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ABSTRACT. The Magnetohydrodynamic (MHD) system of equations governs kinematic fluids that are subjected to a magnetic field. The equation is a combination of the Navier-Stokes equations and Maxwell's equations. Due to the difficulty in solving the MHD system, it has become common to study approximating modifications of the equations, including the MHD-α system, which regularizes the velocity field in exchange for the addition of non-linear terms. Both the kinematic and magnetic parts of the MHD-α system have diffusive terms which dissipate the initial energy of the system. Setting those terms equal to zero returns the Ideal MHD-α system, and the goal of this project is to show that solutions to the MHD-α system with diffusion will converge to the Ideal MHD-αsystem as the diffusion parameters are sent to zero by adapting known results for the analogous problem of determining when solutions to the Navier-Stokes equations will converge to a solution of the Euler equation.

1. Introduction

The viscosity of a fluid is a measure of the internal friction between the fluid particles, resulting in a loss of energy for the system as the particles of the fluid slide past each other. This means a high viscosity fluid dissipates energy quickly and thus seems "thick," while a low viscosity fluid will loses energy slowly and so seems to flow "smoothly". Fluid motion is generally governed by partial differential equations. Viscous fluids are governed by the Navier-Stokes equations, while non-viscous fluids are governed by the Euler equation. In fact, the Navier-Stokes equations reduce to the Euler equation when the viscosity parameter is set to zero.

The Vanishing Viscosity Problem seeks to prove that solutions of the Navier-Stokes equations will converge to the solution of Euler's equation as the viscosity parameter is sent to zero. Establishing the Vanishing Viscosity Problem would mean that idealized fluids with no viscosity can be accurately approximated by fluids that have small viscosity. Though this remains a very difficult open problem in general, there are positive results in some special cases, particularly in the case of circularly symmetric flow (see, for example, [3]).

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For this project we considered a Vanishing Viscosity Problem for the Lagrangian Averaged Magnetohydrodynamic (MHD-α) system, which governs diffusive fluid flow subjected to magnetic fields. In general, the MHD equations are a coupled system of partial differential equations comprised of the Navier-Stokes equations (which governs viscous fluids) and Maxwell's equations (which govern magnetic fields).

There is more than one modification to the MHD equations, and this is due to the fact that our current scientific methods and tools cannot analytically compute or numerically simulate the turbulent behavior of 3D fluids and magnetofluids. There is too large of a range of scales of motions that need to be resolved when the Reynold's number is high. At times, we may only need to compute certain statistical features of the turbulence which can be addressed by having different modifications of the MHD equations (see [2] for a more complete discussion of this topic).

One modification to the MHD is the generalized MHD equations, which replaces the Laplacian with "powers" of the Laplacian. Higher powers dissipate the energy in the system faster, and in [5], the author works with this modification to prove the existence of local classical solutions and several global regularity conditions. In [6], the authors use the generalized MHD to show the existence of global smooth solutions.

Another modification is the Leray-α MHD equations. This version of the equation combines the generalized MHD equations and a regularization of the solution originally used by Leray. In [1], the authors show that the 2D velocity and magnetic fields solution pairs maintained their regularity in two cases. The first case had dissipation that was logarithmically weaker than a full Laplacian and zero magnetic diffusion. The second case was viscous free and had magnetic diffusion logarithmically weaker than a full Laplacian.

The modification of the MHD equations we will be studying is the MHD-α equations. The MHD-α equations provide a closure model of turbulence in infinite channels and pipes because the solutions have agreement with a wide range of Reynolds numbers. This model also theoretically regularizes the underlying equation, thus making the nonlinearity milder and the solutions smoother. In addition, it avoids the unnecessary extra dissipation of the energy of the system. See [2] for a derivation and more details on the origins of the equation. Our goal in this project is to send the diffusion parameters in the MHD-α to zero in a parallel fashion to the Vanishing Viscosity Problem for the Navier-Stokes equations.

The rest of the article is organized as follows: Section 2 is a more complete introduction which delves into the Vanishing Viscosity Problem, including a complete statement of the main result. Section 3 defines unknown terms and builds propositions that we will use to prove the main result in Section 4. Finally, Section 5 contains technical supporting details.

2. Vanishing Viscosity Problems

The main goal of this section is to provide a more in-depth explanation of the Vanishing Viscosity Problem, how it is applied to the MHD-α system, and state our main result. We will also state the PDE's being studied and give definitions of the various differential operators required to state those equations. But we will first provide a simple example illustrating that taking limits of differential equations is not as easy as it seems.

To that end, we consider the differential equation

$$\frac{dy}{dt} = ky(t). \tag{1}$$

If we naively take the limit as k goes to zero, the result is

$$\frac{dy}{dt} = \lim_{k\to 0} \frac{dy}{dt} = \lim_{k\to 0} ky(t) = y(t) \lim_{k\to 0} k = 0, \tag{2}$$

and the general solution to this differential equation is $y(t) = C$.

The actual solution to equation (1) is $\tilde{y}(t) = Ce^{kt}$, which depends on k, making the calculation in (2) invalid because it assumed that y had no k dependence.

However, taking the limit of $\tilde{y}(t)$ as k goes to zero gives

$$\lim_{k\to 0} \tilde{y}(t) = \lim_{k\to 0} Ce^{kt} = C,$$

which is the general solution to the incorrectly derived limit equation (2).

This simple calculation is an example of a general category of problem where we seek to show that solutions to a differential equation which contains a parameter will converge to the solution of the equation derived by naively taking the limit of the original equation. In the case considered here, that parameter is a viscosity parameter and this is called a Vanishing Viscosity Problem.

Before stating our partial differential equations, we will recall the definitions of some differential operators. We start with two differential operators typically introduced in Calculus III, the gradient and the Laplacian. Applied to a scalar valued function $f : \mathbb{R}^2 \to \mathbb{R}$, these are given by

$$\begin{aligned}\nabla f(x_1, x_2) &= \left(\frac{\partial}{\partial x_1} f(x_1, x_2), \frac{\partial}{\partial x_2} f(x_1, x_2) \right), \\ \Delta f(x_1, x_2) &= \frac{\partial}{\partial x_1^2} f(x_1, x_2) + \frac{\partial}{\partial x_2^2} f(x_1, x_2).\end{aligned}$$

These operators can also be applied to a vector function $u = (u_1, u_2)$, returning

$$\begin{aligned}\nabla u &= \begin{bmatrix} \frac{\partial}{\partial x_1} u_1 & \frac{\partial}{\partial x_2} u_1 \\ \frac{\partial^2}{\partial x_1} u_2 & \frac{\partial^2}{\partial x_2} u_2 \end{bmatrix}, \\ \Delta u &= (\Delta u_1, \Delta u_2).\end{aligned} \tag{3}$$

