The Minnesota Journal of Undergraduate Mathematics

The volume of the trace-nonnegative polytope via the Irwin-Hall Distribution

Pietro Paparella[1] and Gregory K. Taylor[2]

[1]University of Washington Bothell
[2]College of William and Mary

The Minnesota Journal of Undergraduate Mathematics

Volume 4 (2018-19 Academic Year)

Sponsored by
School of Mathematics
University of Minnesota
Minneapolis, MN 55455

The Minnesota Journal of Undergraduate Mathematics

Volume 4 (2018-19 Academic Year)

The volume of the trace-nonnegative polytope via the Irwin-Hall Distribution

Pietro Paparella[1] and Gregory K. Taylor* [2]

[1]University of Washington Bothell
[2]College of William and Mary

Abstract. In this work, we find an explicit expression for the volume of the trace-nonnegative polytope, the subset of Euclidean space whose coordinates lie between -1 and 1 and sum to a nonnegative number. The volume of this region provides an upper bound for the volume of a region called the realizable region, the set of vectors which can be realized as the eigenvalues of a nonnegative matrix. This region is of interest for matrix theorists working on the nonnegative inverse eigenvalue problem. To find this expression, we employ a transformation of the Irwin-Hall distribution from probability theory. We conclude by providing a general example of a non-realizable spectrum within the trace-nonnegative polytope and a characterization of the realizability of certain spectra whose entries sum to zero. The paper includes a number of open problems for further inquiry.

1. Introduction and Background

The *real nonnegative inverse eigenvalue problem* (RNIEP) is to find necessary and sufficient conditions on $\sigma = \{\lambda_1, \ldots, \lambda_k\} \subset \mathbb{R}$ so that σ is the spectrum of an entrywise-nonnegative matrix (which from now on we call a *nonnegative matrix*). If A is a nonnegative matrix with spectrum σ, then σ is called *realizable* and A is called a *realizing matrix* for σ. Despite many stringent necessary conditions, the RNIEP remains unsolved when $k > 4$ (for background, see, e.g., [2]; for recent developments, see [5]).

The set $\sigma = \{\lambda_1, \ldots, \lambda_k\} \subset \mathbb{R}$ is said to be *normalized* if

$$\lambda_1 = 1 \geq \cdots \geq \lambda_k.$$

For a normalized set σ, let $x = x_\sigma = \begin{bmatrix} \lambda_2 & \ldots & \lambda_k \end{bmatrix}^\top \in \mathbb{R}^{k-1}$. If $\mathcal{P}^{k-1}$ denotes the set of all projected k-tuples of all normalized spectra of nonnegative matrices, then

$$\mathcal{P}^{k-1} \subseteq \mathcal{T}^{k-1} := \left\{ x \in \mathbb{R}^{k-1} : \|x\|_\infty \leq 1 \text{ and } 1 + \sum_{i=1}^{k-1} x_i \geq 0 \right\}.$$

* *Corresponding author*

This follows from the Perron-Frobenius theorem (see [4] for exposition) and the fact that a realizing matrix has nonnegative trace. The region $\mathcal{T}^{k-1}$, $k \geq 2$, is known as the *trace nonnegative polytope* [6]. It is well-known (see, e.g., [5, 8]) that $\mathcal{P}^{k-1} = \mathcal{T}^{k-1}$ for $2 \leq k \leq 4$.

The purpose of this work is to find an explicit expression for the volume of $\mathcal{T}^n$ ($n \geq 1$). The motivation is three-fold. First, it is clear that this is not a trivial endeavor: one approach is to enumerate the vertices of the polytope and slice it into simplices, at which point the formula for the volume of a simplex can be applied. However, enumerating these vertices is difficult (see, for e.g., [9]). Second, the volume of $\mathcal{T}^n$ gives an upper bound for the volume of $\mathcal{P}^n$ ($n \geq 1$). Lastly, in [5], Johnson and Paparella studied polytopes whose points correspond to projected normalized spectra. These polytopes are proper subsets of all realizable spectra, but the set of all realizable spectra is the union of all such polytopes. In some cases the volume of these polytopes is available; thus, knowing the volume of $\mathcal{T}^n$ gives us a way to quantify of how "big" these polytopes are with respect to the trace nonnegative region which in turn gives an impression of how many realizable spectra are contained within such a polytope.

To compute the volume of this polytope, we employ an affine transformation of the *Irwin-Hall distribution*. This transformation may also be derived from the more general distribution derived in [1].

