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On regrouping convergent series into absolutely convergent series.

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Abstract. We demonstrate how any conditionally convergent series of real numbers can be regrouped as an absolutely convergent series. The regrouping can be expressed using a monotone subsequence of partial sums. The result can be extended to series of real-valued functions under specific conditions; these cases, as well as a constructive counterexample, are presented in detail.

1. Introduction

The study of series and sums is intertwined with Calculus, and understanding their convergence is important when considering series approximation such as Taylor and Laurent series. In the study of simple sums and of series of functions, rearrangement has perhaps gotten more attention, due to Riemann's Rearrangement theorem [2, Theorem 7.13]. In terms of regrouping, very few propositions exist. This paper considers the latter, as how, curiously, convergence may be improved through regrouping.

Absolute convergence can be advantageous in a variety of contexts. Series in themselves have a wide range of applications, and their convergence is crucial in their study. The difference between conditional and absolute convergence is highlighted as a condition for applying theorems or deriving properties. For example in number theory, L-functions have a half-plane of convergence [4, p1] and in complex analysis we study disks of convergence; in both cases one has conditional convergence on the boundary while one has absolute convergence on the interior. Were one to regroup from the boundary appropriately, one could extend the area of absolute convergence. In Fourier analysis, absolute convergence of the Fourier coefficients implies convergence of the Fourier series [6, p42].

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Regrouping is also used for improving convergence rates in applied mathematics, among the tools in the theory of convergence acceleration [8, p15] which dates back as far as Euler. There are further examples of the utility of absolute convergence, such as the Cauchy product of two convergent series converges if at least one of the series is absolutely convergent. In probability, absolute convergence is required to apply the Fubini-Tonelli theorem [7, p207]. It is already known that regrouping cannot diminish the convergence of a series, but our result shows how one can regroup certain conditionally convergent series into absolutely convergent series. Conceptually, it is interesting that regrouping impacts the convergence of a series.

1.1. **Preliminaries.** Recall that an infinite series, $\sum_{n=1}^{\infty} a_n$ is said to be convergent if the sequence of its partial sums, denoted $s_n = \sum_{k=1}^{n} a_k$ tends to a limit, say L. Formally $\sum_{n=1}^{\infty} a_n$ converges to L if for all $\epsilon > 0$ there exists an N such that for all $n \geq N$, $|s_n - L| < \epsilon$. The series is said to be absolutely convergent if the sequence of the partial sums of the series of absolute values of each term also tends to a limit, that is if $\sum_{n=1}^{\infty} |a_n|$ converges. If the series converges but is not absolutely convergent, then it is said to be conditionally convergent. It is well known that absolute convergence implies convergence.

We define a regrouping of a convergent series by placing parentheses around a disjoint collection of a finite number of terms of an infinite series. The new series is called a regrouping of the original, and is related in the sense that both series converge to the same value. We can regroup a regrouping, and thus a regrouping of a regrouping is itself a regrouping. It can be useful to think of a regrouping as corresponding (bijectively) to a subsequence of the sequence of partial sums of the original series.

2. Regrouping

In this section, we will make use of the following result.

Lemma 2.1. *A conditionally convergent series has infinitely many positive and negative terms, therefore if all but a finite number of its terms are of the same sign (positive or negative) then a convergent infinite series is absolutely convergent.*

Proof. In [2], Lemma 7.2 states that if $\sum_{n=1}^{\infty} a_n$ converges conditionally, then the sum of the positive terms $p_n := \frac{|a_n|+a_n}{2}$ and the sum of the negative terms $q_n := \frac{|a_n|-a_n}{2}$, both diverge. □

Theorem 2.2. *Every convergent series of real numbers has an absolutely convergent regrouping.*

Proof. We need to specifically show that every conditionally convergent series has an absolutely convergent regrouping. Let $\sum_{n=1}^{\infty} a_n$ converge conditionally to A. We will consider the sequence of its partial sums, denoted $s_n = \sum_{k=1}^{n} a_k$. Note that s_n also converges to A. Every sequence contains a nondecreasing or nonincreasing subsequence [5, p378], therefore without loss of generality we consider a nondecreasing subsequence of the partial sums. Then, there exist values for $n = \lambda_1 < \cdots < \lambda_t < \ldots$ such that the subsequence of partial sums $s_{\lambda_1} \le \cdots \le s_{\lambda_t} \le \ldots$ is monotone increasing. Now the original series can be regrouped using this monotone subsequence of partial sums:

$$\sum_{n=1}^{\infty} a_n = s_{\lambda_1} + (s_{\lambda_2} - s_{\lambda_1}) + \cdots + (s_{\lambda_p} - s_{\lambda_{(p-1)}}) + \ldots$$

