The Minnesota Journal of Undergraduate Mathematics

The Mathematics Behind an Elimination Game

Cao Zhichun

Victoria Junior College

The Minnesota Journal of Undergraduate Mathematics

Volume 4 (2018-2019 Academic Year)

Sponsored by
School of Mathematics
University of Minnesota
Minneapolis, MN 55455

The Minnesota Journal of Undergraduate Mathematics

Volume 4 (2018-2019 Academic Year)

The Mathematics Behind an Elimination Game

Cao Zhichun

Victoria Junior College

Abstract. The Josephus problem is related to a game where soldiers stand in a circle and are killed based on a given rule. We propose an elimination game that keeps the killing process but positions soldiers in a straight line such that a soldier on an odd-numbered position is uniformly chosen and eliminated each time. After the modification, who survives the longest? Which soldier becomes the last to die? In this article, we employ induction and Markov chain method to algebraically derive each soldier's survival probability. Subsequently, we simulate the game to show that the soldier with the longest life expectancy does not necessarily emerge as the last survivor.

1. The Elimination Game

The Josephus problem is famous in the fields of computer science and mathematics. In the original problem, 41 soldiers stand in a circle and counting starts at a specific soldier. Every second soldier is eliminated in the process of counting until there is one soldier left. The problem is to find the position in the initial circle to avoid elimination. Recursive and non-recursive approaches are often employed in solving the problem [2]. Over the years, the problem has elicited multiple extensions, which can be found in [5], [3], [4]. In the original problem, a circle is formed by the soldiers, leading us to consider if a straight line is formed instead.

We thus modify the Josephus problem to an elimination game. In the game, n soldiers stand in a row with positions numbered from 1 to n. Each time, a soldier is uniformly chosen among all soldiers on odd positions to be eliminated. Positions of remaining soldiers are renumbered after each elimination (i.e., the position number of a soldier decreases by 1 if a soldier before him or her is eliminated, else it is left unchanged). The process ends when there is one soldier left. To better illustrate the elimination process, we use Figure 1 to include all the possible cases when $n = 5$. In Figure 1, each row shows all the possible outcomes after one elimination and each branch in a row is equally likely.

In this article, we introduce the definitions of survival probability and life expectancy to discover the influence of starting positions on the safety of soldiers.

Definition 1.1. For positive integers $n \geqslant k \geqslant 1$, the survival probability $P(k,n)$ is the probability that the soldier starting at position k is the last survivor of the n soldiers.

* *Corresponding author*

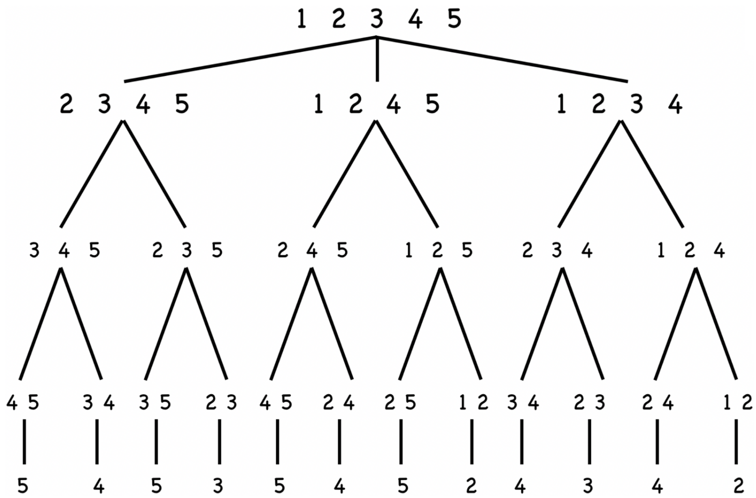


Figure 1. Cases when $n = 5$. Each row shows the outcomes after one elimination and each branch in a row is equally likely. The game ends when one soldier is alive.

Definition 1.2. For positive integers $n \geqslant k \geqslant 1$, the life expectancy $E(k,n)$ is the expected number of eliminations required for the soldier starting at position k among n soldiers to be eliminated.