In addition, We demonstrate that $\mathcal{P}^n \subsetneq \mathcal{T}^n$, for every $n \geq 4$ (i.e., for spectra that contain at least five elements), and provide ancillary results on realizable trace-zero spectra. Finally, we pose the problem of finding an open set in the trace nonnegative polytope containing only non-realizable spectra. The discovery of such an open set would imply that the upper bound for the volume of the realizable region is indeed a strict inequality. Such an open set does not exist for $n \leq 3$ since the trace nonnegative polytope exactly coincides with the realizable region.

2. Computation of the Volume of the Trace Nonnegative Polytope

For $n \in \mathbb{N}$, let $\mathcal{B}^n := \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$. For $i \in \{1,\ldots,n\}$, let $Y_i \sim U[-1,1]$, and let $Y = \sum Y_i$. The fraction of the volume of $\mathcal{B}^n$ that coincides with $\mathcal{T}^n$ equals $P(Y \geq -1)$, i.e.,

$$\operatorname{Vol}(\mathcal{T}^n) = P(Y \geq -1)\operatorname{Vol}(\mathcal{B}^n) = P(Y \geq -1)2^n.$$

For $i \in \{1,\ldots,n\}$, let $X_i \sim U[0,1]$, and let $X = \sum X_i$. The continuous probability distribution for the random variable X is the well-known *Irwin-Hall* (or *uniform-sum*) *distribution* (IHD). The probability density function (PDF) f of the IHD is given by

$$f(x) = \frac{1}{(n-1)!}\sum_{k=0}^{\lfloor x \rfloor}(-1)^k\binom{n}{k}(x-k)^{n-1},$$

and the cumulative distribution function (CDF) F is given by

$$F(x) = \frac{1}{n!}\sum_{k=0}^{\lfloor x \rfloor}(-1)^k\binom{n}{k}(x-k)^{n}.$$

We refer to X as the *Irwin-Hall random variable*.

To compute $P(Y \geq -1)$, we must compute the distribution of the sum of n random variables uniformly distributed on the interval $[a,b]$ where $a=-1$ and $b=1$. We may do this via an affine transformation of an Irwin-Hall random variable. The theory of transformations on random variables is well-established (see, e.g., [7, 12]).

If U and V are random variables where $U = h(V)$ for some monotone, differentiable function h, then

$$f_U(u) = f_V[h^{-1}(u)]\left|\frac{d[h^{-1}(u)]}{du}\right|.$$

Integrating the pdf gives the following formula for the cdf

$$F_U(u) = F_V[h^{-1}(u)].$$

For $i \in \{1,\ldots,n\}$, let $Y_i \sim U[a,b]$ with $a<b$, and let $Y = \sum Y_i$. Let X be as above. Note that Y is an affine transformation of X since

$$Y = h(X) = (b-a)X + na.$$

This is clear when considering the support of each random variable, that is, the subset of $\mathbb{R}$ in which the random variable may take values. The support of X, denoted $\mathcal{S}_X$, is the interval $[0,n]$; and the support of Y, denoted $\mathcal{S}_Y$, is the interval $[na,nb]$. From this, we see that $\operatorname{diam}(\mathcal{S}_Y) = n(b-a) = \operatorname{diam}(\mathcal{S}_X)(b-a)$. Also, the leftmost side of $\mathcal{S}_Y$ lies na units from 0. Both X and Y are identically distributed within their respective intervals.

We can apply the general formula for the transformation of a random variable to derive the PDF and CDF of Y in terms of the PDF and CDF of X:

$$f_Y(y) = \frac{1}{b-a} f_X\left(\frac{y-na}{b-a}\right) \tag{1}$$

$$F_Y(y) = F_X\left(\frac{y-na}{b-a}\right) \tag{2}$$

With these calculations, we may state the probability density and cumulative distribution functions for the sum of n uniform $[a,b]$ random variables.

Theorem 2.1. *If $X_i \sim U[a,b]$ with $a<b$, for $i=1,\ldots,n$, and $X = \sum X_i$, then the probability density function and cumulative distribution function of X are given by*

$$f_X(x) = \frac{1}{(b-a)(n-1)!}\sum_{k=0}^{\lfloor h^{-1}(x)\rfloor}(-1)^k\binom{n}{k}\left[h^{-1}(x)-k\right]^{n-1} \tag{3}$$

and

$$F_X(x) = \frac{1}{n!}\sum_{k=0}^{\lfloor h^{-1}(x)\rfloor}(-1)^k\binom{n}{k}\left[h^{-1}(x)-k\right]^{n}, \tag{4}$$

respectively, where $h^{-1}(x) = \frac{x-na}{b-a}$.

