Distance *d* from center:	432101234

Here we see that the two elements with the same *d* value sum to 4. Additionally, since this is a sequence with odd *l*, the center point is occupied by 2 = ½(a - 1).

Symmetric Sequences are interesting for the following reason: A Summation Sequence is a Symmetric Sequence if and only if its inverse is its codifference. If all pairs of numbers *d* distance from the center sum to (a - 1), when we subtract them from (a - 1), it is akin to swapping those two elements, like we do in inversion. Additionally, if there exists an element in the center point, as long as its value is ½(a - 1), it won't change when performing codifference nor inversion. Thus, in this case, the codifference of the sequence will be the same as its inverse, and it is thus a Symmetric Sequence. To help demonstrate this, we've provided a brief display of operations.

Symmetric Sequence:	003420144
Codifference ((a - 1) - s_n):	441024300 (= 004410243)
Inverse:	441024300 (= 004410243)

Finally, we will discuss the graph theoretical aspects of Symmetric Sequences. In section 2.2, only three of the digraphs are symmetrical, and any axis of symmetry one may find passes through the node with the value of 2. To achieve a Symmetric Sequence by traversing one of these three graphs, a perfectly symmetrical path must be taken. Consider the digraph G3 from Figure 5. Here is a path that would result in Symmetric Sequence:

S_8: 234420012

This path is perfectly symmetric because paths taken through the 2 node, or the axis of symmetry, mirror each other. On the contrary, here is a path through G3 that would result in a Summation Sequence, but not a Symmetric Sequence:

S_9: 244320012

Here, after passing through the 2 node in the fifth position, the path strays from its mirror, losing the symmetric property.

Note that any path through the digraph G1 from Figure 3 will result in a Symmetric Sequence because all paths through the axis of symmetry—the 2 node—are perfectly symmetric.

3 Magical Applications

3.1 Card Operations

We can apply the principles of Summation Sequences to card magic by allowing each card to represent a number in a sequence. For example, below is a small <3,2> Summation Sequence, and its corresponding representation in a packet of cards. Note that, to translate a Summation Sequence to card values, a value of 1 is added to each element of the Summation Sequence. Additionally, A equals 1, J equals 10, Q equals 11, and K equal 12.

0 0 1 2 2

A♡ A♢ 2♤ 3♡ 3♢

Now that we've translated Summation Sequences to playing cards, we can apply certain Summation Sequence operations from Section 2 to card packets to motivate and complement magic effects.

Firstly, the cyclic nature of Summation Sequences allows us to move any number of sequence elements from top to bottom, or from bottom to top. In terms of playing cards, this is identical to cutting the deck. So, before performing certain tricks that depend on the cards being arranged in a way corresponding to a Summation Sequence, the spectator may cut the deck *n* times for any *n*.

Secondly, we can exploit the operation of inversion. As we explained, Summation Sequences can be inverted by being "read backwards." For a magic effect, a magician can begin a trick by performing an overhand shuffle[2] where she peels one single card at a time from the top of the pack to her other hand; this process can be repeated *n* times for any *n* for naturalness. At the end of this shuffle, the pack is simply inverted, and thus retains its unique properties.

Lastly, we will describe how Summation Sequences are well-suited for card magic. At their foundation, they are defined by value. If a magician has a spectator pick a card, and tells them that the suit of the card they are holding is hearts, that is not very impressive because there are only four suits. However, if a magician derives the value of that card, this is more impressive because there are 13 values in a deck. So, by defining our Summation Sequences by the value of the elements in the sequence, we have created an extremely useful type of sequence for card magic.

3.2 Effects

We offer two new card effects that exploit the properties of Summation Sequences which we've explained above.

3.2.1 Dual Discernment

Effect: The magician shuffles the deck, and hands it to the spectator. The spectator is asked to cut the deck as many times as they'd like, while the magician turns away. When satisfied with their cutting, the spectator is instructed to look at and remember the top two cards of the deck. Finally, they are asked to cut the deck one more time, losing their two selections in the middle of the pack. When the magician turns around, she asks the sum of the two cards values. From this information, she is able to identify both selected cards.

Explanation: This simple effect employs the simplest principle of Summation Sequences. Note that, in the effect described, the magician improved the trick by using the playing card operations from the previous section to allow the deck to be shuffled and cut. For the effect itself, we see that the magician is able to know the identity of two adjacent cards by knowing their sum. This is possible because Summation Sequences contain all sums between 0 and $w(a-1)$, where no sum occurs twice. So, once the unique sum is known, the two summands can be derived.

[2]An overhand shuffle is performed by holding the cards between your thumb and fingers of your dominant hand. The other hand then slides packets of cards to itself using its thumb. The non-dominant hand continues to accumulate cards until all cards are taken from the dominant hand, thus mixing the cards.