The value of $E(k,n)$ is calculated as the ratio of sum of rounds of survival to the number of elimination outcomes. For instance, the number of elimination outcomes is 12 when $n = 5$, by adding up the number of rounds of eliminations that the soldier with starting position 2 (i.e., $k = 2$) survives from each of the 12 branches, we obtain

$$E(2,5) = \frac{1+1+2+3+2+3+3+4+2+3+3+4}{12} = \frac{31}{12}.$$

Naturally, we would ask: which soldier lives the longest? Which soldier becomes the last to die? Interestingly, the answers to these two questions are different. Contrary to our intuition, the soldier with the highest $E(k,n)$ does not necessarily coincide with the one with the highest $P(k,n)$. In subsequent sections, we employ induction, Markov chain method and simulation to show the results that the highest $P(k,n)$ is achieved when $k = n$, while the highest $E(k,n)$ is achieved when $k = 2$.

2. Induction

Theorem 2.1. *Let $n \geqslant k \geqslant 1$ be integers,*

$$P(k,n) = \frac{\lfloor \frac{k}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor \lfloor \frac{n+1}{2} \rfloor}.$$

Proof. We will prove Theorem 2.1 by induction. According to the parity of k and n, Theorem 2.1 can be rewritten in the table below.

Obviously, the result holds true for base cases $n = 1$ and $n = 2$.

k, n	odd n	even n
odd k	$\frac{2(k-1)}{n^2-1}$	$\frac{2(k-1)}{n^2}$
even k	$\frac{2k}{n^2-1}$	$\frac{2k}{n^2}$

Moving up to the inductive step, we assume $P(k,n)$ is true for some $n \geqslant 1$, we need to show that $P(k,n+1)$ is true. In this paper, we prove the case when n and k are both even in detail.

Since k is even, there are $\frac{k}{2}$ odd positions in front of k and $(\frac{n-k}{2}+1)$ odd positions behind k. Hence, we can calculate the probability of eliminating an odd-numbered position in front of or behind position k.

$$P(\text{front}) = \frac{\frac{k}{2}}{\frac{n}{2}+1} = \frac{k}{n+2}.$$

$$P(\text{behind}) = \frac{\frac{n-k}{2}+1}{\frac{n}{2}+1} = \frac{n-k+2}{n+2}.$$

Considering $P(k,n+1)$, for the soldier at position k to survive the next elimination and eventually be the last survivor of the $(n+1)$ soldiers, either a soldier standing in front of position k is eliminated and position $(k-1)$ then survives till the last out of n soldiers, or any one soldier standing behind position k is eliminated and position k then survives till the last out of n soldiers. We thus obtain the following:

$$\begin{aligned} P(k,n+1) &= P(\text{front}) \times P(k-1,n) + P(\text{behind}) \times P(k,n) \\ &= \frac{k}{n+2} \times P(k-1,n) + \frac{n-k+2}{n+2} \times P(k,n). \end{aligned}$$

With the assumption that $P(k,n)$ is true,

$$\begin{aligned} P(k,n+1) &= \frac{k}{n+2} \times P(k-1,n) + \frac{n-k+2}{n+2} \times P(k,n) \\ &= \frac{k}{n+2} \times \frac{2(k-1-1)}{n^2} + \frac{n-k+2}{n+2} \times \frac{2k}{n^2} \\ &= \frac{2k}{n^2+2n} \\ &= \frac{2k}{(n+1)^2-1}, \end{aligned}$$

which is true from the table when k is even and $(n+1)$ is odd. The other three cases, with n and k of different parities, can be shown in a similar manner. $\square$

3. Markov Chain Method

To better understand the change of positions of soldiers during the game, we employ Markov chains (see [1]) to model the elimination process, as the state of soldiers is conditional on its present arrangement and independent of the past.

Definition 3.1. For positive integer $n \geqslant k \geqslant 1$, M_n denotes the transition matrix when n soldiers are alive at a point in the game, with dimensions $(n+1)\times(n+1)$. $M_{P_{(k,n)}}$ denotes the matrix of probability distribution for the soldier starting at position k till the end of eliminations among n soldiers. $M_{\{n,i,j\}}$ represents the probability for a soldier to change from position $(j-1)$ to position $(i-1)$ when there are n soldiers alive, where $M_{\{n,i,j\}}$ is the element on the ith row and jth column in M_n.

Definition 3.2. Position 0 is defined as the position for eliminated soldiers during the game.

Definition 3.3. M_{pos} denotes the column matrix representing the soldier starting at position k. When there are n soldiers alive, the column matrix has dimensions $(n+1)\times 1$, where we write 1 on the $(k+1)$th row and 0 on remaining rows to represent the soldier's position.