We are now able to give an expression for the volume of $\mathcal{T}^n$.

Corollary 2.2. *The volume of the n-dimensional trace-nonnegative polytope $\mathcal{T}^n$ is given by*

$$Vol(\mathcal{T}^n) = 2^n \left[1 - \frac{1}{n!} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^k \binom{n}{k} \left(\frac{n-1}{2} - k \right)^n \right].$$

Proof. This formula follows from $Vol(\mathcal{T}^n) = Vol(\mathcal{B}^n)P(X \geq -1) = 2^n(1 - F_X(-1))$, where X is the sum of n uniform $[-1,1]$ random variables. □

3. Non-Realizable Spectra within the Trace Nonnegative Polytope

In this section, we provide non-realizable spectra within the trace nonnegative polytope for all $n \geq 5$. This generalizes a well-known example of such a spectrum for $n = 5$ given by Friedland in [3].

If $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ is realizable, then

$$\rho(\sigma) := \max_i |\lambda_i| \in \sigma \tag{5}$$

and

$$s_1(\sigma) := \sum_{i=1}^{n} \lambda_i \geq 0. \tag{6}$$

As mentioned in the introduction, it is well-known that for $1 \leq n \leq 4$, conditions (5) and (6) are also sufficient for realizability (see, e.g., [5, 8]).

For $n = 5$, the normalized trace-zero spectrum

$$\sigma = \{1, 1, -2/3, -2/3, -2/3\}$$

is not realizable [3]. Indeed, if σ is realizable, then the realizing matrix must be reducible. By the Perron-Frobenius Theorem, there is a partition (σ_1, σ_2) of σ such that each σ_i satisfies (5), (6) and contains the eigenvalue 1 with multiplicity 1. There are only two such partitions:

i) $\sigma_1 = \{1\}$ and $\sigma_2 = \{1, -2/3, -2/3, -2/3\}$;

ii) $\sigma_1 = \{1, -2/3\}$ and $\sigma_2 = \{1, -2/3, -2/3\}$.

In each of these partitions, σ_2 does not satisfy (6), so this spectrum is not realizable.

As the next result shows, this construction generalizes to all odd orders greater than or equal to five.

Theorem 3.1. *Let $n = 2k + 1$ for some integer $k \geq 2$. If*

$$\sigma_n := \{\overbrace{1, \ldots, 1}^{k}, \overbrace{-k/(k+1), \ldots, -k/(k+1)}^{k+1}\},$$

then $s_1(\sigma) = 0$ and σ_n is not realizable.

Proof. We proceed by contradiction. If σ_n is realizable, then the realizing matrix must be reducible. Thus, there exists a partition $(\sigma_1, \ldots, \sigma_k)$ of σ_n such that each σ_i satisfies (5), (6), and 1 appears in σ_i with multiplicity 1. In any such partition, there is some σ_i such that

$$\left\{1, \frac{-k}{k+1}, \frac{-k}{k+1}\right\} \subseteq \sigma_i.$$

Then, since 1 has multiplicity 1 in σ_i, we have

$$s_1(\sigma_i) \leq 1 - \frac{2k}{k+1} = \frac{1-k}{k+1} < 0$$

which contradicts the fact that σ_i satisfies (6). □

Now, we establish a similar construction for even orders. Consider the spectrum

$$\sigma = \{1, 1, -1/5, -3/5, -3/5, -3/5\}.$$

If σ is realizable, then the realizing matrix must be reducible. Thus, there is a partition (σ_1, σ_2) of σ such that σ_i satisfies (5) and (6). This partition is impossible because $\{1, -3/5, -3/5\}$ must be a subset of either σ_1 or σ_2.

We can generalize this construction to all even orders greater than or equal to six.

Theorem 3.2. *Let $n = 2(k+1)$ for some integer $k \geq 2$. If*

$$\sigma_n := \left\{\overbrace{1, \ldots, 1}^{k}, \frac{-1}{2k+1}, \overbrace{\frac{1-2k}{2k+1}, \ldots, \frac{1-2k}{2k+1}}^{k+1}\right\},$$

then $s_1(\sigma) = 0$ and σ_n is not realizable.