Now we turn to defining directional derivatives. Let $u = \langle u_1, u_2\rangle$ and $v = \langle v_1, v_2\rangle$ be vector fields on $\mathbb{R}^2$. Then

$$\begin{aligned}(u\cdot\nabla)v &= \left(u_1\frac{\partial}{\partial_{x_1}} + u_2\frac{\partial}{\partial_{x_2}}\right)\langle v_1, v_2\rangle \\ &= \left\langle\left(u_1\frac{\partial}{\partial_{x_1}} + u_2\frac{\partial}{\partial_{x_2}}\right)v_1, \left(u_1\frac{\partial}{\partial_{x_1}} + u_2\frac{\partial}{\partial_{x_2}}\right)v_2\right\rangle \\ &= \left\langle\left(u_1\frac{\partial v_1}{\partial_{x_1}} + u_2\frac{\partial v_1}{\partial_{x_2}}\right), \left(u_1\frac{\partial v_2}{\partial_{x_1}} + u_2\frac{\partial v_2}{\partial_{x_2}}\right)\right\rangle.\end{aligned}$$

These differential operators are found in the incompressible Navier-Stokes equations, which we recall governs incompressible viscous fluids. The general form of the equation is

$$\begin{aligned}&\partial_t u^\nu - \nu\triangle u^\nu + (u^\nu\cdot\nabla)u^\nu = \nabla p, \\ &u^\nu(0,0) = f(x), \quad \text{div } u^\nu = \text{div } f = 0,\end{aligned}$$

where $u^\nu : I\times M \to \mathbb{R}^n$ is the unknown fluid velocity field, I is a time interval, and $M \subset \mathbb{R}^n$. The fluid pressure (which depends on u^ν) is given by p, $\nu > 0$ is a constant due to the viscosity of the fluid, and the notation u^ν emphasizes that the solution depends on this choice of constant. The requirement that div $u = 0$ makes this the incompressible modification of the Navier-Stokes equations.

The idealized case of a fluid with no internal friction is governed by the Euler equation, which is

$$\begin{aligned}&\partial_t u^0 + (u^0\cdot\nabla)u^0 = \nabla p, \\ &u^0(0) = f, \quad \text{div } u^0 = \text{div } f = 0.\end{aligned}$$

Like the calculation for the simple example outlined at the beginning of the section, the Euler equation can be obtained from the Navier-Stokes equations by taking the naive limit of the Navier-Stokes equations as the viscosity term ν goes to zero. The goal of the Vanishing Viscosity Problem is to prove that solutions u^ν to the Navier-Stokes equations will converge to the solution of the Euler equation u^0 as the viscosity ν goes to zero.

As was mentioned in the introduction, this project focuses on a generalization of the MHD system called the Leray-α Magnetohydrodynamic (MHD-α) system. We start by stating the MHD system which is

$$\begin{aligned}&\partial_t u^\nu + (u^\nu\cdot\nabla)u^\nu - \nu\triangle u^\nu - (b^\eta\cdot\nabla)b^\eta = \nabla p, \\ &\partial_t b^\eta + (u^\nu\cdot\nabla)b^\eta - (b^\eta\cdot\nabla)\,u^\nu - \eta\triangle b^\eta = 0, \\ &\text{div } u = 0, \quad \text{div } b = 0, \\ &u(x,0) = u_0(x), \quad b(x,0) = b_0(x)\end{aligned}$$

where u is fluid viscosity, b is the magnetic field, p is fluid pressure, $\nu \geq 0$ is kinematic viscosity, and $\eta \geq 0$ is magnetic diffusion. Note that setting $b = 0$ returns the Navier-Stokes equations.

The MHD-α equations are

$$\begin{aligned}
&\partial_t w^\nu + (u^\nu \cdot \nabla) w^\nu - \alpha^2 (\nabla u^\nu)^T \Delta u^\nu - \nu \triangle w^\nu + \frac{1}{2}\nabla |b^\eta|^2 = \nabla p + (b^\eta \cdot \nabla) b^\eta,\\
&\partial_t b^\eta + (u^\nu \cdot \nabla) b^\eta - (b^\eta \cdot \nabla) u^\nu - \eta \Delta b^\eta = 0,\\
&w = (1 - \alpha^2 \triangle) u,\\
&\text{div } u^\nu = \text{div } b^\eta = 0,\\
&u^\nu(0,x) = u_0(x), \quad b^\eta(0,x) = b_0(x),
\end{aligned}$$

where $u^\nu : I \times \mathbb{R}^n \to \mathbb{R}^n$ is the fluid velocity, $b^\eta : I \times \mathbb{R}^n \to \mathbb{R}^n$ is the magnetic field, p is the scalar valued fluid pressure, ν, $\eta > 0$ are constants due to kinematic viscosity and magnetic diffusion, respectively, and α is the velocity dissipation exponent. We also recall that $|b^\eta|^2 = b^\eta \cdot b^\eta$ and $(\nabla u^\nu)^T$ is the transpose of the matrix from equation (3).

Setting $\alpha = 0$ returns the standard MHD system, setting $b = b_0 = 0$ returns the Navier-Stokes equations, and setting $\nu = \eta = 0$ returns the diffusion-free modification of the system.

In Chapter 13, Section 6, of [4], the author considered the Vanishing Viscosity Problem for the Navier-Stokes equations with circularly symmetric initial data in the unit ball $D = \{x \in \mathbb{R}^2 : |x| < 1\}$ and requiring the flow to be parallel to the boundary $S^1 = \{x \in \mathbb{R}^2 : |x| = 1\}$ [1]. Proposition 6.1 in [4] shows that, under these assumptions, the solution to the Navier-Stokes equations is also the solution to the Heat equation[2] with the same initial and boundary conditions.

This makes the Vanishing Viscosity Problem for the Navier-Stokes equations equivalent to the well-understood analogous problem for the Heat equation. A more precise discussion of this is beyond the scope of this article; see Proposition 6.2 in [4] for a more details.

The main result of this paper is an adaptation of Proposition 6.1 to the MHD-α setting. Specifically, we prove the following.

Theorem 2.1. *If $u_0, b_0 : D \to \mathbb{R}^2$ are smooth, circularly symmetric, parallel to S^1, and divergence free, then the vanishing viscosity problem for the MHD-α system is equivalent to the vanishing viscosity problem for the Heat equation.*

The proof of this theorem is in Section 4.

3. Definitions and Supporting Results

This section contains definitions and supporting results that will be necessary to prove Theorem 2.1. Several of these results rely on calculations that can be found in Section 5.

We begin with some vector notation. For $x = \langle x_1, x_2 \rangle \in \mathbb{R}^2$, we let $x^\perp = \langle -x_2, x_1 \rangle$. That is, $x^\perp = R_{\pi/2} x$, recalling that $R_{\pi/2}$ is counterclockwise rotation of $\pi/2$ radians. Note that $x^\perp$ is perpendicular to x.

[1] Definitions of circularly symmteric flow and what it means to be parallel to the boundary will be discussed in the next section.

[2] The Heat equation will be discussed in Section 3.2.

We also say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is radial if the value of f at each point x only depends on the distance from that point to the origin. Abusing notation, if f is radial, we will write $f(x) = f(|x|)$.

Next, we recall that a vector field v is circularly symmetric if

$$v(R_\theta x) = R_\theta v(x), \text{ for all } x \in D,$$

where $\theta \in [0, 2\pi]$ and R_θ is the rotation matrix that rotates the vector x counterclockwise θ radians.

We now state our first proposition.

