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1. Introduction

Earth's oceans are essential in controlling global temperatures by absorbing heat and distributing it around the globe. The heat capacity of water is extremely high when compared to other materials such as metals, meaning that it can absorb large amounts of heat without resulting in much temperature change. With climate change, the oceans have constantly been increasing in temperature, causing the melting of icecaps, changing marine ecosystems, and the possible disruption of the ocean currents.

Ocean currents carry this extra heat throughout the entire planet, absorbing most of it and decreasing the effect on the landmasses. This circulation, called thermohaline circulation, leads warm shallow waters to flow towards the poles, cooling down and increasing in density. These then flow into the deep ocean and flow back towards the equator, heating again and once more flowing back up to the surface, creating a "conveyor belt" movement throughout the oceans, especially the Atlantic Ocean [1]. These changing ocean currents can be modeled mathematically to predict their trends. However, this is a very subtle and detailed process represented by a series of differential equations, combined to form the General Circulation Models, or GCMs [2]. Generally, these models are too complex to analyze mathematically. A more manageable alternative to work with these complex systems is to use simplified models, such as Stommel's two-box model [3].

2. Two Box Model

Stommel's model consists of two vessels, one with high temperature and salinity and one with low temperature and salinity, representing the equatorial and polar waters of the North Atlantic Ocean, respectively [3]. In Figure 1, the two vessels are connected by an overflow and capillary through which fluid flows with rate q, and diffusion occurring between the vessels and the surrounding tanks of constant temperature and salinity. Both vessels are surrounded by tanks with constant salinity and temperature, separated by

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porous walls to allow for flow between the vessel and the tank adjacent to it. The change in temperature and salinity from the two vessels can be represented by a system of differential equations, which has multiple solutions.

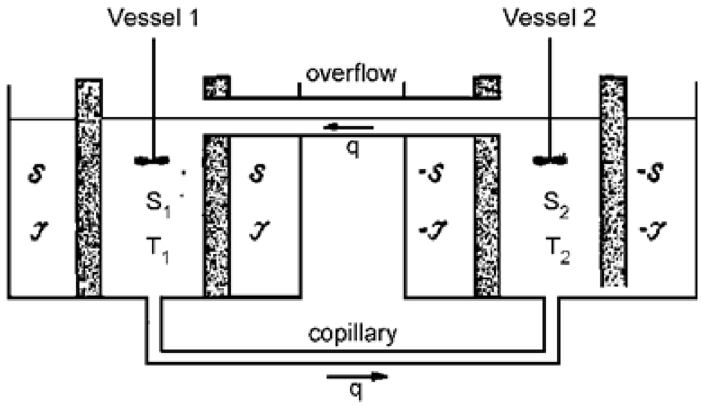


FIGURE 1. Two box model presented in Stommel's paper [3].

Using the structure of the model shown in Figure 1, it is possible to define vessel 1 as starting with higher temperature and salinity and vessel 2 as starting with lower temperature and salinity, representing the equatorial and polar parts of the North Atlantic Ocean respectively [3]. The surrounding tank for vessel 1 has constant salinity $\mathcal{S}$ and constant temperature $\mathcal{T}$, and to be able to replicate a free convective system, vessel 2 is surrounded by a tank of constant salinity $-\mathcal{S}$ and constant temperature $-\mathcal{T}$, which forces a change in the water density [3]. This change in water density is what leads a convective system to be formed between the two vessels. The entire system is defined by four different differential equations, one for each of the salinity and temperature variables.