Define each term of the regrouping $\sum_{t=1}^{\infty} \Lambda_t$ as

- $\Lambda_1 = s_{\lambda_1}$ for $t = 1$
- $\Lambda_t = (s_{\lambda_t}$ - $s_{\lambda_{(t-1)}})$ for $t > 1$

$$\implies \quad \sum_{n=1}^{\infty} a_n = \sum_{t=1}^{\infty} \Lambda_t$$

Given that the subsequence of partial sums is monotone increasing, each term in the regrouping Λ_t is nonnegative except potentially the first term. Therefore, $\sum_{t=1}^{\infty} \Lambda_t$ is absolutely convergent by Lemma 2.1, and it defines an absolutely convergent regrouping of the conditionally convergent series $\sum_{n=1}^{\infty} a_n$. If the subsequence of partial sums is monotone decreasing, then each term in the regrouping (except perhaps the first term) is nonpositive, which also produces an absolutely convergent regrouping. □

3. Regrouping series of functions

We now consider conditionally convergent series of functions. The issue is a convergent series such as $F(x) = \sum_{n=1}^{\infty} f_n(x)$, where there exists at least one element x in the domain of F such that $\sum_{n=1}^{\infty} f_n(x)$ does not converge absolutely. We start by presenting some elementary properties, and then several cases where such a series can be regrouped as an absolutely convergent series are discussed. In the next section we will present a counterexample.

3.1. Elementary Properties.

Remark. If a series is absolutely convergent, then every regrouping formed by inserting parentheses will also be absolutely convergent.

Proof. Let $\sum_{n=1}^{\infty} f_n$ be absolutely convergent, and consider an arbitrary regrouping $\sum_{m=1}^{\infty} g_m$. If $g_m = (f_p + f_{p+1} + \cdots + f_{p+t})$ then $|g_m| \leq |f_p| + |f_{p+1}| + \cdots + |f_{p+t}|$. By assumption, $\sum_{n=1}^{\infty} |f_n|$ converges, therefore $\sum_{m=1}^{\infty} |g_m|$ also converges by the Comparison Test [2, p146]. □

The proofs of the following remarks are immediate.

Remark. If $\sum_{n=1}^{\infty} f_n(x)$ has an absolutely convergent regrouping on a domain D, then the series has an absolutely convergent regrouping on every subset of D.

Remark. Consider a convergent series of periodic functions $f_n : D \to \mathbb{R}$ which all have the same period p. Then, $F(x) = \sum_{n=1}^{\infty} f_n$ also has period p. Let $K = D \cap [0, p]$. If $\sum_{n=1}^{\infty} f_n$ has an absolutely convergent regrouping on K, then the same regrouping yields an absolutely convergent series on D.

Remark. Let D be a subset of the reals, E be any set, $f_n : D \to \mathbb{R}$ be a sequence of functions and $t : E \to D$ be a surjective map. Consider the composition $h_n(e) = f_n(t(e))$ for all $n \in \mathbb{N}$, for all $e \in E$. Then the following hold:

(1) If $\sum_{n=1}^{\infty} f_n$ converges, so does $\sum_{n=1}^{\infty} h_n$

(2) If $\sum_{n=1}^{\infty} f_n$ converges conditionally, absolutely or uniformly, then the same holds for $\sum_{n=1}^{\infty} h_n$

(3) If t and f_n are continuous for all n, then h_n is continuous for all n

Proposition 3.1. *Let $D \subset \mathbb{R}$ and consider $\sum_{n=1}^{\infty} f_n$ with partial sums denoted s_n. Suppose there exists a subsequence of the partial sums, $\{s_{n_k}\}$ such that*

$$|s_{n_{k+1}} - s_{n_k}| \leq M_k$$

for some constant M_k. Also assume $\sum_{k=1}^{\infty} M_k < \infty$. Then, $\sum_{n=1}^{\infty} f_n$ has an absolutely convergent regrouping.

