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Abstract. In this note, I study the result and proof of the classical Salem-Zygmund Theorem. I apply the method to random orthogonal polynomials on the unit circle. The goal is to find the distribution of M, the sup norm of suitably defined random polynomial orthogonal on the unit circle. In my proof, I use Bernstein and Chebyshev inequalities to achieve this goal. I find that for fixed large κ, the probability of $M > \kappa$ drops significantly as the degree n of the orthogonal polynomial grows.

1. Introduction

Eigenfunctions of discrete Schrödinger operator in mathematical physics can be rewritten in terms of polynomials orthogonal on the unit circle. The recursion coefficients for the latter are related to potential in the Schrödinger operator. Making the recursion coefficients random is not only a natural choice for condensed matter physics in the Anderson model, but also related to the theory of semiconductors.

In this section, I will recall the famous Salem-Zygmund theorem and other results including Bernstein and Chebyshev inequalities. I will also introduce orthogonal polynomials on unit circle that will be used in my proof of the main result.

Salem-Zygmund Theorem. *Consider the random trigonometric polynomial*

$$T_n = \sum_{j=-n}^{n} a_j e^{ijx} b_j,\ x \in \mathbb{R},\ a_j \in \mathbb{C},$$

where b_j's are subnormal, independent, and identically distributed random variables. Then

$$P\left(\|T_n\|_{L^\infty(\mathbb{R})} \geq 3 \left(\sum_{j=-n}^{n} |a_j|^2 log(Cn\kappa)\right)^{1/2} \right) \leq \frac{2}{\kappa},$$

where C is a positive absolute constant.

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Remark. The above result comes from the book *Some Random Series of Functions* by Jean-Pierre Kahane [2]. For the formal proof, see Theorem 1, page 55. The proof uses Bernstein's inequality.

Bernstein's Inequality [1]. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree at most* n *with complex coefficients. Then* $\|p'\|_{L^\infty(\mathbb{T})} \leq n\|p\|_{L^\infty(\mathbb{T})}$, *where* $\mathbb{T}$ *denotes the unit circle.*

The Salem-Zygmund theorem is important because it gives an estimate for the distribution of $\|T_n\|_{L^\infty(\mathbb{R})}$. The probability $P(\|T_n\|_{L^\infty(\mathbb{R})} \geq 3(\sum_{-n}^{n} |a_j|^2 log(Cn\kappa))^{1/2})$ drops significantly if κ is large.

Chebyshev's inequality. *Let* $(\Omega, \mathcal{A}, P)$ *be a probability space and let* X *be a random variable. Then for any* $\epsilon > 0$, $P(|X| \geq \epsilon) \leq \frac{E|X|^r}{\epsilon^r}$ *provided that* $E|X|^r < \infty$, $0 < r < \infty$.

In the next section, I will apply the method used in the Salem-Zygmund Theorem to random orthogonal polynomials on the unit circle. I will use Bernstein's inequality and Chebyshev's inequality to obtain the result.

1.1. **Orthogonal polynomials on unit circle.** Let $\mathbb{D}$ be the open unit disk, $\{z : |z| < 1\}$, in $\mathbb{C}$ and let μ be an arbitrary nontrivial (i.e., its support is an infinite set) probability measure (i.e., μ is nonnegative and normalized by $\mu(\mathbb{T}) = 1$) on $\mathbb{T}$, the unit circle $\{z : |z| = 1\}$ parametrized by $z = e^{i\theta}$. Define the inner product on the Hilbert space $\mathcal{H} = L^2(\mathbb{T}, d\mu)$ by $\langle f, g\rangle_\mu = \int_0^{2\pi} \overline{f(e^{i\theta})} g(e^{i\theta}) d\mu(\theta)$.