$$\frac{dT_1}{dt} = c(\mathcal{T} - T_1) - |q|T_1 + |q|T_2 \tag{1}$$

$$\frac{dT_2}{dt} = c(-\mathcal{T} - T_2) + |q|T_1 - |q|T_2 \tag{2}$$

$$\frac{dS_1}{dt} = d(\mathcal{S} - S_1) - |q|S_1 + |q|S_2 \tag{3}$$

$$\frac{dS_2}{dt} = d(-\mathcal{S} - S_2) + |q|S_1 - |q|S_2 \tag{4}$$

The parameters c, d, and q are constants throughout these four equations, where q represents the flow between the two vessels, c is a constant for temperature, and d is a constant for salinity, and none have specified values. The system of equations can be simplified by combining the equations. By adding equations (1) and (2), a third equation is reached that only depends on the sum of T_1 and T_2, which decays exponentially to 0. The same behavior is observed for the salinity equations (3) and (4). This derivation was originally performed by Stommel and is being replicated here [3].

$$\frac{d(T_1+T_2)}{dt} = c(\mathcal{T}-T_1) + c(-\mathcal{T}-T_2) - |q|T_1 + |q|T_2 + |q|T_1 - |q|T_2 \tag{5}$$

$$\frac{d(T_1+T_2)}{dt} = -c(T_1+T_2) \tag{6}$$

$$\frac{d(S_1+S_2)}{dt} = -d(S_1+S_2) \tag{7}$$

Since equations (6) and (7) decay exponentially to 0 as time approaches infinity, this allows for the consideration of only the solutions where $S_1 = -S_2 = S$ and $T_1 = -T_2 = T$. With this, the system can be simplified further to the two differential equations below [3].

$$\frac{dT}{dt} = c(\mathcal{T}-T) - 2|q|T$$

$$\frac{dS}{dt} = d(\mathcal{S}-S) - 2|q|S$$

To simplify these even further, they can be nondimensionalized to create a system that only depends on two parameters, where $y = T/\mathcal{T}$, $x = S/\mathcal{S}$, $\delta = \frac{d}{c}$, $\tau = ct$, and $\lambda = (\frac{c}{4\rho_0\alpha\mathcal{T}})$, where ρ_0 is the density of saltwater, based on the definition used by Stommel [3]. Also, the parameter α represents the average thermal contraction coefficient, typically represented with a value of $\alpha = 1.5 \times 10^{-4}$ deg^{-1}.

$$\frac{dx}{d\tau} = \delta(1-x) - \frac{x}{\lambda}|y-Rx| \tag{8}$$

$$\frac{dy}{d\tau} = 1 - y - \frac{y}{\lambda}|y-Rx| \tag{9}$$

The parameters λ and δ are constants, while R represents the ratio of the change in salinity and the change in temperature between the two vessels [3]. This last parameter is found from the density equation of the ocean and is defined as $R = \frac{\beta\mathcal{S}}{\alpha\mathcal{T}}$, with α and β being constants [3]. The parameter R can be thought as a quantification of the relation between salinity and temperature and how they affect the flow between the vessels. The parameters $\mathcal{S}$ and $\mathcal{T}$ represent the temperature and salinity of the tanks that surround both vessels. The parameter β represents the saline expansion and has a value of $\beta = 8 \times 10^{-4}$ psu^{-1} [1]. R depends on both the variation of salinity and the temperature between the equatorial and polar components of the North Atlantic Ocean, and in recent years has changed significantly [4].

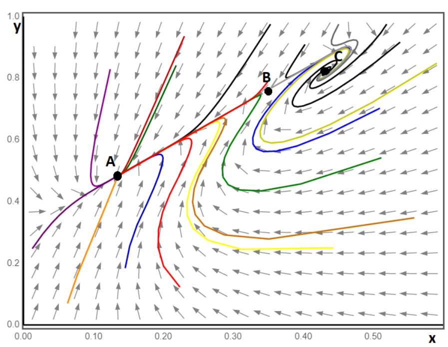
Another variable, called f, is defined originally by the equation $f = \frac{2q}{c}$, and is then nondimentionalized to $\lambda f = -y + Rx$. It represents the direction and the intensity of the flow between the two vessels. An overall trend can be seen for all values of f, with positive values representing the heat flow from higher to lower temperatures area and negative values representing the opposite flow. The positive f is related to a system where salinity is the defining element for the capillary flow, and negative f represents that temperature is the main element for the flow. This variation in f-values and the solutions also depends on R, which changes with the temperature and salinity of the ocean. The solutions of the system of differential equations are shown below.