Proof. We apply the Weierstrass M-test to the sequence $\{s_{n_{k+1}} - s_{n_k}\}$. Then, $\sum_{k=1}^{\infty}(s_{n_{k+1}} - s_{n_k})$ is uniformly absolutely convergent, and together with s_{n_1} defines an absolutely convergent regrouping for $\sum_{n=1}^{\infty} f_n$ following the method in Theorem 2.2. □

Remark. For the following result, it suffices that there exist a uniformly convergent regrouping, in other words, that there be a subsequence of partial sums that converges uniformly. To simplify the exposition, we will assume general uniform convergence.

Theorem 3.2. *Let $D \subset \mathbb{R}$, and let $f_n : D \to \mathbb{R}$ with $\sum_{n=1}^{\infty} f_n$ converging uniformly to a function F on D. Then, the series has an absolutely convergent regrouping.*

Proof. We will apply Proposition 3.1, with constants $M_k = \frac{1}{2^k}$. By uniform convergence, there is an n_1 such that

$$|F - s_n| < \frac{1}{4}$$

for all $n \geq n_1$. Again by uniform convergence, there exists an $n_2 > n_1$ such that

$$|F - s_n| < \frac{1}{8}$$

for all $n \geq n_2$. Now

$$|s_{n_1} - s_{n_2}| \leq |s_{n_1} - F| + |s_{n_2} - F| < \frac{1}{2}$$

Without loss of generality, pick $n_3 > n_2$ such that

$$|F - s_{n_3}| < \frac{1}{16} \quad \text{for all}\, n \geq n_3 \implies |s_{n_3} - s_{n_2}| < \frac{1}{4}$$

This process yields a subsequence of partial sums $\{s_{n_k}\}$, with $|s_{n_{k+1}} - s_{n_k}| < \frac{1}{2^k}$. Since $\sum_{k=1}^{\infty} \frac{1}{2^k} \to 1$ then by Proposition 3.1 we can define an absolutely convergent regrouping of $\sum_{n=1}^{\infty} f_n$. □

3.2. **Consequences of Theorem 3.2.** Uniform convergence is very useful for regrouping into absolute convergence. The following corollaries illustrate a range of cases where uniform convergence is used to obtain an absolutely convergent regrouping.

Corollary 3.3. *If D is a finite set and $\sum_{n=1}^{\infty} f_n$ converges to F, then it has an absolutely convergent regrouping.*

Proof. A series of functions which converges pointwise on a finite set converges uniformly. Apply Theorem 3.2 □

Corollary 3.4. *If $D = D_1 \cup D_2$, and if $\sum_{n=1}^{\infty} f_n$ converges absolutely to F on D_1 and uniformly on D_2 then the series has an absolutely convergent regrouping on D.*

Proof. By Theorem 3.2 the series has an absolutely convergent regrouping on D_2. Then, by Lemma 3.1 we can apply that regrouping over all of D. □

Corollary 3.5. *If a real-valued function has a Taylor series representation then we can regroup it to form an absolutely convergent series on the whole interval of convergence.*

Proof. Since the series converges absolutely on the open interval of convergence $(c-R, c+R)$, there are at most two points (the endpoints) of conditional convergence. Then, apply 3.3. □

Corollary 3.6. *Suppose $\sum_{n=1}^{\infty} f_n$ converges to a continuous function F on a compact set D and that for all n, f_n is monotone increasing. Then, the series has an absolutely convergent regrouping.*

Proof. We invoke Polya's theorem [1, p173] which states that if we have a sequence of monotone increasing functions g_n, and if $g(x) = \lim_n (g_n(x))$ is continuous, then the convergence is uniform. Since the partial sums of a series of monotone increasing functions are also monotone increasing, we conclude that $\sum_{n=1}^{\infty} f_n$ converges to F uniformly. By Theorem 3.2 an absolutely convergent regrouping exists. □

Corollary 3.7. *Given a series $\sum_{n=1}^{\infty} f_n$ of measurable functions f_n which converges on a set A of finite measure, then for all $\epsilon > 0$, there exists a subset B of measure ϵ, $B \subset A$ such that the series has an absolutely convergent regrouping on $A \setminus B$*