Because 1, z, z^2, ... are linearly independent in $\mathcal{H}$, we use the Gram-Schmidt process to define the monic orthogonal polynomials for μ by $\phi_n(z) = z^n - P_n[z^n]$, where P_n is projection onto $\{1, ..., z^{n-1}\}^\perp$. The orthonormal polynomials are $\varphi_n = \frac{\phi_n}{\|\phi_n\|_\mu}$. Thus, $\phi_n(z) = z^n +$ lower order, and $\varphi_n(z) = k_n z^n +$ lower order, where $k_n = \|\phi_n\|_\mu^{-1}$. Following the *Szegő Difference Equation*, we define ϕ_n^*, the reversed polynomial, by $\phi_n^*(z) = z^n \overline{\phi_n(1/\bar{z})}$. Notice that $|\phi_n| = |\phi_n^*|$ for $z \in \mathbb{T}$.

Lastly, by the *Szegő Recursion*, for any nontrivial probability measure μ on $\mathbb{T}$, we have a sequence $\{a_j\}_{j=0}^{\infty}$ of numbers in $\mathbb{D}$ so that

$$\begin{cases} \phi_{j+1}(z) = z\phi_j(z) - \overline{a_j}\phi_j^*(z), & \phi_0(z) = 1, \\ \phi_{j+1}^*(z) = \phi_j^*(z) - a_j z \phi_j(z), & \phi_0^*(z) = 1, \end{cases} \tag{1}$$

where $|a_j| < 1$. Notice that the recursion is equivalent to

$$\begin{pmatrix} \phi_{j+1} \\ \phi_{j+1}^* \end{pmatrix} = A_j(z) \begin{pmatrix} \phi_j \\ \phi_j^* \end{pmatrix},$$

where matrix $A_j(z) = \begin{pmatrix} z & -\overline{a_j} \\ -a_j z & 1 \end{pmatrix}$[3]. This recursion provides a bijection between the set $\{a_j\} \in \mathbb{D}^\infty$ and the set of all nontrivial probability measures on $\mathbb{T}$.

2. Main Result

Theorem 2.1. *Consider a large parameter $n \in \mathbb{N}, n > 1$ and define the recursion coefficients $\{a_j\}$ in (1) by*

$$a_j = \frac{1}{\sqrt{n \log n}} b_j, \quad j = 0, \ldots, n,$$

where $b_j = e^{i\beta_j}$ and $\{\beta_j\}$'s are independent random variables each uniformly distributed over $[0, 2\pi]$. Define $M = \|\phi_n(z)\|_{L^\infty(\mathbb{T})}$. Then, there is a constant c such that we have a bound

$$P(M > \kappa) \le \frac{2\pi}{n^{(\log\kappa - \log 2 - c - 1)}}. \tag{2}$$

Remark. When $\log\kappa > 1 + c + \log 2$, the right-hand side goes to 0 as n goes to infinity.

Remark. The choice of a_j makes the uniform norm of the polynomial bounded with high probability. Some variations of the choice of recursion coefficients is possible (for example, some decay in n). However, the choice of $a_j = \frac{1}{\sqrt{n \log n}} b_j$ is the most natural one. It is important to realize that without randomness, the decay one needs for the polynomial to be bounded is much stronger: $\frac{1}{n}$.

Proof. From recursion, we have

$$\begin{aligned}\phi^*_{j+1}(z) &= \phi^*_j(z) - a_j z \phi_j(z) = \phi^*_j(z) - a_j z \overline{\phi^*_j(z)} z^j \\ &= \phi^*_j(z)\left(1 - z^{j+1} a_j \frac{\overline{\phi^*_j(z)}}{\phi^*_j(z)}\right) = \phi^*_j(z)(1 - z^{j+1} a_j e^{-2i\alpha_j})\end{aligned}$$

where $z \in \mathbb{T}$ and α_j is a real-valued random variable that depends on $\beta_0, \ldots, \beta_{j-1}$ only. Computing the conditional expectation and letting $\lambda = \log n$, we have

$$|\phi^*_{j+1}|^\lambda = |\phi^*_j(1 - z^{j+1} a_j e^{-2i\alpha_j})|^\lambda,$$

$$E(|\phi^*_{j+1}|^\lambda) = E_{\beta_0,\ldots,\beta_{j-1}}\Big(|\phi^*_j|^\lambda \cdot E(|1 - z^{j+1} a_j e^{-2i\alpha_j}|^\lambda | \{\beta_0, \ldots, \beta_{j-1}\})\Big). \tag{3}$$