$$\lambda f = -y + Rx$$

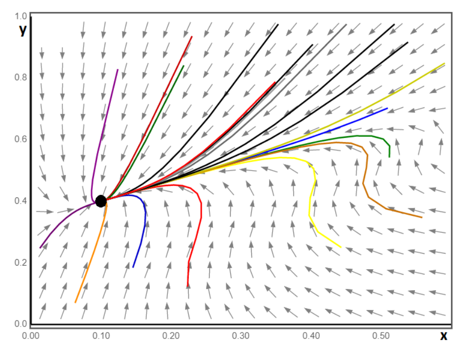
$$x = \frac{1}{1 + \frac{|f|}{\delta}}$$

$$y = \frac{1}{1 + |f|}$$

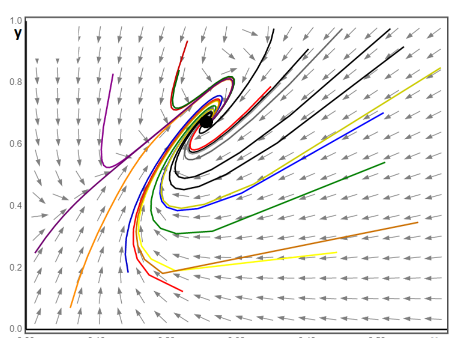
By solving the differential equations, three equilibrium solutions were found for the original values of $\delta = \frac{1}{6}$, $\lambda = \frac{1}{5}$, and $R = 2$, each represented by a different f-value [3]. Since the publishing of Stommel's paper in 1961, these parameters have been used in a large amount of research done on this model [3, 5, 6, 7]. These were for values of -1.1, -0.30, and 0.23, with two stable solutions at $f = -1.1$ and $f = 0.23$, represented in Figure 2(a) by A and C, respectively, and an unstable solution at $f = -0.30$, represented in Figure 2(a) by B. The $f = 0.23$ means that the surface flow of temperature and salinity moves from tank 1 to tank 2, representing that the flow of heat from the higher temperature equatorial oceans to the colder polar oceans, which is the current state of the ocean current. The stable equilibrium at $f = -1.1$ represents a current model of the ocean, where the low



(A) $R = 2$



(B) $R = 1$



(C) $R = 3$

FIGURE 2. Vector field of the system at different R-values, showing the different equilibrium solutions their respective R-values.

heat, low salinity water from the poles flows towards the equator on the surface, represented by an overflow from tank 2 to tank 1. This would lead to extremely large climatic changes, including the cooling of the planet and the changing of marine ecosystems since the life forms would have to adapt to the colder surface water.

Each R-value has a corresponding slope field and solutions attached to it. These three values were chosen as $R = 2$ corresponds to value used in the original paper by Stommel, while $R = 1$ and $R = 3$ were chosen as examples to model what happens with the equilibria when the value of R changes, one above and one below $R = 2$. For each new value for it, a new slope field was made to examine the solutions as stable or unstable. Figure 2(a) shows the slope field for the case of R=2, and it is possible to see two stable equilibrium solutions with one stable node at A and one stable spiral at C. The unstable equilibrium solution is labeled as B on the graph. Figure 2(c) shows the slope field with R=3, and it is now possible to see that there is only one equilibrium solution, which is a stable spiral. For the case of R=1, shown in Figure 2(b), there is again only one solution to the system of differential equations, resulting in a stable node.

3. Resilience of the System

Resilience is the characteristic that systems can absorb an imposed change and keep working, even if in a different basin of attraction from a different equilibrium state, and two different methods are further discussed [8, 9]. The resilience model used in this work is the method introduced by Cessi, where the chosen way to impose a change is to start from one of the stable equilibrium solutions and change the parameters of the system for enough time to force the original solution to switch to the basin of attraction of the other equilibrium solution [7]. Using this resilience model, the system of differential equations described by (5) and (6) will be analysed by changing the value of R while maintaining the other parameter values constant.