Proof. By Egorov's theorem, the series will converge uniformly on $A \setminus B$, and we conclude by invoking Theorem 3.2. □

Corollary 3.8. *Let $\sum_{n=1}^{\infty} f_n$ converge to F on $[a, b]$ with partial sums s_n. If there is a subsequence of s_n that is uniformly bounded and equicontinuous, then $\sum_{n=1}^{\infty} f_n$ has an absolutely convergent regrouping.*

Proof. By the Arzela-Ascoli theorem, there is a subsequence of the subsequence that converges uniformly to F. Use that subsequence to define the regrouping for the series, and apply Theorem 3.2. □

Corollary 3.9. *(Certain inner products). Given two series of functions* $\sum_{n=1}^{\infty} f_n$ *and* $\sum_{n=1}^{\infty} g_n$*, their inner product is the series* $\sum_{n=1}^{\infty} f_n g_n$*. There are natural instances when the inner product is uniformly convergent, and hence has an absolutely convergent regrouping by Theorem 3.2. Two cases of interest are found in Dirichlet's Test and Abel's Test c.f. [1, p318]. By way of contrast, the convergent alternating series* $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ *has a divergent inner product with itself.*

3.3. **Regrouping via a monotone subsequence of partial sums.** Although Theorem 2.2 follows from Theorem 3.2, we presented it separately partly because it was at the root of our motivation. It involves finding an absolutely convergent regrouping specifically through a monotone subsequence of partial sums. We will refer to such a regrouping as the monotone case. It is clear that if a series of functions $\sum_{n=1}^{\infty} f_n$ converges to F and has a monotone subsequence of partial sums $\{s_{n_k}\}$, then this sequence defines an absolutely convergent regrouping by the argument of Theorem 2.2. Note that the presence of such a monotone subsequence of partial sums does not mean that one is working in the uniformly convergent case. There are easy examples of convergent series of functions which can be regrouped using the method of the monotone case, but fail to be uniformly convergent. Thus, one can sometimes regroup a series even in the absence of uniform convergence or an adequate monotone subsequence of partial sums. The following two results, the countable case and the Hölder case, both follow from the monotone case.

3.4. **A countable domain.**

Theorem 3.10. *Every convergent series of functions defined on a countable domain D has an absolutely convergent regrouping on D.*

The following diagonalization argument is due to T. Kenney and G. Lukacs.

Proof. Enumerate D by $\mathbb{N}$. If the series converges absolutely, then there is nothing to show. Suppose $\sum_{n=1}^{\infty} f_n(x)$ converges conditionally to $F(x)$ for at least one $x \in \mathbb{N}$. We'll first find an absolutely convergent regrouping for the series evaluated at a fixed point, starting with 1, as in Corollary 3.3 and Theorem 2.2. Let $g_{1,m}$ denote the absolutely convergent regrouping of the series evaluated at 1. That is,

$$\sum_{n=1}^{\infty} f_n(x) = \sum_{m=1}^{\infty} g_{1,m}(x) \qquad \forall x \in \mathbb{N}$$

and $\sum_{m=1}^{\infty} g_{1,m}(1)$ converges absolutely. The first term of this regrouping, $g_{1,1}$ is our first term of the regrouping for all $\mathbb{N}$. We will proceed to regroup $\sum_{m=1}^{\infty} g_{1,m}(2)$ but starting with the second term, since we want to keep $g_{1,1}$ fixed. That is, we want to find an absolutely convergent regrouping for $\sum_{m=2}^{\infty} g_{1,m}(2)$
Let $g_{2,k}$ denote the absolutely convergent regrouping for the series evaluated at 2, past the first fixed term. Recall that by Lemma 2.1, we don't mind that the first term isn't regrouped, since it's just a finite sum. We have

$$\sum_{n=1}^{\infty} f_n(x) = g_{1,1}(x) + \sum_{k=1}^{\infty} g_{2,k}(x) \qquad \forall x \in \mathbb{N}$$