Proposition 3.1. *Let v be a vector field. Then v is circularly symmetric if and only if there exists radial functions S_0 and S_1 such that*

$$v(x) = S_0(|x|)x^\perp + S_1(|x|)x.$$

Proof. We first assume that v is circularly symmetric. Since x and $x^\perp$ are linearly independent, we have that $v(x) = f_0(x)x^\perp + f_1(x)x$, for some $f_0, f_1 : \mathbb{R}^2 \to \mathbb{R}$, and our goal is to show that f_0 and f_1 must be radial.

We start by observing that

$$v(R_\theta x) = f_0(R_\theta x)(R_\theta x)^\perp + f_1(R_\theta x)R_\theta x, \tag{4}$$

and

$$R_\theta v(x) = R_\theta f_0(x)x^\perp + R_\theta f_1(x)x. \tag{5}$$

Equation (5) can be rearranged as

$$R_\theta v(x) = f_0(x)R_\theta\left(x^\perp\right) + f_1(x)R_\theta x,$$

because $f_1(x)$ and $f_2(x)$ are scalars, and thus commute with the matrix R_θ. Since rotation matrices commute, we have that

$$R_\theta\left(x^\perp\right) = R_\theta R_{\pi/2}x = R_{\pi/2}R_\theta x = (R_\theta x)^\perp, \tag{6}$$

and so

$$R_\theta v(x) = f_0(x)(R_\theta x)^\perp + f_1(x)R_\theta x. \tag{7}$$

Since v is circularly symmetric, equations (4) and (7) are equal to each other, and we can conclude through linear algebra that

$$f_1(R_\theta x) = f_0(x),$$
$$f_2(R_\theta x) = f_1(x).$$

By Proposition 5.1, f_0 and f_1 are radial, which completes this direction of the proof.

For the other direction, we first assume that

$$v(x) = S_0(|x|)x^\perp + S_1(|x|)x,$$

and we need to show that $v(R_\theta x) = R_\theta v(x)$ for any θ. To begin, we have

$$v(R_\theta x) = S_0(|R_\theta x|)(R_\theta x)^\perp + S_1(|R_\theta x|)R_\theta x.$$

The magnitude of a vector is not affected by rotation, so $|R_\theta x| = |x|$, and we can rewrite the previous equation as

$$v(R_\theta x) = S_0(|x|)(R_\theta x)^\perp + S_1(|x|)R_\theta x. \tag{8}$$

Next, recalling that $S_0(|x|)$ and $S_1(|x|)$ are scalars (and thus commute with R_θ), we have

$$\begin{aligned} R_\theta v(x) &= R_\theta S_0(|x|)x^\perp + R_\theta S_1(|x|)x \\ &= S_0(|x|)R_\theta x^\perp + S_1(|x|)R_\theta x \\ &= S_0(|x|)(R_\theta x)^\perp + S_1(|x|)R_\theta x, \end{aligned} \tag{9}$$

where the last equality used equation (6).

Comparing equations (8) and (9) shows that

$$v(R_\theta x) = R_\theta v(x),$$

which completes the proposition. □

Our next result further classifies circularly symmetric vector fields subject to a boundary condition. But first we recall that a vector field is parallel to the boundary of a region R if the vector field is parallel to the tangent vectors of the points on the boundary.

In our case, the region R is $\overline{D} = \{v \in \mathbb{R}^2 : |v| \le 1\}$ and so the boundary of R is the unit circle, S^1. From calculus, we know that for any $y \in S^1$, the tangent vector at y is

$$T(y) = y^\perp. \tag{10}$$

Now we are ready to state our next proposition.

Proposition 3.2. *Let v be a smooth divergence free circularly symmetric vector field on $\overline{D}$ and assume v is parallel to S^1. Then*

$$v(x) = S_0(|x|)x^\perp,$$

where S_0 is a smooth radial function.

Proof. Proposition 3.1 allows us to write $v(x) = S_0(|x|)x^\perp + S_1(|x|)x$ for all x, and so the proof will be completed if we show that $S_1(|x|) = 0$ for all $x \in \overline{D}$. From equation (10), we know that for any $y \in S^1$, $T(y) = y^\perp$, and so

$$v(y) = S_0(|y|)y^\perp.$$

This completes the proof for v restricted to the boundary of D. To complete the Proposition, we let $x \in D$ be arbitrary. This means we have to go back to $v = S_0(|x|)x^\perp + S_1(|x|)x$. Then by definition,

$$\text{div } S_0(|x|)x^\perp = -\frac{\partial}{\partial x_1} S_0\left(\sqrt{x_1^2 + x_2^2}\right)x_2 + \frac{\partial}{\partial x_2} S_0\left(\sqrt{x_1^2 + x_2^2}\right)x_1.$$

By Proposition 5.2, we have

$$\text{div } S_0(|x|)x^\perp = -\frac{x_1 x_2 S_0'(|x|)}{|x|} + \frac{x_2 x_1 S_0'(|x|)}{|x|} = 0. \tag{11}$$

Turning to the other term, we have

$$\text{div } S_1(|x|)x = S_1(|x|) + \frac{x_1^2 S_1'(|x|)}{|x|} + S_1(|x|) + \frac{x_1^2 S_1'(|x|)}{|x|} = 2S_1(|x|) + S_1'(|x|)|x|. \tag{12}$$

Using (11) and (12), we have that since div $v = 0$,

$$2S_1(|x|) + S_1'(|x|)|x| = 0.$$

If we substitute t for $|x|$, this becomes the first order differential equation

$$2S_1(t) + S_1'(t)\,t = 0,$$

and separating variables gives

$$\begin{aligned} &S_1'(t)t = -2S_1(t) \\ \iff &\frac{S_1'(t)}{S_1(t)} = \frac{-2}{t} \\ \iff &\frac{1}{S_1(t)}\frac{dS_1(t)}{dt} = \frac{-2}{t}. \end{aligned}$$

Integrating both sides gives

$$\ln S_1(t) = -2\ln t + C,$$

and then taking the exponential of both sides returns

$$S_1(t) = Ct^{-2}.$$

Substituting $|x|$ back in for t, we finally have

$$S_1(|x|) = C|x|^{-2}.$$

Now, to find the constant C, we have to use an initial condition. We don't know the condition at $t = 0$, but we do know from the beginning of the argument that when $|x| = 1$, $S_1(|x|) = S_1(1) = 0$, and we can use that information to solve for the constant. This gives

$$S_1(1) = C1^{-2} = 0$$

which requires $C = 0$, and that means that $S_1(|x|) = 0$ for all $x \in \overline{D}$. This completes the proof. □

Our next set of results involve showing that many of the nonlinear terms of the MHD-α equations are conservative vector fields. To do so, we will rely on the just proven Proposition 3.2.

3.1. **Conservative Vector Fields.** Before proceeding, we recall that a vector field v is conservative if there exists a scalar valued function p such that $v = \nabla p$.

