Weak visibility preserving functions

Jack J. Billings and Neil R. Nicholson

North Central College

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Abstract. A point P in a set S of lattice points is weakly visible in S if no other point in S lies on the line segment from the origin to P. A function whose domain is S with co-domain consisting of lattice points is said to be weak visibility preserving if every point P that is weakly visible in S is also weakly visible in $f[S]$. Using previous work on lattice point visibility as the foundation for our approach, we explore the topic of weak visibility preserving functions. Two particular types of functions are investigated, with one always being weak visibility preserving and the other never so.

1. Introduction

Weak lattice point visibility is often described in terms of someone trying to photograph a marching band. Band member A is said to be weakly visible by the photographer if no other band member stands in the line of sight between the photographer and A. It is known that there are positions for the photographer to stand in order to weakly view every band member [1, 2, 3, 7], with a lower bound on how close such a position could be [9]. Moreover, both necessary and sufficient conditions for a particular band member being weakly viewable from a specific external position exist, provided that the band formation is a rectangular array [10].

Along with other thorough investigations [4, 6], the concept of a static, fixed arrangement of band members has been widely investigated. While interesting questions remain [5], we investigate here the mathematical equivalent of what marching bands do: they march from one position to another. In particular, if a marching band member is weakly visible in the original marching band formation and the formation translates itself in some way, will that same member be weakly visible after the translation?

2. Background

Throughout this paper, unless otherwise noted, all points are assumed to be lattice points in the first quadrant. We say a point P in some set S of points is *weakly visible in S* from a point $Q \notin S$ if no other point in S lies on the segment PQ. The $r \times s$ rectangular array $\{1, 2, ...r\} \times \{1, 2, ..., s\}$ of points, called $\Delta_{r,s}$, has the following property.

Corresponding author

Theorem 2.1. [6] *A point* $(x,y) \in \Delta_{r,s}$ *is weakly visible from the origin if and only if* $gcd(x,y) = 1$.

Because we only investigate weak visibility from the origin, any reference to a point being weakly visible means that that point is weakly visible from the origin. If P is some weakly visible point in a collection S of points, an injective function $f : S \to \mathbb{Z}^+ \times \mathbb{Z}^+$ is said to *preserve the weak visibility of* P if $f(P)$ is weakly visible in $f[S]$. If f preserves the weak visibility of every point in S, then f is called *weak visibility preserving on* S.

Theorem 2.1 immediately yields a sufficient condition for knowing that a function is weak visibility preserving, regardless of the set S it is defined on.

Lemma 2.2. *Let* $f : S \to \mathbb{Z}^+ \times \mathbb{Z}^+$ *be an injective function with the property that if* (a,b) *is a relatively prime pair of positive integers, then* $f(a,b)$ *is a relatively prime pair of positive integers. Then,* f *is weak visibility preserving on any set* S.

An example of such a function is $f(x,y) = (x^m, y^n)$, for $m, n \in \mathbb{Z}^+$. If $gcd(x,y) = 1$, then $gcd(x^m, y^n) = 1$.

We focus here on two basic types of functions defined on $\Delta_{r,s}$, resulting in the following classifications. In each of the following statements, $m, n \in \mathbb{Z}^+$.

(1) For certain values values of r and s, $f(x,y) = (x+m, y+n)$ is not weak visibility preserving on $\Delta_{r,s}$.

(2) On $\Delta_{r,s}$, the function $f(x,y) = (mx, ny)$ is weak visibility preserving.

A key component of the proofs of these results is a corollary to the following result.

Theorem 2.3. [10] *The point* $Q = (x,y) \in \Delta_{r,s}$ *is not weakly visible by the point* $P = (a,b)$ *if and only if all of the following conditions hold:*

(1) $gcd(a-x, b-y) > 1$,

(2) $r - x \geq (a-x)/gcd(a-x, b-y)$, *and*

(3) $s - y \geq (b-y)/gcd(a-x, b-y)$.