Proof. We proceed by contradiction. If σ_n is realizable, then the realizing matrix must be reducible. Thus, there is a partition $(\sigma_1, \ldots, \sigma_k)$ of σ_n such that each σ_i satisfies (5) and (6). Any such partition contains a σ_i with

$$\left\{1, \frac{1-2k}{2k+1}, \frac{1-2k}{2k+1}\right\} \subseteq \sigma_i,$$

but

$$s_1(\sigma_i) \leq 1 + 2\frac{1-2k}{2k+1} = \frac{3-2k}{2k+1} < 0,$$

a contradiction. □

Since $\mathcal{P}^n \subset \mathcal{T}^n$, it follows that $\mathrm{Vol}(\mathcal{P}^n) \leq \mathrm{Vol}(\mathcal{T}^n)$ and it is natural to consider whether this inequality is strict. This can be settled by investigating the following problem.

Problem 3.3. Determine whether $\mathcal{T}^n \backslash \mathcal{P}^n$ contains an open-set.

This problem is nontrivial. The spectra exhibited in the last section have trace equal to 0. That is, $s_1(\sigma) = 0$. This property allowed us to construct reducible spectra for which we had greater control over the reduced spectra. In order to find an open set in $\mathcal{T}^n \backslash \mathcal{P}^n$, one would need to find non-realizable spectra whose trace are strictly positive which appears to be quite difficult.

In the next section, we consider a class of trace-zero spectra which may be viewed as a generalization of the notions in the constructions above.

4. A Characterization of Partitionable Trace-Zero Spectra

Here, we present a necessary and sufficient condition on the realizability of certain trace-zero spectra which can be decomposed as in Theorems 3.1 and 3.2. First, we prove the following lemma.

Lemma 4.1. *Let σ be a realizable spectrum. If $\sigma = \sigma_1 \cup \cdots \cup \sigma_k$, where each σ_i is realizable, then $s_1(\sigma_i) \leq s_1(\sigma)$.*

Proof. Note that $s_1(\sigma) = \sum_{i=1}^{k} s_1(\sigma_i)$. Since each σ_i is realizable, we have that each $s_1(\sigma_i) \geq 0$. So $s_1(\sigma_i) \leq s_1(\sigma)$ for each i. □

This is a rather simple idea, but it proves to be very useful when considering the realizability of trace-zero spectra. Before we state the main result, we introduce a certain type of spectrum.

Definition 4.2. A normalized spectrum σ is called a *Suleĭmanova spectrum* if $s_1(\sigma) \geq 0$ and the only positive eigenvalue is 1.

Remark 4.3. Friedland [3] and Perfect [11] proved that every Suleĭmanova spectrum is realizable via companion matrices (for other proofs, see references in [3]). Recently, Paparella [10] gave a constructive proof via *permutative matrices*.

Theorem 4.4. *Let $\sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{R}$ be a trace-zero spectrum which satisfies* (5); (6); *$\rho(\sigma)$ appears with multiplicity k; and $\lambda < 0$ for every $\lambda \neq \rho(\sigma)$. Then σ is realizable if and only if it is the union of Suleĭmanova spectra.*

Proof. By assumption, we have $\sigma = \{1, \ldots, 1, \lambda_{k+1}, \ldots, \lambda_n\}$, where $\lambda_i < 0$ for $i \in \{k+1, \ldots, n\}$. Thus, there exists a partition $(\sigma_1, \ldots, \sigma_k)$ of σ such that each σ_i is realizable and 1 occurs with multiplicity 1 in each σ_i. Lemma 4.1 implies that $s_1(\sigma_i) \leq s_1(\sigma) = 0$. Because $\lambda_i < 0$, if $\lambda_i \neq 1$, each σ_i is a Suleĭmanova spectrum.

Conversely, if σ is the union of Suleĭmanova spectra, say $(\sigma_1, \ldots, \sigma_k)$ then each σ_i is realizable as noted in Remark 8. Suppose each σ_i has realizing matrix A_i, then σ is realized by the block diagonal matrix

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & A_k \end{bmatrix}.$$

□

5. Conclusion

We conclude by introducing a generalization of our original problem. The *nonnegative inverse eigenvalue problem* is to determine necessary and sufficient conditions such that $\sigma = \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C}$ is the spectrum of a nonnegative matrix. In addition to satisfying (5) and (6), σ must be *self-conjugate*, i.e., $\bar{\lambda} \in \sigma$ if $\lambda \in \sigma$.