Proposition 3.3. *Let $v(x) = R(|x|)x^{\perp}$ and let $u(x) = S(|x|)x^{\perp}$ for smooth R and S. Then $(u\cdot\nabla)v = (v\cdot\nabla)u = T(|x|)\,x$ for some radial function T.*

Proof. Recalling that if $x = \langle x_1, x_2 \rangle$, then $x^{\perp} = \langle -x_2, x_1 \rangle$, we have

$$(u \cdot \nabla) v = \left(-x_2 S(|x|) \frac{\partial}{\partial_{x_1}} + x_1 S(|x|) \frac{\partial}{\partial_{x_2}} \right) \langle -R(|x|) x_2, R(|x|) x_1 \rangle .$$

Using the radial derivatives found in Proposition 5.2, we will substitute and distribute to get

$$\begin{aligned}(u \cdot \nabla) v = \Bigg\langle & -x_2 S(|x|) \left(\frac{-x_1 x_2 R'(|x|)}{|x|} \right) + x_1 S(|x|) \left(-R(|x|) - \frac{x_2^2 R'(|x|)}{|x|} \right), \\ & -x_2 S(|x|) \left(R(|x|) + \frac{x_1^2 R(|x|)}{|x|} \right) + x_1 S(|x|) \left(\frac{x_1 x_2 R'(|x|)}{|x|} \right) \Bigg\rangle .\end{aligned}$$

After simplifying, the result is

$$(u \cdot \nabla) v = \langle -x_1 S(|x|) R(|x|), -x_2 S(|x|) R(|x|) \rangle = -S(|x|) R(|x|) \langle x_1, x_2 \rangle = T(|x|) x,$$

where T is defined by the last equality. Swapping the roles of u and v in the above calculation shows that $(v \cdot \nabla) u = -R(|x|) S(|x|) x$ which completes the proof. □

Next, we will show that any radial function multiplied by the vector x is also a conservative vector field.

Proposition 3.4. *Let f be an integrable radial function and let $p(x) = -\int_{|x|}^{1} f(\rho)\rho d\rho$. Then $f(|x|)x = \nabla p$.*

Proof. Let F be the anti-derivative of $f(\rho)\rho$. Then

$$\begin{aligned}\partial_{x_i} p(x) &= \partial_{x_i} \left(-\int_{|x|}^{1} f(\rho) \rho d\rho \right) = \partial_{x_i} (-[F(1) - F(|x|)]) = \partial_{x_i} (F(|x|) - F(1)) \\ &= \frac{F'(|x|) x_i}{|x|} = \frac{f(|x|) |x| x_i}{|x|} = f(|x|) x_i,\end{aligned}$$

where we again used Proposition 5.2. So $\nabla p = \left(\partial_{x_1} p, \partial_{x_2} p \right) = (f(|x|) x_1, f(|x|) x_2) = f(|x|) x_0$. □

The next result shows that the Laplacian of a circularly symmetric vector field remains circularly symmetric. This will be important later when working with the Heat equation.

Proposition 3.5. *If $v(x) = S(|x|) x^{\perp}$ for a smooth S, then $\Delta v = R(|x|) x^{\perp}$, where R is a radial function.*

Proof. We know that $x = \langle x_1, x_2\rangle$ and $x^\perp = \langle -x_2, x_1\rangle$. Then,

$$\begin{aligned}
\frac{\partial}{\partial x_1} S(|x|)(-x_2) &= \frac{\partial}{\partial x_1} S\left(\sqrt{x_1^2+x_2^2}\right)(-x_2) \\
&= S'\left(\sqrt{x_1^2+x_2^2}\right)(-x_2)\cdot\frac{1}{2}\left(\sqrt{x_1^2+x_2^2}\right)^{-\frac{1}{2}}\cdot 2x_1 \\
&= -\frac{x_1x_2S'\left(\sqrt{x_1^2+x_2^2}\right)}{\sqrt{x_1^2+x_2^2}} \\
&= -\frac{x_1x_2S'(|x|)}{|x|},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial x_2} S(|x|)(-x_2) &= \frac{\partial}{\partial x_2} S\left(\sqrt{x_1^2+x_2^2}\right)(-x_2) \\
&= -\frac{x_2(x_2)S'(|x|)}{|x|} - (1)S(|x|) \\
&= -\frac{x_2^2S'(|x|)}{|x|} - S(|x|).
\end{aligned}$$

For the second derivatives, using the quotient rule gives

$$\begin{aligned}
\frac{\partial^2}{\partial {x_1}^2} S(|x|)(-x_2) &= \frac{|x|\left[-x_2S'(|x|) - \frac{x_1x_2S''(|x|)x_1}{|x|}\right] - \frac{-x_1x_2S'(|x|)}{|x|}\left(\frac{x_1}{|x|}\right)}{|x|^2} \\
&= -\frac{x_2S'(|x|)}{|x|} - \frac{x_1^2x_2S''(|x|)}{|x|^2} + \frac{x_1^2x_2S'(|x|)}{|x|^4},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial {x_2}^2} S(|x|)(-x_2) &= -\frac{S'(|x|)(-x_2)}{|x|} + \frac{|x|\left[-2x_2S'(|x|) + \frac{(-x_2^2)S''(|x|)x_2}{|x|}\right] - \frac{x_2^2S'(|x|)}{|x|}\left(\frac{x_2}{|x|}\right)}{|x|^2} \\
&= -\frac{3x_2S'(|x|)}{|x|} - \frac{x_2^3S''(|x|)}{|x|^2} - \frac{x_2^3S'(|x|)}{|x|^4}.
\end{aligned}$$

Adding these two results together gives

$$\begin{aligned}
\Delta S(|x|)(-x_2) &= -\frac{4x_2S'(|x|)}{|x|} - \frac{x_1^2x_2S''(|x|)}{|x|^2} - \frac{x_1^2x_2S'(|x|)}{|x|^4} - \frac{x_2^3S''(|x|)}{|x|^2} - \frac{x_2^3S'(|x|)}{|x|^4} \\
&= -\frac{4x_2S'(|x|)}{|x|} - \frac{S''(|x|)(-x_2)}{|x|^2}\left[x_1^2+x_2^2\right] - \frac{S'(|x|)(-x_2)}{|x|^2}\left[x_1^2+x_2^2\right] \\
&= -\frac{4x_2S'(|x|)}{|x|} + x_2S''(|x|) - \frac{-x_2S'(|x|)}{|x|^2}.
\end{aligned}$$

Setting

$$R(|x|) = \frac{4S'(|x|)}{|x|} - S''(|x|) - \frac{-S'(|x|)}{|x|^2},$$

we get

$$\Delta S\left(|x|\right)\left(-x_2\right) = R\left(|x|\right)\left(-x_2\right). \tag{13}$$

To compute the second entry, we need $\Delta S(|x|)(x_1)$. By symmetry, this will only differ from equation (13) by replacing x_2 with x_1 and removing the minus sign. So we have

$$\Delta S\left(|x|\right)\left(x_1\right) = \frac{4x_1 S'(|x|)}{|x|} + x_1 S''(|x|) + \frac{x_1 S'(|x|)}{|x|^2} = R(|x|)x_1. \tag{14}$$

Combining equations (13) and (14) gives

$$\begin{aligned}\Delta v &= \langle R\left(|x|\right)\left(-x_2\right), R\left(|x|\right)\left(x_1\right)\rangle \\ &= R\left(|x|\right)\langle -x_2, x_1\rangle \\ &= R\left(|x|\right)x^{\perp},\end{aligned}$$

which completes the proof. □

Now we finally address the nonlinear terms unique to the MHD-α system. Because these terms are not in the Navier-Stokes equations, these results are not adapted from the results in [4].