4. Method and Results

Starting with the system at $R = 2$, the coordinates of the stable spiral, which were found to be at $(0.43205, 0.82028)$, were used as the initial condition. Then, the system was switched to the new parameter, the new value of R, for a certain period of time. The coordinates of where the new system was at after this time was then returned to the original system of equations, where $R = 2$, and was let freely flow to see if the system would return to the stable spiral. This process was repeated to determine the maximum time length the system can remain with the new parameter values before it does not return to the stable spiral once the system is switched back to $R = 2$, instead tending towards the stable node. This procedure was repeated for several R values ranging from $R = 6$ to $R = 2.5$, where $R = 2.5$ is the minimum value of R where a switch between equilibrium solutions happens. The pattern observed between all of the new systems is that the larger the difference from $R = 2$, the shorter time the system has before it switches to the new equilibrium state.

Figure 3 shows the minimum time a perturbation needs to last to be able to switch between equilibrium solutions for each different value of ΔR. This result matches with the

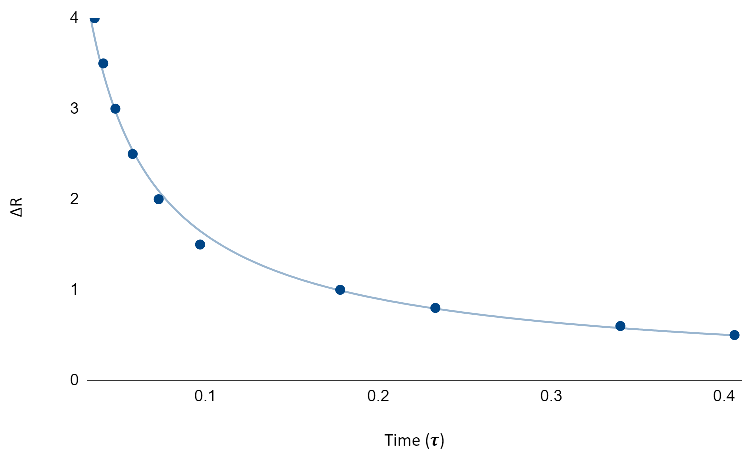


FIGURE 3. The minimum time length of a perturbation required to switch equilibrium solutions based on the difference from $R = 2$.

information found by Cessi, which also showed the resilience of the system to be in the form of a power function [7]. The larger disturbances, or variations, need much shorter times to cause a shift between solutions.

5. Discussion and Conclusion

Recently, the temperature of the surface of the planet has increased substantially, causing the value of R to change. Climate change leads to an increase in global temperature, which leads to ice melting. Both glaciers and sea ice have a lower salinity than seawater, so as temperature rises and they melt, there is an increase in the amount of freshwater present, which decreases the salinity of the ocean [1, 10]. With the increase in temperature and the decrease in salinity, the value of R decreases, leading to a weakening of the North Atlantic current. Research done by the National Oceanic and Atmospheric Administration indicates that there has been a significant increase in ocean temperature in the last few decades [4]. Additionally, there is evidence that the North Atlantic ocean current is already starting to slow down [11, 12]. The data indicates that the value of R has begun to decrease since temperatures are increasing due to global warming. The slowing of the ocean currents leads to less heat being carried to the North Atlantic region, causing a cooling of the region and a heating of the equatorial region [13].

In this paper, we have explored the response of Stommel's model as the parameter R changes. We observed that a sufficiently large change in R induces the model to switch to a new steady state. We saw the system remains in the new state, even if R is returned to its original value. We quantified the relation between the magnitude of the change in R and amount of time that the change must be maintained to reach the tipping point. Although this model does not lend itself to quantification of actual ocean circulation, it illustrates the urgency of mitigating the factors leading to climate change to avoid the possibility of tipping the Earth's climate system to a different state from which it will be difficult to return.

6. Acknowledgments

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