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Abstract. We first give simple examples of pairs of functions with constant Wronskians. Next, we consider three differentiable (not necessarily analytic) real functions such that their Wronskians, when taken in pairs, are constants, and show the following possibilities: (i) constants are zero, (ii) functions are linearly dependent through those constants.

1. Introduction

For n real functions $f_1,...,f_n$ of x, that are $n-1$ times differentiable on an interval I, their Wronskian is defined by the determinant

$$W(f_1,\dots,f_n) = \begin{vmatrix} f_1 & \dots & f_n \\ f_1' & \dots & f_n' \\ . & \dots & . \\ . & \dots & . \\ . & \dots & . \\ f_1^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}.$$

The Wronskian of two differentiable functions f and g is the determinant $W(f,g) = fg' - gf'$. If the functions $f_1,...,f_n$ are linearly dependent, then the Wronskian vanishes. Therefore, the functions are linearly independent on I if their Wronskian W does not vanish identically on I. We note the following result of Peano [5] which says that, if $W = 0$ for all $x \in I$, then the functions may or may not be linearly dependent. For example, the functions x^2 and $x|x|$ have continuous derivatives and their Wronskian vanishes everywhere, but are not linearly dependent in any neighborhood of 0. It can be easily verified that this holds more generally for functions x^p and $x^{p-1}|x|$ for $p \geq 2$. That the identical vanishing of n analytic functions is a necessary and sufficient condition for their linear dependence, was pointed by Peano [5] and proven by Bôcher [2].

As the identical vanishing of the Wronskian of two analytic functions implies their linear dependence, it is interesting to examine functions (not necessarily analytic) whose Wronskians are constants (not necessarily zero). Examples of such functions in pairs are $(\cos x, \sin x)$, (e^x, e^{-x}), and $(1, x)$. These pairs of functions are solutions of the following

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second order linear homogeneous differential equations $y'' + y = 0$ (simple harmonic motion), $y'' - y = 0$ and $y'' = 0$ respectively. This is supported by Abel's identity (see Redheffer and Port [7]): $W' = -p(x)W$, where W is the Wronskian of the solutions of the differential equation $y'' + p(x)y' + q(x)y = 0$. So, if W is constant, then $p(x) = 0$. In this paper we consider such functions (*not necessarily analytic*) with constant Wronskians and prove the following result.

2. Main Result

Theorem 2.1. *Let f, g, h be twice differentiable functions on an interval I such that $W(f,g) = k_1$, $W(g,h) = k_2$ and $W(h,f) = k_3$ where k_1, k_2, k_3 are constants. Then the following hold. (i) $k_1 = k_2 = k_3 = 0$, (ii) f, g, and h are related by*

$$k_1 h + k_2 f + k_3 g = 0.$$

Remark: Case (i) of Theorem 2.1 seems pretty intriguing and does not give us any idea as to the linear dependence or independence of the three functions, and therefore needs further investigation. A trivial example for this possibility is $f(x) = x^2$, $g(x) = x|x|$ and $h(x) = -x|x|$ for $x \in \mathbb{R}$. However, we could not find an example when f, g and h would be linearly independent.

The proof of Theorem 2.1 is based on the following lemma.

Lemma 2.2. *Any three twice differentiable functions f, g and h satisfy the identity (known as Jacobi identity, see Arnold* [1]*):*

$$W(W(f,g),h) + W(W(g,h),f) + W(W(h,f),g) = 0. \tag{1}$$

Though this result is known (see Lawvere [4] and Poinsot [6]), we were not aware of it during the preparation of this article. So, we include its proof for the sake of completeness. A direct computation shows that

$$W(W(f,g),h) = fg'h' - gf'h' - hfg'' + hgf''.$$

Permuting f, g, h twice in the above equation and then adding the three equations yields equation (1), proving the lemma.

Proof Of Theorem 2.1 Using the hypothesis in Lemma 2.2 we get

$$\begin{vmatrix} k_1 & h \\ 0 & h' \end{vmatrix} + \begin{vmatrix} k_2 & f \\ 0 & f' \end{vmatrix} + \begin{vmatrix} k_3 & g \\ 0 & g' \end{vmatrix} = 0,$$

i.e.

$$k_1 h' + k_2 f' + k_3 g' = 0.$$

When integrated, it gives

$$k_1 h + k_2 f + k_3 g = C, \tag{2}$$

where C is an arbitrary constant. We have the following possibilities: I. $k_1 = k_2 = k_3 = 0$ which implies $C = 0$ and corresponds to part (i) of the Theorem. II. $C = 0$ and one of k_1, k_2, k_3 is non-zero, and this corresponds to part (ii) of the Theorem. III. $C \neq 0$ which implies that one of k_1, k_2, k_3 must be non-zero, so let $k_1 \neq 0$. Let us pursue the possibility III. First we note from equation (2) that

$$h = -\frac{k_2}{k_1} f - \frac{k_3}{k_1} g + \frac{C}{k_1}. \tag{3}$$

Substituting the value of h from (3) in the hypothesis $W(h, f) = k_3$ and using $W(f, g) = k_1$, and then simplifying we obtain $\frac{C}{k_1} f' = 0$. As $C \neq 0$, we conclude that f is constant, say C_1. Here we note that $C_1 \neq 0$ because if $C_1 = 0$, there would be a conflict with $k_1 = 0$. Next, substituting $f = C_1$ in $W(f, g) = k_1$ we find $C_1 g' = k_1$ which integrates to

$$g = \frac{k_1}{C_1} x + C_2, \tag{4}$$

where C_2 is an arbitrary constant. Now substituting $f = C_1$ in $W(h, f) = k_3$ provides $-C_1 h' = k_3$. Integrating it we get

$$h = -\frac{k_3}{C_1} x + C_3. \tag{5}$$

Finally, substituting results from (4) and (5) into $W(g, h) = k_2$ and simplifying we see that $C_3 = -\frac{k_2}{k_1} C_1 - \frac{k_3}{k_1} C_2$. Consequently, we obtain

$$f = C_1, g = \frac{k_1}{C_1} x + C_2, h = -\frac{k_3}{C_1} x - \frac{k_2}{k_1} C_1 - \frac{k_3}{k_1} C_2.$$

Using these values of f, g and h in (2) shows $C = 0$, contradicting the possibility III. This completes the proof. □

3. Concluding Remark

The Wronskian is a Lie bracket (as is the cross product of 3-dimensional vectors).

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