The definitions can be best illustrated by an example where $n = 5$. We have a 6×6 matrix for M_5 as position 0 is where eliminated soldiers end up with.

$$M_5 = \begin{bmatrix} 1 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In M_5, $M_{\{5,1,4\}} = \frac{1}{3}$, suggesting that the soldier starting at position 3 has a $\frac{1}{3}$ probability to change to position 0 after one elimination, since soldiers on positions 1, 3 and 5 have equal probability to be eliminated.

For the soldier starting at position 3 out of 5 soldiers, M_{pos} is a 6×1 matrix, and the element on the 4th row and 1st column is 1. Hence,

$$M_{pos} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

By inductive steps, the generalized transition matrix in 1 can be derived. Note that if there are even number of soldiers at the start of the game, the last soldier does not disrupt the probability for soldiers in front to change between positions.

$$\begin{aligned} M_{2n} &= M_{2n-1} \\ &= \begin{bmatrix} 1 & \frac{1}{n} & 0 & \frac{1}{n} & \cdots & 0 & \frac{1}{n} & 0 \\ 0 & \frac{n-1}{n} & \frac{1}{n} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{n-1}{n} & \frac{1}{n} & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{n-2}{n} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{n-1}{n} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{n} & \frac{n-1}{n} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{1}$$

By matrix multiplication of Markov chains,

$$\begin{aligned} M_{P_{(k,n)}} &= M_2 M_3 \cdots M_{n-1} M_n M_{pos} \\ &= \prod_{i=2}^{n} M_i M_{pos}. \end{aligned} \tag{2}$$

The formula 2 derived can calculate survival probability of any soldier at the end of the game. However, we are also curious to find out what positions each soldier can take in the middle of the game.

Definition 3.4. Let m be the number of eliminations that have taken place. For $1 \le m \le n-1$, $P(k,m,n)$ is the probability that the soldier starting at position k survives after m eliminations among n soldiers, and $M_{P_{(k,m,n)}}$ denotes the matrix of probability distribution for the soldier starting at position k after m eliminations among n soldiers.

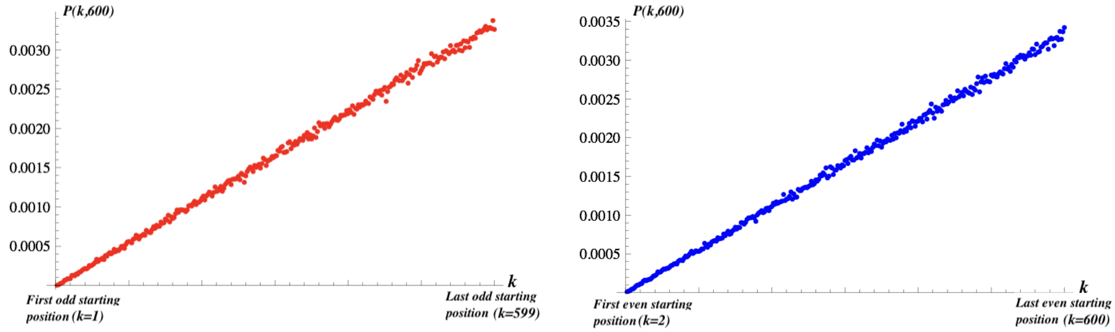
Formula 3 is derived in a similar form as 2. Since there are $(n-m)$ soldiers alive after m eliminations,

$$\begin{aligned} M_{P_{(k,m,n)}} &= M_{n-m+1} M_{n-m+2} \cdots M_{n-1} M_n M_{pos} \\ &= \prod_{i=n-m+1}^{n} M_i M_{pos}. \end{aligned} \tag{3}$$

4. Simulation

Setting $n = 600$, we use Python to simulate the game. We are curious if the experimental results of survival probability corresponds to our probability formulae found in previous sections.

From Figure 2, after separating positions numbers based on the parity, we observe a linear trend of survival probability conforming to Theorem 2.1, validating our result that the

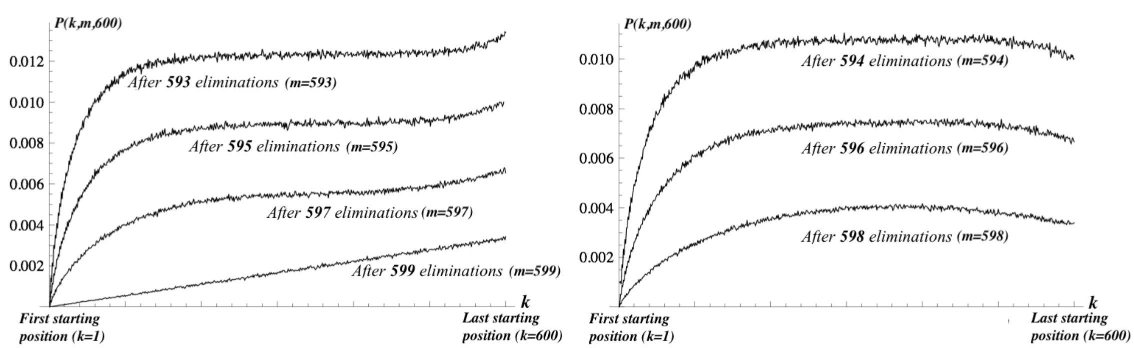


(A) (B)