and in particular $g_{1,1}(2) + \sum_{k=1}^{\infty} g_{2,k}(2)$ converges absolutely. By Lemma 3.1, we know that $g_{1,1}(1) + \sum_{k=1}^{\infty} g_{2,k}(1)$ also converges absolutely. Again, we pull out the first term, $g_{2,1}$ and fix it for the whole domain. This is our second term of the regrouping for the series over $\mathbb{N}$. Next we find the absolutely convergent regrouping for $\sum_{m=3}^{\infty} g_{2,m}(3)$. We proceed in this manner by induction. Suppose $g_{n-1,s}$ is the absolutely convergent regrouping for the series on $D_{n-1} = \{1, 2, ..., n-1\}$ that is,

$$\sum_{n=1}^{\infty} f_n(x) = g_{1,1}(x) + \cdots + g_{n-2,1}(x) + \sum_{s=1}^{\infty} g_{n-1,s}(x)$$

converges absolutely on D_{n-1}. Then, we know that there exists a regrouping $g_{n,t}$ such that

$$g_{1,1}(x) + \cdots + g_{n-1,1}(x) + \sum_{t=1}^{\infty} g_{n,t}(x)$$

converges absolutely on $D_{n-1} \cup \{n\}$. □

3.5. **Hölder continuous series of functions.**

Definition 3.11. A real valued function f is said to be Hölder continuous if there exist nonnegative $K, \alpha \in \mathbb{R}$ such that for all x, y in the domain of f,

$$|f(x) - f(y)| \leq K|x-y|^{\alpha}$$

The proofs of the following remarks are immediate.

Remark. Given two Hölder continuous functions with the same α and respective constants K_1 and K_2, their sum is Hölder continuous using the same α, the associated constant will be $K = \max\{K_1, K_2\}$ and $K < K_1 + K_2$.

Remark. If a function f_n is Hölder continuous with α and K, then $|f_n|$ is also Hölder continuous for α and K.

Proposition 3.12. *Consider a convergent series of functions f_n that are all Hölder continuous on a domain D. Suppose that α is fixed for all f_n and that $\sum_{n=1}^{\infty} K_n$ is finite. Then, we can find an absolutely convergent regrouping for $\sum_{n=1}^{\infty} f_n(x)$ on D.*

Proof. To form an absolutely convergent regrouping with this series of functions, begin by choosing a fixed y in D. Since a series of functions evaluated at a fixed domain point is simply a series of real numbers, we can find an absolutely convergent regrouping at y using Theorem 2.2. Call it

$$\sum_{m=1}^{\infty} g_m(y)$$

Let P_m be the Hölder constants for each individual function g_m. By Remark 3.5, P_m is smaller than a finite sum of the K_n. Since $\sum_{n=1}^{\infty} K_n$ converges by our assumption, then $\sum_{m=1}^{\infty} P_m$ also converges. Let $P = \sum_{m=1}^{\infty} P_m$. Now we consider the regrouping $g_m(x)$ for any $x \in D$. By the Hölder inequality,

$$\begin{aligned} &|g_m(x) - g_m(y)| \leq P_m|x-y|^\alpha \\ \implies &|g_m(x)| - |g_m(y)| \leq P_m|x-y|^\alpha \\ \implies &|g_m(x)| \leq |g_m(y)| + P_m|x-y|^\alpha \end{aligned}$$

Taking the infinite sum on both sides, we get

$$\sum_{m=1}^{\infty} |g_m(x)| \leq P|x-y|^\alpha + \sum_{m=1}^{\infty} |g_m(y)| \quad (1)$$

Then, $\sum_{m=1}^{\infty} |g_m(x)|$ is bounded for all $x \in D$, therefore $\sum_{m=1}^{\infty} g_m(x)$ is an absolutely convergent regrouping for $\sum_{n=1}^{\infty} f_n(x)$ on D. □

Example 3.13. As an example, consider the clearly conditionally convergent series of functions f_n where

$$f_n : [0,1] \to \mathbb{R} \quad f_n(x) = \frac{x}{2^n} + \frac{(-1)^{n+1}}{n}$$

and

$$\sum_{n=1}^{\infty} \frac{x}{2^n} + \frac{(-1)^{n+1}}{n} \to x + \ln(2)$$

Taking $\alpha = 1$ and $K_n = \frac{1}{2^n}$, f_n is Hölder continuous (in fact, Lipschitz) as

$$\left| \frac{x}{2^n} - \frac{y}{2^n} + \frac{(-1)^{n+1}}{n} - \frac{(-1)^{n+1}}{n} \right| \leq \frac{1}{2^n}|x-y|$$

Since $\sum_{n=1}^{\infty} \frac{1}{2^n} \to 1$, we satisfy the requirement that $\sum_{n=1}^{\infty} K_n$ be finite, thus we can regroup this series to be absolutely convergent by Proposition 3.12.