Without loss of generality, we may write

$$\sigma = \{1, \lambda_1, \ldots, \lambda_r, \mu_1 \pm \nu_1 \mathrm{i}, \ldots, \mu_c \pm \nu_c \mathrm{i}\},$$

where

(i) $\operatorname{Im}(\lambda_i) = 0$, for all $i \in \{1, \ldots, r\}$;

(ii) $|\lambda_i| \leq 1$, for all $i \in \{1, \ldots, r\}$;

(iii) $\nu_i \neq 0$, for all $i \in \{1, \ldots, c\}$; and

(iv) $|\mu_i + \nu \mathrm{i}| = \sqrt{\mu_i^2 + \nu_i^2} \leq 1$ for all $i \in \{1, \ldots, c\}$.

With the above in mind, (6) can be written as

$$1 + \sum_{i=1}^{r} \lambda_i + 2 \sum_{i=1}^{c} \mu_i \geq 0.$$

Let $\mathcal{B}_\infty^n := \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$ and $\mathcal{B}_2^n := \{x \in \mathbb{R}^n : \|x\|_2 \leq 1\}$. We identify $a + b\mathrm{i}$ with $(a, b) \in \mathbb{R}^2$.

Problem 5.1. For $n \geq 1$, find the volume of the *trace-nonnegative region*

$$\mathcal{TN}_n := \left\{ (\lambda_1, \ldots, \lambda_r) \times (\mu_1, \nu_1, \ldots, \mu_c, \nu_c) \in \mathcal{B}_\infty^r \times \mathcal{B}_2^{2c} : 1 + \sum_{i=1}^{r} \lambda_i + 2 \sum_{i=1}^{c} \mu_i \geq 0 \right\}$$

Remark 5.2. Corollary 2.2 solves Problem 5.1 when $c = 0$.

References

[1] David M. Bradley and Ramesh C. Gupta. On the distribution of the sum of n non-identically distributed uniform random variables. *Ann. Inst. Statist. Math.*, 54(3):689–700, 2002.

[2] Patricia D. Egleston, Terry D. Lenker, and Sivaram K. Narayan. The nonnegative inverse eigenvalue problem. *Linear Algebra Appl.*, 379:475–490, 2004. Tenth Conference of the International Linear Algebra Society.

[3] Shmuel Friedland. On an inverse problem for nonnegative and eventually nonnegative matrices. *Israel Journal of Mathematics*, 29(1):43–60, 1978.

[4] Roger A Horn and Charles R Johnson. *Matrix analysis*. Cambridge university press, 2012.

[5] Charles R. Johnson and Pietro Paparella. Perron spectratopes and the real nonnegative inverse eigenvalue problem. *Linear Algebra Appl.*, 493:281–300, 2016.

[6] C. Knudsen and J. J. McDonald. A note on the convexity of the realizable set of eigenvalues for nonnegative symmetric matrices. *Electron. J. Linear Algebra*, 8:110–114, 2001.

[7] Lawrence Leemis. *Probability*. Lightning Source, 2011.

[8] Raphael Loewy and David London. A note on an inverse problem for nonnegative matrices. *Linear and Multilinear Algebra*, 6(1):83–90, 1978/79.

[9] J. J. McDonald and M. Neumann. The Soules approach to the inverse eigenvalue problem for nonnegative symmetric matrices of order $n \leq 5$. In *Algebra and its applications (Athens, OH, 1999)*, volume 259 of *Contemp. Math.*, pages 387–407. Amer. Math. Soc., Providence, RI, 2000.

[10] Pietro Paparella. Realizing suleǐmanova-type spectra via permutative matrices. *Electron. J. Linear Algebra*, 31:306–3012, 2016.

[11] Hazel Perfect. Methods of constructing certain stochastic matrices. *Duke Math. J.*, 20:395–404, 1953.

[12] Dennis Wackerly, William Mendenhall, and Richard Scheaffer. *Mathematical statistics with applications*. Cengage Learning, 7 edition, 2007. Revised reprint of the 1973 original by Mendenhall and Scheaffer.

Student biographies

Gregory Taylor: (*Corresponding author:* gtaylo9@uic.edu) At the time of this work, Greg Taylor was an undergraduate at The College of William & Mary. Currently, he is a graduate student at UIC. His research interests are in algebraic geometry, and his hobbies include running and playing guitar.