Figure 2. Results of simulation of probability for a soldier to be the last to die starting with (A) odd and (B) even positions respectively after 100,000 iterations. Survival probability is low for soldiers starting at a front position, and increases as the starting position number increases.

last soldier has the highest survival probability (i.e., the highest $P(k,n)$ is achieved when $k = n$).

However, surviving till the last is a special case of 599 eliminations with 600 soldiers. We are curious to see how survival probability varies for the last few soldiers alive in order to discover the significance of each elimination on $P(k,m,n)$ as the game approaches the end. Thus, we let the number of eliminations range from 593 to 598 and simulate the game to obtain Figure 3.



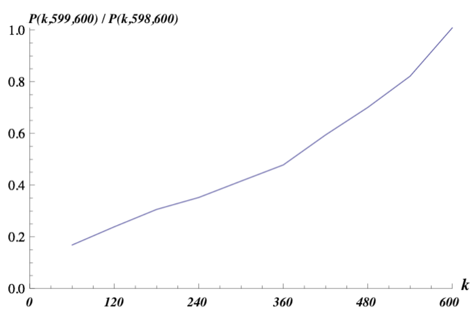
(A) After odd eliminations (B) After even eliminations

Figure 3. Results of simulation of probability for a soldier to remain alive after (A) odd-numbered and (B) even-numbered eliminations against position number with 600 soldiers at the start of the game after 100,000 iterations. Linearity of probability to survive is lost when more than one soldier is alive and the direction of ending part of each curve depends on whether the elimination number is odd or even.

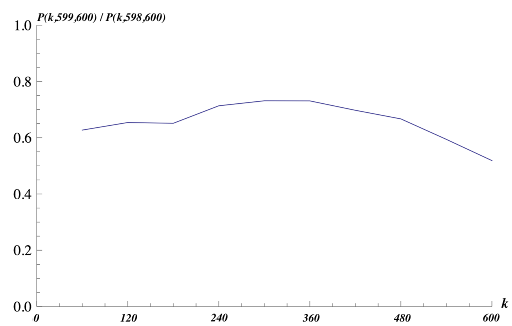
From Figure 3, the linear shape of curves is lost when more than one soldier is alive. The curves shift down as elimination number increases. This makes intuitive sense as area under the curve represents the number of soldiers alive. More interestingly, the orientation of tail for each curve is dependent on the parity of elimination number, suggesting fluctuating survival probability.

Additionally, we observe that the fluctuation in $P(k,m,600)$ is not equally drastic for soldiers with different starting positions (i.e., values of k) as m changes. For $1 \leq k \leq 50$, the survival probabilities of soldiers are consistently low throughout the game. For $51 \leq k \leq 200$, these soldiers are comparatively safer during the early part of the game. However, for the final few eliminations, this group of soldiers are more prone to be eliminated than the ones where $201 \leq k \leq 600$.

To investigate this, we plot $\frac{P(k,599,600)}{P(k,598,600)}$ and $\frac{P(k,598,600)}{P(k,597,600)}$ for every 60th soldier to observe the general trend.



(A)



(B)

Figure 4. Results of simulation of probability for a soldier to survive one elimination given he or she survives the previous elimination with 600 soldiers at the start of the game after 100,000 iterations. Only the 60th soldiers are chosen.

The increasing ratio in Figure 4a suggests it is less likely for soldiers near the front to survive the 599th elimination, confirming that at the end of the elimination game, soldiers starting at positions at the front have higher probability to be eliminated than soldiers starting at the back. This shows bias of the elimination process towards soldiers at the front.

However, if we look at Figure 4b, the ratio becomes generally a constant. This is because all the soldiers starting at 60th positions are now on odd-numbered positions after the 597th elimination, which causes their survival probability to be adjusted and balanced in the course of the game.

Besides survival probability, we simulate the game to calculate $E(k,n)$ for all soldiers at the end of the game.