We plug in $z = e^{i\theta}$ and $a_j = \rho e^{i\beta_j}$, where $\rho = \frac{1}{\sqrt{n \log n}}$. Notice that

$$\begin{aligned}|1 - z^{j+1} a_j e^{-2i\alpha_j}|^\lambda &= |1 - e^{i\theta(j+1)} a_j e^{-2i\alpha_j}|^\lambda = |1 - e^{i\theta(j+1)} \rho e^{i\beta_j} e^{-2i\alpha_j}|^\lambda \\ &= |1 - \rho e^{i(\theta(j+1) - 2\alpha_j + \beta_j)}|^\lambda.\end{aligned}$$

For our convenience, let ξ_j be a random variable, $\xi_j = \theta(j+1) - 2\alpha_j + \beta_j$. Given α_j, $e^{i\xi_j}$ is uniformly distributed over $\mathbb{T}$ for all θ and j. Then, we have

$$|1 - z^{j+1} a_j e^{-2i\alpha_j}|^\lambda = |1 - \rho e^{i\xi_j}|^\lambda$$

and (3) implies $E(|\phi^*_{j+1}|^\lambda) = \gamma E(|\phi^*_j|^\lambda)$, $\gamma = (2\pi)^{-1} \int_0^{2\pi} |1 - \rho e^{i\phi}|^\lambda d\phi$. Iterating this identity and using $\phi^*_0 = 1$, we get

$$E(|\phi^*_0|^\lambda) = 1, \ldots, E(|\phi^*_{n+1}|^\lambda) = \gamma^{n+1}.$$

Let us estimate γ. Notice that

$$\begin{aligned}|1-\rho e^{i\phi}|^{\lambda} &= |(1-\rho e^{i\phi})^{\frac{\lambda}{2}}|^2 = |\exp(\frac{\lambda}{2}\log(1-\rho e^{i\phi}))|^2 = |\exp(\frac{\lambda}{2}[(-\rho e^{i\phi})+O(\rho^2)])|^2\\ &= |e^{-\frac{\lambda}{2}\rho e^{i\phi}}\cdot e^{O(\rho^2\lambda)}|^2 = |(1-\frac{\lambda}{2}\rho e^{i\phi}+O(\rho^2\lambda^2))\cdot(1+O(\rho^2\lambda))|^2\\ &= |1-\frac{\lambda}{2}\rho e^{i\phi}|^2\cdot(1+O(\rho^2\lambda^2))\end{aligned}$$

provided that $\lambda\rho<1$. Therefore,

$$\begin{aligned}\gamma &= \frac{1}{2\pi}\int_0^{2\pi}|1-\frac{\lambda}{2}\rho e^{i\phi}|^2\cdot(1+O(\rho^2\lambda^2))d\phi\\ &= \frac{1+O(\rho^2\lambda^2)}{2\pi}\int_0^{2\pi}|1-\frac{\lambda}{2}\rho(\cos\phi+i\sin\phi)|^2d\phi\\ &= \frac{1+O(\rho^2\lambda^2)}{2\pi}\int_0^{2\pi}\left(1-\lambda\rho\cos\phi+\frac{\lambda^2}{4}\rho^2\cos^2\phi+\frac{\lambda^2}{4}\rho^2\sin^2\phi\right)d\phi\\ &= (1+O(\rho^2\lambda^2))(1+\frac{1}{2\pi}\cdot 2\pi\cdot\frac{\lambda^2}{2}\rho^2) = (1+O(\rho^2\lambda^2))(1+\frac{\lambda^2\rho^2}{2}) = 1+O((\lambda\rho)^2).\end{aligned}$$