Corollary 2.4 follows immediately.

Corollary 2.4. *Let* $f(x,y) = (x+m, y+n)$ *for* $m, n \in \mathbb{Z}^+$. *Then, the point* $f(x,y) = (x+m, y+n)$ *is not weakly viewable in* $f[\Delta_{r,s}]$ *if and only if all of the following conditions hold:*

(1) $gcd(x+m, y+n) > 1$,

(2) $x - 1 \geq (x+m)/gcd(x+m, y+n)$, *and*

(3) $y - 1 \geq (y+n)/gcd(x+m, y+n)$.

3. Main Results

In this section we prove our two main theorems, Theorems 3.1 and 3.3. Contextually, the first theorem proves that a rigid translation of a rectangular marching band is not weak

visibility preserving while the latter theorem shows that scaling a rectangular marching band formation does preserve weak visibility.

Theorem 3.1. *For r and s sufficiently large, the function f defined by $f(x, y) = (x + m, y + n)$ (where m, n are non-negative integers with at least one nonzero) is not weak visibility preserving on $\Delta_{r,s}$.*

Proof. Consider first the case when m and n are both nonzero. Let p be a prime dividing neither m nor n. We prove that $f(x, y) = (x + m, y + n)$ is not weak visibility preserving. In pursuit of this, we aim to construct x and y such that x and y are relatively prime, but $x + m$ and $y + n$ are not.

Because

$$mp = -m(1 - p) + m,$$

we have that $p \,|\, [-m(1 - p) + m]$. Then, for any $a \in \mathbb{Z}$, $p \,|\, [-m(1 - p) + ap + m]$. Because $p \geq 2$, choose $a \in \mathbb{Z}$ such that

$$2pm + p - p^2 m \leq a(p^2 - p),$$

or equivalently,

$$\frac{-m(1 - p) + ap + m}{p} \leq -m(1 - p) + ap - 1.$$

Define $x = -m(1 - p) + ap$ and we have that both $p \,|\, (x + m)$ and

$$\frac{x + m}{p} \leq x - 1. \tag{1}$$

Note that p is not a divisor of x, else it necessarily must divide m. Thus, there exist $b_1, b_2 \in \mathbb{Z}$ such that

$$b_1 p + b_2 x = 1.$$

This yields

$$(1 + n)b_1 p = (1 + n)(-b_2)x + (1 + n),$$

so that $p \,|\, [((1 + n)(-b_2)x + 1) + n]$.
Then, for any $b \in \mathbb{Z}$, we have that $p \,|\, [(((1 + n)(-b_2) + bp)x + 1) + n]$. As before, since $p \geq 2$, we can choose b so that

$$x(1 + n)(-b_2)(1 + p) + 1 + n \leq bx(p^2 - p),$$

or equivalently

$$\frac{((1+n)(-b_2)+bp)x+1+n}{p} \leq ((1+n)(-b_2)+bp)x.$$

Define $y = ((1+n)(-b_2)+bp)x+1$ and we see that

$$\frac{y+n}{p} \leq y-1. \tag{2}$$

Note that $p|(y+n)$, giving $\gcd(x+m,y+n) \geq p$. This, along with Equations 1 and 2, proves via Corollary 2.4 that $f(x,y)$ is not weakly visible in $f[\Delta_{r,s}]$ for sufficiently large r and s. Moreover, note that if d divides both x and y, by the definition of y, then d must divide 1. Hence, x and y are relatively prime, ultimately proving the result when $m, n > 0$.

Consider now the case when exactly one of m or n is nonzero. Without loss of generality, assume that $n = 0$. When $m > 1$, the proof of the proceeding case holds, simply taking the prime p to not divide m. When $m = 1$, the point $(2,3)$ is weakly viewable while $f(2,3) = (3,3)$ is not. $\qquad\square$

To prove the second main result, we utilize the following lemma.