From Figure 5, the soldier starting at position 2 has the highest life expectancy (i.e., the higest $E(k,n)$ is achieved when $k = 2$). Moreover, $E(k,n)$ fluctuates as position number increases initially. More specifically, it increases as odd starting positions increase, while decreases as even starting positions increase. The change in values of $E(k,n)$ is significant for the first ten soldiers approximately. However, it quickly approaches to 300 for soldiers

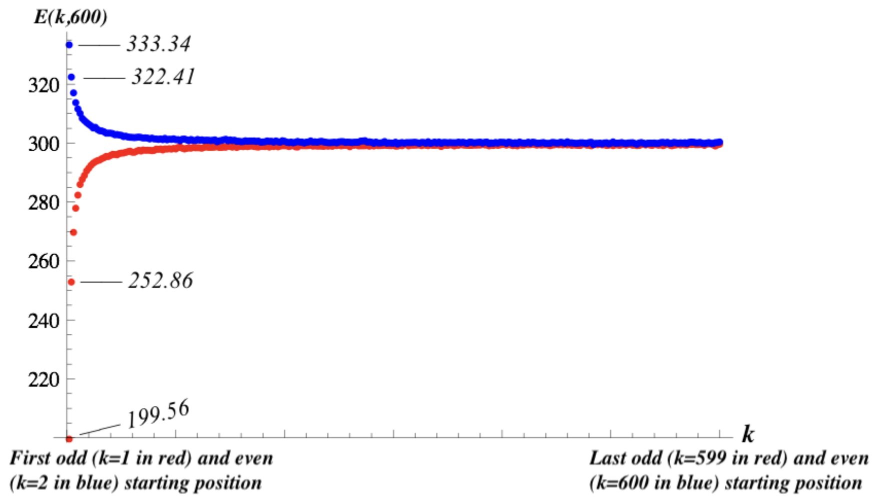


Figure 5. Results of simulation of life expectancy of soldiers starting from odd (plotted in red) and even (plotted in blue) positions with 600 soldiers at the start of the game after 100,000 iterations. $E(k,600)$ increases when k increases and k is odd, and $E(k,n)$ decreases when k increases and k is even. For large values of k, $E(k,600) \approx 300$.

starting at the back. This number is exactly half the number of soldiers at the start of the game.

5. Conclusion

In this article, we devise an elimination game where n soldiers stand in a row with positions numbered from 1 to n. Each time, a soldier is uniformly chosen among all soldiers on odd positions to be eliminated. Positions of remaining soldiers are renumbered after each elimination.

From previous sections, we have shown that the last soldier tends to become the last to die, while the soldier starting at position two tends to live the longest. This result is interesting as two seemingly closely associated questions, namely the starting positions that give soldiers the highest survival probability and life expectancy, actually lead to different answers in the context of the elimination game. Furthermore, we have found that, when starting position number is separated into odd and even numbers, survival probability $P(k,n)$ increases linearly as position number k increases. Meanwhile, life expectancy $E(k,n)$ fluctuates rapidly for soldiers starting at the front and gradually approaches to $\frac{n}{2}$ for soldiers starting at the back.

6. Acknowledgements

I would like to thank Professor Lee Ching Hua from National University of Singapore and Professor Steven J. Miller from Williams College for their insight and expertise that

greatly assisted the research, although any errors are my own and should not tarnish the reputations of these esteemed persons.

References

[1] Geoffrey Grimmett and David Stirzaker. *Probability and random processes*. Oxford university press, 2001.

[2] Lorenz Halbeisen and Norbert Hungerbühler. The Josephus problem. *J. Théor. Nuombres Bordeaux*, 9:303–318, 1997.

[3] Frank Ruskey and Aaron Williams. The feline Josephus problem. *Theory of Computing Systems*, 50(1):20–34, 2012.

[4] Armin Shams-Baragh. Formulation of the extended Josephus problem. In *National Computer Conference*, 2002.

[5] Shivam Sharma, Raghavendra Tripathi, Shobha Bagai, Rajat Saini, and Natasha Sharma. Extension of the Josephus problem with varying elimination steps. *DU Journal of Undergraduate Research and Innovation*, 1(3):211–218, 2015.

Student biographies

Cao Zhichun: (*Corresponding author:* cao.zhichun.2017@vjc.sg) Cao Zhichun is a Senior at Victoria Junior College in Singapore. He is interested in studying Mathematics and Computer Science at the undergraduate and postgraduate levels.