Lemma 3.6. *Let $v(x) = S(|x|)x^{\perp}$ for a smooth S. Then*

$$\nabla_v (1-\Delta)v - \alpha^2 (\nabla v)^T \Delta v = \nabla p,$$

where p is a scalar function.

Proof. By linearity,

$$\nabla_v (1-\Delta)v = \nabla_v v - \nabla_v (\Delta v),$$

and so we will show that, for some scalar functions p_0, p_1, and p_2,

$$\nabla_v v = \nabla p_0, \tag{15a}$$

$$-\nabla_v (\Delta v) = \nabla p_1, \tag{15b}$$

$$-\alpha^2 (\nabla v)^T \Delta v = \nabla p_2.$$

Setting $p = p_0 + p_1 + p_2$ will then complete the proof.

Beginning with equation (15*a*), by Proposition 3.3, we have that

$$(v\cdot\nabla)v = \nabla p_0,$$

for some scalar function p_0.

Turning to equation (15*b*), Proposition 3.5 gives that $\Delta v = R(|x|)x^{\perp}$, and so by Proposition 3.3 we have that

$$-(v\cdot\nabla)\Delta v = Q(|x|)x,$$

where Q is another radial function. Due to Proposition 3.4, we know that any radial function multiplied with the vector x is a conservative vector field, and so

$$(v\cdot\nabla)\Delta v = \nabla p_1,$$

for some scalar function p_1.

We now turn our attention to $-\alpha^2 (\nabla v)^T \Delta v$. Because $-\alpha^2$ is a constant, it will be ignored in the following calculations for notation simplicity. By Proposition 5.3 (and recalling that $v(x) = S(|x|)x^\perp$), we have

$$(\nabla v)^T = \begin{bmatrix} \frac{-x_1 x_2 S'(|x|)}{|x|} & S(|x|) + \frac{x_1^2 S'(|x|)}{|x|} \\ -S(|x|) + \frac{-x_2^2 S'(|x|)}{|x|} & \frac{x_1 x_2 S'(|x|)}{|x|} \end{bmatrix}.$$

Recalling that $\Delta v = R(|x|)x^\perp$, we have

$$(\nabla v)^T \Delta v = \begin{bmatrix} \frac{-x_1 x_2 S'(|x|)}{|x|} & S(|x|) + \frac{x_1^2 S'(|x|)}{|x|} \\ -S(|x|) + \frac{-x_2^2 S'(|x|)}{|x|} & \frac{x_1 x_2 S'(|x|)}{|x|} \end{bmatrix} \begin{bmatrix} -x_2 R(|x|) \\ x_1 R(|x|) \end{bmatrix} = \langle I_1, I_2 \rangle, \tag{16}$$

where

$$\begin{aligned} I_1 &= \frac{x_1 x_2^2 S'(|x|) R(|x|)}{|x|} + x_1 S(|x|) R(|x|) + \frac{x_1^3 S'(|x|) R(|x|)}{|x|}, \\ I_2 &= x_2 S(|x|) R(|x|) + \frac{x_2^3 S'(|x|) R(|x|)}{|x|} + \frac{x_1^2 x_2 S'(|x|) R(|x|)}{|x|}. \end{aligned}$$

Then for I_1 we have

$$\begin{aligned} I_1 &= x_1 \left(\frac{\left(x_1^2 + x_2^2\right) S'(|x|)R(|x|)}{|x|} + S(|x|)R(|x|) \right) \\ &= \left(S(|x|)R(|x|) + S'(|x|)R(|x|)|x|\right) x_1 = N(|x|)x_1, \end{aligned} \tag{17}$$

where the last equality defines $N(|x|)$. For I_2, we have

$$I_2 = x_2 \left(S(|x|)R(|x|) + \frac{\left(x_1^2 + x_2^2\right) S'(|x|)R(|x|)}{|x|} \right) = N(|x|)x_2. \tag{18}$$

Using (17) and (18) in (16) gives

$$(\nabla v)^T \Delta v = N(|x|)x.$$

Once again, because of Proposition 3.4, we can assert that $N(|x|)x$ is conservative and thus

$$\alpha^2 (\nabla v)^T \Delta v = \nabla p_2,$$

for some scalar function p_2. This completes the proof. □

3.2. **Heat Equation Results.** The goal of this section is to establish results related to the Heat equation, but we begin by recalling some properties of reflection operators from linear algebra. For any unit vector ω, the operator Φ_ω is the reflection across the line generated by ω. The reflection matrix associated to Φ_ω is

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix},$$

where θ is the angle the unit vector ω makes with the positive x-axis.

Φ_ω can also be defined directly. For any vector $x \in \mathbb{R}^2$, since ω and $\omega^\perp$ are linearly independent, x can be written as $x = a\omega + b\omega^\perp$, where $a = x \cdot \omega$ and $b = x \cdot \omega^\perp$. Then

$$\Phi_\omega\left(a\omega + b\omega^\perp\right) = a\omega - b\omega^\perp.$$

Now we are ready to prove the following.

Proposition 3.7. *Let ω be a unit vector in $\mathbb{R}^2$ and let v be a vector field. Then $v(x) = S(|x|)x^\perp$ if and only if $v(\Phi_\omega x) = -\Phi_\omega v(x)$.*

Proof. We begin by assuming $v(x) = S(|x|)x^\perp$ and we will show that $v(\Phi_\omega x) = -\Phi_\omega v(x)$.

Evaluating v at $\Phi_\omega x$ gives

$$v(\Phi_\omega x) = S(|\Phi_\omega x|)(\Phi_\omega x)^\perp = S(|x|)\left((x \cdot \omega)\omega - \left(x \cdot \omega^\perp\right)\omega^\perp\right)^\perp, \tag{19}$$

where we used the definition of Φ_ω and that, since Φ_ω is an isometry, $S(|\Phi_\omega x|) = S(|x|)$ for any x.

Similarly,

$$\begin{aligned} -\Phi_\omega v(x) &= -\left(\left(S|x|x^\perp \cdot \omega\right)\omega - \left(S|x|x^\perp \cdot \omega^\perp\right)\omega^\perp\right) \\ &= S(|x|)\left(\left(-x^\perp \cdot \omega\right)\omega + \left(x^\perp \cdot \omega^\perp\right)\omega^\perp\right). \end{aligned} \tag{20}$$

By Proposition 5.4, we know that

$$\left((x \cdot \omega)\omega - \left(x \cdot \omega^\perp\right)\omega^\perp\right)^\perp = \left(\left(-x^\perp \cdot \omega\right)\omega + \left(x^\perp \cdot \omega^\perp\right)\omega^\perp\right), \tag{21}$$

and so equations (19) and (20) are equal.