Example 3.14. Here is an example where we can use Proposition 3.2, but not Proposition 3.12 (since for $n > 1$, the Hölder constants associated to each f_n, are $K_n > \frac{1}{n}$). Let $D = [\frac{1}{2}, \frac{3}{4}]$ and consider first the partial sums $s_n(x) = (-1)^n x^n$. Let $s_0 = 0$. Now we define the functions as

$$f_n(x) = s_n(x) - s_{n-1}(x)$$

then $\sum_{n=1}^{\infty} f_n(x)$ clearly converges uniformly to $F(x) = 0$ on D, with n^{th} partial sum $s_n \neq 0$ for all $x \in D$, and for all $n \geq 1$.

4. Constructing a counterexample

Lemma 4.1. *Given an interval $[a,b]$ of length $2L$, a term $x \in [a,b]$ and a sequence of real numbers $\{y_n\}$ with $0 \leq y_n \leq L$, then for all m, there are real numbers $\{z_1, \ldots, z_n\}$ such that each $|z_n| = y_n$ and $\left(x + \sum_{n=1}^{m} z_n\right) \in [a,b]$*

Proof. We proceed inductively. If $x \in [a, a+L]$, we make the positive choice, $z_1 = y_1$, and $x + z_1 \leq a + 2L = b$. Similarly, if $x \in [a+L, b]$, then we add a negative term. We can continue this process m times, since each $0 \leq y_n \leq L$. □

The following is based on an argument suggested by P. Selinger. It uses the terms from the harmonic series, but the result also holds working from an arbitrary conditionally convergent series.

Theorem 4.2. *There exists a convergent series of functions on a domain D such that no regrouping yields an absolutely convergent series on D.*

Proof. First we define a set of sequences obtained from the infinite Cartesian product

$$A := \{-1, 1\} \times \left\{\frac{-1}{2}, \frac{1}{2}\right\} \times \cdots \times \left\{\frac{-1}{n}, \frac{1}{n}\right\}$$

With the product topology, this is a Cantor space which can be embedded in $\mathbb{R}$. Consider the subset D of A, which contains all the sequences (a_n) which yield convergent series $\sum_{n=1}^{\infty} a_n$. Define the canonical projection for each n:

$$f_n : D \to \mathbb{R} \qquad f_n((a_1, a_2, \dots)) = a_n$$

Clearly, by our choice of D, the series of functions $\sum_{m=1}^{k_1} f_m((a_n))$ converges for all $(a_n) \in D$. However, there is no absolutely convergent regrouping over all D. This is shown by contradiction. Suppose that such a regrouping did exist, and call it $g_s((a_n))$ where k_i corresponds bijectively to the subsequence of partial sums of $\sum_{m=1}^{\infty} f_m((a_n))$ such that

$$g_1 = \sum_{m=1}^{k_1} f_m((a_n)) \qquad g_2 = \sum_{m=k_1+1}^{k_2} f_m((a_n)) \dots$$

To arrive at a contradiction, we construct a sequence (a_n) in D, such that $\sum_{s=1}^{\infty} g_s((a_n))$ has a regrouping $\sum_{n=1}^{\infty} c_n$ which is not absolutely convergent. By Lemma 3.1, this implies that the regrouping $\sum_{s=1}^{\infty} g_s((a_n))$ was not absolutely convergent either. Our construction is via a sequence $b_n \in I_n$, where

$$\begin{aligned} I_1 &= [1,2] \\ I_n &= \left[\frac{-1}{n-1}, \frac{-1}{n}\right] \quad \text{for } n \text{ even} \\ I_n &= \left[\frac{1}{n}, \frac{1}{n-1}\right] \quad \text{for } n \text{ odd}, n \geq 3 \end{aligned}$$

The first few intervals are $[1,2], [-1,-1/2], [1/3,1/2]$. The intervals I_n have length $\frac{1}{n(n-1)}$, alternate from the positive to negative sides of the origin, and tend to the origin as $n \to \infty$. We will choose (a_n) in order to construct a sequence (b_n) of its partial sums with the following three properties:

(1) Each $b_n \in I_n$

(2) The last term in each b_n will be the last term of $g_s((a_n))$ for some s.