Lemma 3.2. *Suppose the function f is given by $f(x,y) = (mx, ny)$, where $m, n \in \mathbb{Z}^+$. A point $P \in \Delta_{r,s}$ is weakly visible if and only if $f(P)$ is weakly visible.*

Proof. Let $f : \Delta_{r,s} \to \mathbb{Z}^+ \times \mathbb{Z}^+$ by $f(x,y) = (mx, ny)$, where $m, n \in \mathbb{Z}^+$. Let $P \in \Delta_{r,s}$ be the point (x_0, y_0) and first assume that P is not weakly visible. This implies that there are $t \geq 1$ points on the interior segment OP, where O is the origin. Let $Q = (x_1, y_1)$ be the first of these points to the right of O. Then $y_0 = (t+1)y_1$ and $x_0 = (t+1)x_1$.

Since $f(P) = (mx_0, ny_0)$, it follows that both $mx_0 = m(t+1)x_1$ and $ny_0 = n(t+1)y_1$. Because $f(Q) = (mx_1, ny_1)$, $f(Q)$ lies on the interior of the line segment $Of(P)$, proving that $f(P)$ is not weakly visible.

Next assume that $f(P) = (mx_0, ny_0)$ is not weakly visible. Because of this, there are $t \geq 1$ points in $f[\Delta_{r,s}]$ which lie on the interior of the line segment $Of(P)$. Let $f(Q) = (mx_1, ny_1)$ be the first of these points to the right of O. Then $mx_0 = m(t+1)x_1$ and $ny_0 = n(t+1)y_1$, or equivalently, $x_0 = (t+1)x_1$ and $y_0 = (t+1)y_1$. Hence $\gcd(x_0, y_0) = t+1 > 1$, implies that P is not weakly visible by Theorem 2.1. $\qquad\square$

Lemma 3.2 immediately yields our second main result.

Theorem 3.3. *The function f given by $f(x,y) = (mx, ny)$, where $m, n \in \mathbb{Z}^+$, is weak visibility preserving on $\Delta_{r,s}$.*

It is natural to consider a general linear shift of $\Delta_{r,s}$ given by the function $f(x,y) = (ax + m, by + n)$ for $a, b, m, n \in \mathbb{Z}^+$. Even though $f = h \circ g$, where $g(x,y) = (ax, by)$ and $h(x,y) = (x+m, y+n)$ with g weak visibility preserving and h not weak visibility preserving, we cannot immediately conclude that f is not weak visibility preserving. We know that h is not weak visibility preserving on a rectangular array $\Delta_{r,s}$. However, we do not know that

the image of g contains any of the points for which h does not preserve weak visibility. Thus, we conclude with the following conjecture.

Conjecture 3.4. The function given by $f(x,y) = (ax + m, by + n)$, where $a, b, m, n \in \mathbb{Z}^+$, is not weak visibility preserving on $\Delta_{r,s}$ for some values of r and s.

4. FURTHER DIRECTIONS

In this paper we only considered linear transformations of a rectangular array of points. What happens if the original set of lattice points is not required to be a rectangle? What happens under non-linear types of transformations?

In addition, while functions of the type $f(x,y) = (x + m, y + n)$ were shown to not preserve weak visibility, there is the possibility that restrictions on the size of $\Delta_{r,s}$ relative to m and n may yield a weak visibility preserving function. What sorts of restrictions would do this? Moreover, with or without restricting any of the variables, do certain functions preserve weak visibility on a higher percentage of points in the original array?

In all the mentioned situations (rectangular vs. non-rectangular, linear vs. non-linear), probabilistic investigations abound as well ("A band member standing in position A is more likely to be seen after the band moves than a band member standing in position B"). Could answers to these questions provide insights to the multitude of weak visibility questions that remain open [5]?

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Student biographies

Jack J. Billings: (*Corresponding author:* jackjbill@gmail.com) Jack graduated from North Central College in 2018 with a B.S. in Mathematics and Computer Science. Currently, he works as a software engineer in Chicago, IL. When not solving technical problems, he enjoys running around and climbing rocks.