Now, we will prove the other direction. We begin by assuming that $v(\Phi_\omega x) = -\Phi_\omega v(x)$, and we will show that $v(x) = S(|x|)x^\perp$. We will assume that v has the form $v(x) = f_1(x)x^\perp + f_2(x)x$. Then we have

$$\begin{aligned} v(\Phi_\omega x) &= f_1(\Phi_\omega x)(\Phi_\omega x)^\perp + f_2(\Phi_\omega x)(\Phi_\omega x) \\ &= f_1(\Phi_\omega x)\left((x \cdot \omega)\omega - (x \cdot \omega^\perp)\omega^\perp\right)^\perp + f_2(\Phi_\omega x)\left((x \cdot \omega)\omega - (x \cdot \omega^\perp)\omega^\perp\right). \end{aligned}$$

Next, we will look at the other side of the equality, which is

$$\begin{aligned} -\Phi_\omega v(x) &= -f_1(x)\Phi_\omega\left(x^\perp\right) - f_2(x)\Phi_\omega(x) \\ &= f_1(x)\left((-x^\perp \cdot \omega)\omega + (x^\perp \cdot \omega^\perp)\omega^\perp\right) - f_2(x)\left((x \cdot \omega)\omega - (x \cdot \omega^\perp)\omega^\perp\right) \\ &= f_1(x)\left((x \cdot \omega)\omega - \left(x \cdot \omega^\perp\right)\omega^\perp\right)^\perp - f_2(x)\left((x \cdot \omega)\omega - (x \cdot \omega^\perp)\omega^\perp\right), \end{aligned}$$

where we used equation (21). Because $v(\Phi_\omega x) = -\Phi_\omega v(x)$, we can say that

$$\begin{aligned} f_1(\Phi_\omega x) &= f_1(x), \\ f_2(\Phi_\omega x) &= -f_2(x), \end{aligned} \tag{22}$$

for all $\omega \in S^1$.

Starting with f_2, we set $\omega^* = \frac{x}{|x|}$, and we have that $f_2(\Phi_{\omega^*}x) = f_2(x)$. By equation (22), $f_2(\Phi^*_\omega x) = -f_2(x)$, which means $f_2(x) = -f_2(x)$ for all x, and so we conclude that $f_2(x) = 0$, because zero is the only number equal to its own negative.

An argument analogous to the proof of Proposition 5.1 shows that f_1 is a radial function, and this concludes the proof. □

Now we recall the incompressible Heat equation, which is given by

$$\begin{aligned} &\partial_t u = \triangle u, \\ &u(0,x) = u_0, \\ &\text{div } u_0 = \text{div } u = 0. \end{aligned} \tag{23}$$

We will consider this problem for $u : \overline{D} \to \mathbb{R}^2$, with the assumption that the flow is parallel to the boundary.

Proposition 3.8. *Let u_0 be circularly symmetric vector field on the unit disk, parallel to the boundary, and divergence free. If u solves the Heat equation given in (23), then for any unit vector ω, the vector field $v(t,x) = -\Phi_\omega u(t, \Phi_\omega x)$ also solves (23).*

Proof. We will first show that $\partial_t v = \Delta v$. Since partial derivatives commute with matrix multiplication, we have

$$\partial_t v(t,x) = -\Phi_\omega (\partial_t u)(t, \Phi_\omega x), \tag{24}$$

and

$$\Delta v(t,x) = -\Phi_\omega \Delta (u(t, \Phi_\omega x)).$$

Because Φ_ω is an orthogonal transformation, we can use Proposition 5.5 and we get

$$\Delta v(t,x) = -\Phi_\omega \Delta (u(t, \Phi_\omega x)) = -\Phi_\omega (\Delta u)(t, \Phi_\omega x). \tag{25}$$

Since u solves the Heat equation (and thus $\partial_t u = \triangle u$), equations (24) and (25) show that $\partial_t v = \Delta v$.

Next we will show that $v(0,x) = u_0(x)$. Since $v(t,x) = -\Phi_\omega u(t, \Phi_\omega x)$ for any t, evaluating at $t = 0$ gives

$$v(0,x) = -\Phi_\omega u(0, \Phi_\omega x) = -\Phi_\omega u_0(\Phi_\omega x).$$

From Proposition 3.7, we know that $u_0(\Phi_\omega x) = -\Phi_\omega u_0(x)$. Since Φ_ω is its own inverse, multiplying on each side by $-\Phi_\omega$ gives

$$v(0,x) = -\Phi_\omega u_0(\Phi_\omega x) = -\Phi_\omega (-\Phi_\omega u_0(x)) = u_0(x),$$

which completes the argument.

A calculation showing that v is divergence free can be found in Proposition 5.6, so the last step in the argument is proving that v is parallel to S^1.

To be parallel to the boundary,

$$u(t, \Phi_\omega x) = C(\Phi_\omega x)^\perp,$$

where C is a constant. If we apply $-\Phi_\omega$ to both sides, we get

$$-\Phi_\omega u(t, \Phi_\omega x) = C\left[-\Phi_\omega (\Phi_\omega x)^\perp\right] = Cx^\perp,$$

which shows that both v and u are both solutions, but they are just flowing around the boundary in different directions. □

We have the following very useful corollary.

Corollary 3.9. *If u solves the Heat equation on $\overline{D}$ with circularly symmetric initial data u_0 that is parallel to the boundary, then $u(x) = S(|x|)x^{\perp}$ for some radial function S.*

Proof. By the Existence and Uniqueness Theorem for differential equations, since u and v from Proposition 3.8 both solve the same differential equation and have the same initial and boundary conditions,

$$u(t,x) = v(t,x) = -\Phi_{\omega} u\left(t, \Phi_{\omega} x\right).$$

Multiplying both sides by the inverse of Φ_{ω} gives

$$\Phi_{\omega}{}^{-1} u\left(t,x\right) = \Phi_{\omega}{}^{-1}\left(-\Phi_{\omega} u\left(t, \Phi_{\omega} x\right)\right).$$

Since Φ_{ω} is a reflection matrix, $\Phi_{\omega} = \Phi_{\omega}{}^{-1}$, and so we have

$$\Phi_{\omega} u\left(t,x\right) = -u\left(t, \Phi_{\omega} x\right).$$

So by Proposition 3.7, this means $u(x) = S(|x|)x^{\perp}$. □

4. Proof of Theorem 2.1

In this section we finally prove Theorem 2.1.

Proof. Let u_0 and b_0 be circularly symmetric vector fields on the unit disk, parallel to the boundary, and divergence free. Then let (u^{ν}, b^{η}) be the known solution to

$$\begin{aligned} &\partial_t u = \nu \Delta u, \\ &\partial_t b = \eta \Delta b, \\ &\text{div } u^{\nu} = \text{div } b^{\eta} = 0, \\ &u^{\nu}(0,x) = u_0(x), \quad b^{\eta}(0,x) = b_0(x). \end{aligned} \tag{26}$$

By Corollary 3.9, u^{ν} and b^{η} are circularly symmetric. Recalling that $w^{\nu} = (1-\alpha^2\Delta)u^{\nu}$, we apply $(1-\alpha^2\Delta)$ to both sides of the u equation and get

$$\partial_t w^{\nu} = \nu \Delta w^{\nu}, \tag{27a}$$

$$\partial_t b^{\eta} = \eta \Delta b^{\eta}. \tag{27b}$$

By Lemma 3.6 and Proposition 3.3, we have that

$$\begin{aligned} &\left(u^{\nu} \cdot \nabla\right) w^{\nu} - \alpha^2 \left(\nabla u^{\nu}\right)^T \Delta u^{\nu} = \nabla p_1, \\ &\left(b^{\eta} \cdot \nabla\right) b^{\eta} = \nabla p_2, \end{aligned}$$

for some scalar functions p_1 and p_2. This means equation (27a) is equivalent to

$$\partial_t w^{\nu} + \left(u^{\nu} \cdot \nabla\right) w^{\nu} - \alpha^2 \left(\nabla u^{\nu}\right)^T \Delta u^{\nu} = \nu \Delta w^{\nu} + \left(b^{\eta} \cdot \nabla\right) b^{\eta} + \nabla(p_1 - p_2). \tag{28}$$