(3) The last term in each b_n will be less or equal to $\frac{1}{2n(n+1)}$

Condition (2) ensures that $\sum_{n=1}^{\infty} c_n$ is a regrouping of $\sum_{n=1}^{\infty} g_s((a_n))$, while condition (3) allows us to invoke Lemma 4.1.

We define the b_n inductively, starting with b_1. Let $a_1 = 1$. The first partial sum is inside the interval I_1 of length 1, and all subsequent terms satisfy condition (3): $|a_n| \leq \frac{1}{2}$ (for $n > 1$), so we can apply Lemma 4.1 to the interval I_1 where $x = 1$ and $\{y_1, y_2, ...\} = \{1/2, 1/3, ...\}$.

The uniform regrouping has infinitely many regrouped terms, therefore there is a k_{p_1} with the property that for all $m > k_{p_1}$, $|a_m| \leq \frac{1}{2(1+1)} = \frac{1}{4}$. Now apply Lemma 4.1 to obtain $b_1 = 1 + a_2 + ... + a_{k_{p_1}}$ so that $b_1 \in I_1$. It is clear that b_1 satisfies conditions (1), (2), and (3).

Suppose now that we have defined $b_1, ..., b_n$, and now define b_{n+1} for the inductive step. Without loss of generality assume n is odd so that we are on the positive side of the origin. We want to move over into I_{n+1}. We work with terms that begin where k_{p_n} ends. By condition (3), each of those terms $a_m \leq \frac{1}{2n(n+1)}$, therefore is smaller than half the length of I_{n+1}. A finite number of negative terms will get us close to I_{n+1} because the harmonic series diverges. There is no danger of "jumping past" I_{n+1} because each term is smaller than the length of I_{n+1}. So, there is a finite sum

$$s_t = \sum_{n=1}^{t} a_n \in I_{n+1}$$

Now find $k_{p_{n+1}} > t$ so that $m > k_{p_{n+1}} \implies |a_m| \leq \frac{1}{2(n+1)(n+2)}$, and apply Lemma 4.1 with $x = s_t$ and $\{y_1, ..., y_l\} = \{a_{t+1}, ...a_{k_{p_{n+1}}}\}$ to get b_{n+1}.

Define a sequence c_n by $c_1 = b_1$, $c_n = (b_n - b_{n-1})$ for $n > 1$. Then,

$$\sum_{n=1}^{\infty} a_n = b_1 + (b_2 - b_1) + \cdots = b_1 + \sum_{n=1}^{\infty} (b_n - b_{n-1}) := \sum_{m=1}^{\infty} c_m$$

$\sum_{m=1}^{\infty} c_m$ is a regrouping of the absolutely convergent regrouping $\sum_{s=1}^{\infty} g_s((a_n))$ on D, therefore by Lemma 3.1 it should be absolutely convergent as well. However, $|c_1| = |b_1| \geq 1$ and for each $n > 1$,

$$|c_n| = |b_n - b_{n-1}| \geq |b_n| \geq \frac{1}{n}$$

therefore by comparison with the harmonic series, $\sum_{m=1}^{\infty} c_m$ cannot converge absolutely, although it does converge, since $\sum_{m=1}^{\infty} c_m$ is a telescoping series with n^{th} partial sum equal to b_n which goes to 0 as $n \to \infty$. We have a contradiction, therefore we conclude that the initial regrouping $\sum_{s=1}^{\infty} g_s((a_n))$ was not absolutely convergent for all $(a_n) \in D$. □

Remark. Given $a < b$, consider $[a,b]$, a compact connected interval on the real line. We can transfer the canonical projections f_n in Theorem 4.2 from D to a subset of $[a,b]$. Indeed, if D_0 is the set of sequences whose canonical projections yield a series which converges to zero, then we have a subset of $[a,b]$ on which the series converges conditionally to the (continuous) zero function. We can extend this series to all of $[a,b]$ by choosing a fixed series from D_0, and applying it on $[a,b] \setminus D_0$. This yields a series converging conditionally to zero on $[a,b]$. By 4.2, this series of functions does not have an absolutely convergent regrouping. Note that by Theorem 3.2 these functions do not converge uniformly.