Note that the magician works with two selected cards in this case because, for the sake of memorization, the Summation Sequence has a window of two; a window of three would allow for three selected cards, a window of four would allow for four selected cards, and so on. Of course, using a window of two does still require some memorization, but we have provided examples of Summation Sequences below that should be a short enough length that allows for this. Also, it's important to note that the sum, given to you by the spectator, only allows you to know the values of the two cards which were the summands. However, below we offer sequences to also derive the suits or colors, thus improving the effect.

To derive both value and color in an effect, consider the following <7,2> sequence, where odd cards are red and even cards are black:

A♡ A♢ 5♡ 6♧ 6♤ 7♢ 7♡ 3♢ 5♢ 4♧ 3♡ 2♤ 2♧

To derive both value and suit, it's best we use a simpler sequence, such as a length nine <5,2> sequence. The one shown below follows a pattern, in order: Clubs, Hearts, Spades, Diamonds. (Helpful mnemonic device: CHaSeD) However, one must just remember that this pattern is broken if one of the spectator's cards is a three; the three is always the three of hearts. Remembering this pattern is sufficient to derive the two respective suits of the selected cards after calculating the two values.

A♧ A♡ 2♤ 5♢ 5♧ 4♡ 4♤ 2♢ 3♡

3.2.2 Magic Mirror

Effect: The magician begins by mixing a packet of cards, and instructs the two spectators to agree on a number between one and four inclusive, but not to tell her this number; for the purposes of the effect, let us say this number is three. The magician hands the packet of cards to one of the spectators, say Spectator 1, and turns her back. She instructs Spectator 1 to deal the agreed upon number of cards down onto the table, one at a time, then removing the next two cards as selections. Lastly, Spectator 1 deals the rest of the cards down onto the table. Now, Spectator 2 repeats this exact process, and obtains two cards as selections as well. Finally, the magician asks just one of the spectators to reveal the sum of their two cards, and, from this, determines the identity of all four selections.

Explanation: This effect exploits the properties of the special type of Summation Sequence discussed in section 2.4, which we have called a Symmetric Sequence. One such example is the packet:

A♡ A♢ 2♤ 7♢ 5♧ A♤ 4♢ 7♧ 3♡ A♧ 6♧ 7♤ 7♡

Here, the value of the sum of any two cards that are the same distance from the middlemost card is 8.

To mix the cards at the beginning of the trick, one can perform the sequence-preserving overhand shuffle described in section 3.1. Next, the spectators are asked to choose a number, n, between one and four inclusive. The lowest number in the set of numbers that can be chosen is always one, but the highest number, m, must be such that:

$$m \leq \tfrac{1}{2}\, l - 2$$

For the effect above, where $m = 4$ and $l = 13$:

$$4 \leq \tfrac{1}{2}\,(13) - 2$$

$$4 \leq 6.5 - 2$$

$$4 \leq 4.5$$

In the above effect, the number three was chosen, so Spectator 1 deals three cards, removes 7♢ 5♧, and then deals the rest of the cards, leaving us with the following packet:

7♡ 7♤ 6♧ A♧ 3♡ 7♧ 4♢ A♤ 2♤ A♢ A♡

Next, Spectator 2 repeats the process, dealing three cards, removing A♧ 3♡, and deals the rest of the cards for continuity.

Now, after Spectator 1 or 2 tells us the sum of their two cards, we can exploit the properties of Symmetric Sequences and Summation Sequences to identify all four cards. To explain how this is the case, let us observe the selections' origins in the packet:

A♡ A♢ 2♤ **7♢ 5♧** A♤ 4♢ 7♧ **3♡ A♧** 6♧ 7♤ 7♡

Note that each individual selection has a symmetrical mate that is the same distance *d* from the middlemost card. As we described in section 2.4, two numbers the same distance from the middlemost number always have the same sum; in this case, that sum is 8. Further, 4 cards, each having a symmetrical mate, will have a total sum of 16.

Now, suppose Spectator 2 tells us that the sum of their cards is 4 (as A + 3 = 4). Using the properties of Symmetric Sequences we just described, we know that Spectator 1's cards sum to 12, as 16 - 4 = 12.

Finally, using the general properties of Summation Sequences as discussed in 3.2.1, one can identify all four cards.

4 Conclusion

We have provided an introduction to a new type of Universal Cycle, Summation Sequences, and provided some basic mathematical results regarding these structures via both a combinatorial and a graph theoretical approach. In addition, we have shown how these sequences can be used to great effect in card tricks. Much work, however, remains to be done. In particular, extending the graph theoretical methods sketched above to Summation Sequences with window *w* greater than 2 seems particularly promising. In addition, we hope to determine a formula that gives, for any alphabet *a* and window *w*, the exact number of <*a, w*> Summation Sequences. Finally, there is no doubt that solving these additional mathematical problems will lead to additional tools that can be used to construct even more compelling mathematical card tricks.

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