Again using Proposition 3.3, we have that $(u\cdot\nabla)b = (b\cdot\nabla)u$, and so equation (27b) becomes

$$\partial_t b^\eta + (u^\nu\cdot\nabla)\, b^\eta = \eta\Delta b^\eta + (b^\eta\cdot\nabla)\, u^\nu. \tag{29}$$

Combining equations (28) and (29), we have

$$\begin{aligned}
&\partial_t w^\nu + (u^\nu\cdot\nabla)\, w^\nu - \alpha^2 (\nabla u^\nu)^T \Delta u^\nu + \frac{1}{2}\nabla|b^\eta|^2 = \nu\Delta w^\nu + (b^\eta\cdot\nabla)\, b^\eta + \nabla p,\\
&\partial_t b^\eta + (u^\nu\cdot\nabla)\, b^\eta = \eta\Delta b + (b^\eta\cdot\nabla)\, u^\nu,
\end{aligned}$$

where we set $p = p_1 - p_2 + \frac{1}{2}|b^\eta|^2$. Since we already know u^ν and b^η satisfy the appropriate boundary and initial conditions, this means the pair (u^ν, b^η) from equation (26) are also the solutions to the MHD-α system. □

As we discussed in Section 2, this means the Vanishing Viscosity Problem for the MHD-α is equivalent to the Vanishing Viscosity Problem for the Heat equation. This problem is very well understood in the context of the Heat equation, with positive known results in many standard settings (like Sobolev spaces). More details can be found in [4].

5. Appendix: Vector Calculus Computations

This appendix includes the sometimes tedious computations we performed that would distract from the main point of various arguments. Our first set of results involve rotation matrices.

Proposition 5.1. *Let $f:\mathbb{R}^2\to\mathbb{R}$. If*

$$f(R_\theta x) = f(x)$$

for any rotation matrix R_θ, then f is a radial function.

Proof. To see this, we will shift from Cartesian coordinates to polar coordinates where $\langle x_1, x_2\rangle = \langle r\cos\phi, r\sin\phi\rangle$. Then $R_\theta x = \langle r\cos(\phi+\theta), r\sin(\phi+\theta)\rangle$, and so $f(x) = f(R_\theta x)$ implies

$$f(r,\phi) = f(r\cos(\phi), r\sin(\phi)) = f\left(r\cos(\phi+\theta), r\sin(\phi+\theta)\right) = f(r,\phi+\theta)$$

for any angle θ. By substituting in $\theta = -\phi$, we get $f(r,\phi) = f(r,0)$ which illustrates f is independent of the angle of rotation. □

The next proposition details the tedious calculations for differentiating radial functions that we use extensively in Section 3.1.

Proposition 5.2. *Let $S:\mathbb{R}^2\to\mathbb{R}$ be a radial function. Then*

$$\partial_{x_i} S(|x|) = \frac{x_i S'(|x|)}{|x|}$$

for $i = 1, 2$.

Proof. Recalling that $|x| = \sqrt{x_1^2 + x_2^2}$, we have

$$\begin{aligned}\frac{\partial}{\partial x_i} S(|x|) &= \frac{\partial}{\partial x_i} S\left(\sqrt{x_1^2 + x_2^2}\right) \\ &= S'\left(\sqrt{x_1^2 + x_2^2}\right) \cdot \frac{1}{2}\left(\sqrt{x_1^2 + x_2^2}\right)^{-\frac{1}{2}} \cdot 2x_i \\ &= \frac{x_i S'\left(\sqrt{x_1^2 + x_2^2}\right)}{\sqrt{x_1^2 + x_2^2}} \\ &= \frac{x_i S'(|x|)}{|x|}.\end{aligned}$$

□

This result was used in the beginning of the proof of Lemma 3.6.

Proposition 5.3. *Let $v(x) = S(|x|)x^{\perp}$. Then*

$$(\nabla v)^T = \begin{bmatrix} \frac{-x_1 x_2 S'(|x|)}{|x|} & S(|x|) + \frac{x_1^2 S'(|x|)}{|x|} \\ -S(|x|) + \frac{-x_2^2 S'(|x|)}{|x|} & \frac{x_1 x_2 S'(|x|)}{|x|} \end{bmatrix}.$$

Proof. We start by using equation (3) and get

$$\nabla v = \begin{bmatrix} \frac{\partial}{\partial x_1} S(|x|)(-x_2) & \frac{\partial}{\partial x_2} S(|x|)(-x_2) \\ \frac{\partial}{\partial x_1} S(|x|)(x_1) & \frac{\partial}{\partial x_2} S(|x|)(x_1) \end{bmatrix}.$$

Taking the derivatives gives

$$\begin{bmatrix} \frac{-x_1 x_2 S'(|x|)}{|x|} & -S(|x|) + \frac{-x_2^2 S'(|x|)}{|x|} \\ S(|x|) + \frac{x_1^2 S'(|x|)}{|x|} & \frac{x_1 x_2 S'(|x|)}{|x|} \end{bmatrix}.$$

Taking the transpose finishes the proof. □

Our next result is an exercise in linear algebra that will be useful in the proof of Proposition 3.7.

Proposition 5.4. *Let $x \in \mathbb{R}^2$ and ω be a unit vector. Then*

$$\left((x \cdot \omega)\,\omega - \left(x \cdot \omega^{\perp}\right)\omega^{\perp}\right)^{\perp} = \left(-x^{\perp} \cdot \omega\right)\omega + \left(x^{\perp} \cdot \omega^{\perp}\right)\omega^{\perp}. \tag{30}$$

Proof. Recalling that $x^{\perp} = R_{\pi/2}x$, we have that

$$\begin{aligned} R_{\pi/2}\left[(x \cdot \omega)\,\omega - \left(x \cdot \omega^{\perp}\right)\omega^{\perp}\right] &= (x \cdot \omega) R_{\pi/2}\omega - \left(x \cdot \omega^{\perp}\right) R_{\pi/2} R_{\pi/2}\omega \\ &= (x \cdot \omega)\,\omega^{\perp} - \left(x \cdot \omega^{\perp}\right) R_{\pi}\omega = (x \cdot \omega)\,\omega^{\perp} + \left(x \cdot \omega^{\perp}\right)\omega, \end{aligned} \tag{31}$$

where we recall that $R_{\theta}R_{\omega} = R_{\theta+\omega}$ and $R_{\pi} = -1$.

Now, if we compare the right side of equation (30) and the conclusion of equation (31), the problem is completed if we show $(-x^{\perp}\cdot\omega) = (x\cdot\omega^{\perp})$ and $(x^{\perp}\cdot\omega^{\perp}) = (x\cdot\omega)$.