4.1. **An absolutely convergent regrouping via Dini's Theorem.** We now give an example of a series of functions which can be regrouped to be absolutely convergent, yet fails to have a monotone subsequence of partial sums. We will invoke Dini's theorem, which states that a monotone increasing series of continuous functions which converges to a continuous function on a compact space must converge uniformly.

Proposition 4.3. *There exists a convergent series of functions with an absolutely convergent regrouping, but no monotone subsequence of partial sums.*

Proof. We work with the functions f_n on [0,1]

$$f_n = nx(1-x^2)^{(n)}, n \in \mathbb{N}$$

as studied in [2, p168], where it is shown that $f_n \to 0$, but not uniformly so, because their integrals tend to $\frac{1}{2}$. For the same reason, no subsequence of the f_n converges uniformly.

Let D be the compact set $[0,1] \cup \{2\}$ and let k_n be defined as follows:

$$k_1 = \begin{cases} f_1 & x \in [0,1] \\ 1 & x = 2 \end{cases}$$

$$n > 1, \quad k_n = \begin{cases} f_n - f_{n-1} & x \in [0,1] \\ \frac{(-1)^{n+1}}{n} & x = 2 \end{cases}$$

Clearly all the k_n are continuous. If $s_n = \sum_{m=1}^{n} k_m$ then $s_n = f_n$ on $[0,1]$ and is the n^{th} partial sum of the alternating harmonic series at 2. Thus the series $\sum_{n=1}^{\infty} k_n$ converges to 0 on $[0,1]$, to $\ln(2)$ at 2 and is conditionally convergent. Note that it is absolutely convergent at each point of $[0,1]$ because the zero of k_n in $[0,1]$ lies at $\frac{1}{\sqrt{n+1}}$ which means that given any $x \in (0,1]$, k_n is positive for all n with only a finite number of exceptions (all but finitely many of the $\frac{1}{\sqrt{n+1}}$ lie to the right of x). Therefore all but finitely many of the $f_n(x) - f_{n-1}(x)$

are negative. By Corollary 3.5, $\sum_{n=1}^{\infty} k_n$ has an absolutely convergent regrouping, for example the standard one used for the alternating harmonic series. However, no regrouping can contain a monotone subsequence of partial sums, as then $s_m\big|_{[0,1]}$ has a monotone subsequence. But these partial sums are just the f_n as noted above. So there would be a monotone subsequence of the f_n converging to the zero function on $[0,1]$. By Dini's theorem this would imply that a subsequence of the f_n converges uniformly which we know to be false. □

Remark. If one examines the (standard) regrouping of $\sum_{n=1}^{\infty} k_n$ (the functions as defined above) dictated by the alternating harmonic series,

$$(k_1 + k_2) + (k_3 + k_4) + \dots$$

and passes to their absolute values

$$|k_1 + k_2| + |k_3 + k_4| + \dots$$

one knows that one has convergence. Interestingly, the function to which this sum converges cannot be continuous, as shown next by again invoking Dini's Theorem.

Proposition 4.4. *Let k_n be continuous, and suppose $\sum_{n=1}^{\infty} k_n$ converges on a compact space K. Suppose the convergence is not uniform. Also suppose that any regrouping $\sum_{m=1}^{\infty} g_m$ does not converge uniformly either (this happens in our example above). If a regrouping $\sum_{m=1}^{\infty} g_m$ is absolutely convergent, then the function to which it converges cannot be continuous, yet it is continuous on a dense subset of K.*

Proof. Let $\sum_{m=1}^{\infty} |g_m|$ converge to a function G. The convergence of the partial sums is obviously monotone. If the limit function G is continuous, then Dini's theorem applies because all of the $|g_m|$ are continuous and their partial sums are continuous. This says that $\sum_{m=1}^{\infty} |g_m|$ is uniformly convergent and therefore $\sum_{m=1}^{\infty} g_m$ is uniformly convergent which is false. Therefore, G cannot be continuous. It has to be continuous on a dense subset of K as a consequence of the Baire Category Theorem [3, p158]. □

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