Thus, if $\lambda\rho<1$, then there is a constant c so that

$$E(|\phi_n^*|^{\lambda})\le\exp(c\lambda^2\rho^2 n).$$

Next, we integrate over the unit circle. Recall that $z=e^{i\theta}$ and $\rho=(n\log n)^{-1/2}$. Since $|\phi_n^*|$ is continuous on all variables, by Fubini's Theorem:

$$\int_{-\pi}^{\pi}E|\phi_n^*(z)|^{\lambda}d\theta = E\int_{-\pi}^{\pi}|\phi_n^*(z)|^{\lambda}d\theta\le 2\pi\cdot\exp\left(\frac{c\lambda^2}{\log n}\right).$$

The function $|\phi_n^*|$ is continuous on $\mathbb{T}$. Thus, $M=|\phi_n^*(\tilde{z})|$ for some $\tilde{z}=e^{i\tilde{\theta}}$, $\tilde{\theta}\in[-\pi,\pi]$. Then within some interval I containing $\tilde{\theta}$, we must have $|\phi_n^*(e^{i\theta})|\ge\frac{M}{2}$, $\forall\theta\in I$. To get a good bound for the length of I, first note that by Bernstein inequality, $|\phi_n^{*'}(e^{i\theta})|\le n\cdot M$. Thus

$$|\phi_n^*(e^{i\tilde{\theta}})-\phi_n^*(e^{i\theta})| = |\int_{\theta}^{\tilde{\theta}}\phi_n^{*'}(e^{i\theta})d\theta|\le\int_{\theta}^{\tilde{\theta}}|\phi_n^{*'}(e^{i\theta})|d\theta\le n\cdot M\cdot|\tilde{\theta}-\theta|.$$

We choose $|\tilde{\theta}-\theta|\le\frac{1}{2n}$, $|I|=\frac{1}{n}$, then $|\phi_n^*(e^{i\theta})|\ge\frac{M}{2}$, $\theta\in I$ and

$$\frac{1}{n}\left(\frac{M}{2}\right)^{\lambda}\le\int_I|\phi_n^*(z)|^{\lambda}d\theta\le\int_{-\pi}^{\pi}|\phi_n^*(z)|^{\lambda}d\theta.$$

Therefore,

$$E\left(\frac{1}{n}\left(\frac{M}{2}\right)^{\lambda}\right) \leq 2\pi \cdot \exp\left(\frac{c\lambda^2}{\log n}\right),$$
$$E(M^{\lambda}) \leq 2^{\lambda} \cdot 2\pi n \cdot \exp\left(\frac{c\lambda^2}{\log n}\right).$$

By choosing $\lambda = \log n$, we satisfy $\lambda\rho = \frac{\sqrt{\log n}}{\sqrt{n}} < 1$. Then, we use *Chebyshev's Inequality* to have

$$\begin{aligned} P(M > \kappa) = P(M^{\lambda} > \kappa^{\lambda}) \leq \frac{E(M^{\lambda})}{\kappa^{\lambda}} &\leq \frac{2^{\lambda} \cdot 2\pi n \cdot \exp(c\frac{\lambda^2}{\log n})}{\kappa^{\lambda}} \\ &= \exp(\log(2^{\lambda} \cdot 2\pi n) + c\log n - \log n \log \kappa) \\ &= \exp(\log(2\pi) + (\log 2 + c + 1 - \log\kappa)\log n) \\ &= \frac{2\pi}{n^{(\log\kappa - \log 2 - c - 1)}}, \end{aligned}$$

which is (2). □

Remark. I used random variables with uniform distribution on the interval $[0, 2\pi]$. I believe that analogous results can be obtained for other random variables.

Remark. The Main Result is analogous to the Salem-Zygmund Theorem because not only do the statements in two theorems, i.e. $P\left(\|T_n\|_{L^\infty(\mathbb{R})} \geq 3(\sum_{j=-n}^{n}|a_j|^2 log(Cn\kappa))^{1/2}\right) \leq \frac{2}{\kappa}$ and $P(\|\phi_n(z)\|_{L^\infty(\mathbb{T})} > \kappa) \leq \frac{2\pi}{n^{(\log\kappa - \log 2 - c - 1)}}$ share similarity, but the proof of the Salem-Zygmund Theorem can also be used to prove the theorem in Main Result.

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