For the first term, we have

$$\left(-x^{\perp}\cdot\omega\right) = (R_{\pi}R_{\pi/2}x\cdot\omega) = (R_{3\pi/2}x\cdot\omega) = \left(R^{T}_{\pi/2}x\cdot\omega\right) = (x\cdot R_{\pi/2}\omega) = \left(x\cdot\omega^{\perp}\right),$$

and for the second term we have

$$\left(x^{\perp}\cdot\omega^{\perp}\right) = (R_{\pi/2}x\cdot R_{\pi/2}\omega) = (R^{T}_{\pi/2}(R_{\pi/2}x)\cdot\omega) = (R_{3\pi/2}(R_{\pi/2}x)\cdot\omega) = (R_{2\pi}x\cdot\omega) = (x\cdot\omega),$$

which completes the proof. □

Our next proposition involves the Laplacian and is central to the proof of Proposition 3.8.

Proposition 5.5. *Let $u:\mathbb{R}^2\to\mathbb{R}$ and let $A=(a_{ij})$ be a two-by-two orthogonal matrix. Then*

$$\Delta u(Ax) = (\Delta u)(Ax).$$

Proof. Setting $x=\langle x_1,x_2\rangle$, we have that

$$u(Ax) = u(a_{11}x_1 + a_{12}x_2, a_{21}x_1 + a_{22}x_2).$$

We will next take the ∂_{x_1} derivative of $u(Ax)$, and using the chain rule gives

$$\partial_{x_1}u(Ax) = u_{x_1}(Ax)\cdot a_{11} + u_{x_2}(Ax)\cdot a_{21}.$$

After taking a second ∂_{x_1} derivative, the result is

$$\partial_{x_1x_1}u(Ax) = a_{11}^2u_{x_1x_1}(Ax) + a_{11}a_{21}u_{x_1x_2}(Ax) + a_{21}a_{11}u_{x_2x_1}(Ax) + a_{21}^2u_{x_2x_2}(Ax).$$

Similarly,

$$\partial_{x_2x_2}u(Ax) = a_{12}^2u_{x_1x_1}(Ax) + a_{12}a_{22}u_{x_1x_2}(Ax) + a_{22}a_{12}u_{x_2x_1}(Ax) + a_{22}^2u_{x_2x_2}(Ax).$$

Adding these results, we get

$$\Delta(u(Ax)) = (a_{11}^2 + a_{12}^2)u_{x_1x_1}(Ax) + (2a_{11}a_{21} + 2a_{12}a_{22})u_{x_1x_2}(Ax) + (a_{21}^2 + a_{22}^2)u_{x_2x_2}(Ax). \quad (32)$$

Since A is an orthogonal matrix,

$$\begin{aligned} a_{11}a_{21} + a_{22}a_{12} &= 0,\\ a_{11}^2 + a_{12}^2 &= 1,\\ a_{21}^2 + a_{22}^2 &= 1. \end{aligned}$$

So equation (32) reduces to

$$\Delta(u(Ax)) = (1)u_{x_1x_1}(Ax) + (0)u_{x_1x_2}(Ax) + (1)u_{x_2x_2}(Ax) = u_{x_1x_1}(Ax) + u_{x_2x_2}(Ax) = (\Delta u)(Ax),$$

which completes the argument. □

Our last result involves the divergence operator.

Proposition 5.6. *Let u be a circularly symmetric divergence free vector field. Then $\Phi_\omega(u(\Phi_\omega x))$ is divergence free for any unit vector ω.*

Proof. We begin by recalling from (3.2) that the matrix representation of any reflection matrix is of the form

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}.$$

This means

$$\Phi_\omega x = \langle ax_1 + bx_2, bx_1 - ax_2 \rangle,$$

and

$$\Phi_\omega u(x_1, x_2) = \langle au_1(x_1, x_2) + bu_2(x_1, x_2), bu_1(x_1, x_2) - au_2(x_1, x_2) \rangle,$$

where $u(x_1, x_2) = (u_1(x_1, x_2), u_2(x_1, x_2))$.

And so we get

$$\begin{aligned} \Phi_\omega(u(\Phi_\omega x)) = \langle\, & au_1(ax_1 + bx_2, bx_1 - ax_2) + bu_2(ax_1 + bx_2, bx_1 - ax_2), \\ & bu_1(ax_1 + bx_2, bx_1 - ax_2) - au_2(ax_1 + bx_2, bx_1 - ax_2) \rangle. \end{aligned}$$

Taking the divergence of $\Phi_\omega(u(\Phi_\omega x))$ gives

$$\text{div } (\Phi_\omega(u(\Phi_\omega x))) = I + J, \tag{33}$$

where

$$\begin{aligned} I =& \partial_{x_1} (au_1(ax_1 + bx_2, bx_1 - ax_2) + bu_2(ax_1 + bx_2, bx_1 - ax_2)), \\ J =& \partial_{x_2} (bu_1(ax_1 + bx_2, bx_1 - ax_2) - au_2(ax_1 + bx_2, bx_1 - ax_2)). \end{aligned}$$

By the chain rule,

$$\begin{aligned} I &= a\Big(a\partial_{x_1} u_1(\Phi_\omega x) + b\partial_{x_2} u_1(\Phi_\omega x)\Big) + b\Big(b\partial_{x_1} u_1(\Phi_\omega x) - a\partial_{x_2} u_1(\Phi_\omega x)\Big), \\ J &= b\Big(a\partial_{x_1} u_2(\Phi_\omega x) + b\partial_{x_2} u_2(\Phi_\omega x)\Big) - a\Big(b\partial_{x_1} u_2(\Phi_\omega x) - a\partial_{x_2} u_2(\Phi_\omega x)\Big), \end{aligned}$$

and so

$$\begin{aligned} I + J =& a^2\partial_{x_1} u_1(\Phi_\omega x) + ab\partial_{x_2} u_1(\Phi_\omega x) + b^2\partial_{x_1} u_1(\Phi_\omega x) - ab\partial_{x_2} u_1(\Phi_\omega x) \\ & + ab\partial_{x_1} u_2(\Phi_\omega x) + b^2\partial_{x_2} u_2(\Phi_\omega x) - ab\partial_{x_1} u_2(\Phi_\omega x) + a^2\partial_{x_2} u_2(\Phi_\omega x) \\ =& a^2\partial_{x_1} u_1(\Phi_\omega x) + b^2\partial_{x_1} u_1(\Phi_\omega x) + b^2\partial_{x_2} u_2(\Phi_\omega x) + a^2\partial_{x_2} u_2(\Phi_\omega x). \end{aligned}$$

Plugging this back into equation (33), we get

$$\text{div } (\Phi_\omega(u(\Phi_\omega x))) = (a^2 + b^2)(\partial_{x_1} u_1(\Phi_\omega x) + \partial_{x_2} u_2(\Phi_\omega x)).$$

Since u is divergence free, $\partial_{x_1} u_1(\Phi_\omega x) + \partial_{x_2} u_2(\Phi_\omega x) = 0$, and therefore,

$$\text{div } (\Phi_\omega u) = a^2\,(0) + b^2\,(0) = 0.$$

□

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