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Abstract. A combinatorial game is a two player game that has a well-defined ruleset and no element of chance. This means that assuming optimal play, the winner can be determined before the first turn is taken. In many well-known games, this gives the first player an advantage. In an effort to address this, we observe how both players' strategies would change when the second player is given an extra turn at the start to compensate for having to go second. We focus on examples of impartial games under this change, and how it affects the strategic play of the games.

1. Introduction

In two player combinatorial games, the order that you play in determines whether or not a winning strategy is available to you [1]. One of the ways to get around this issue is a bidding game, where both players bid on the opportunity to play at every move [5, 7]. In many ways this succeeds in mitigating the advantage given by turn order. However, it requires an additional element of complexity, akin to a second game, which could make it difficult for casual players. Another alternative is to make turn order random, so that players must account for the fact that they do not know when they will next be able to play [8]. While this is a relatively simple alteration to make to the game, it has far-reaching implications in terms of strategy, and introduces an element of chance that we prefer to avoid.

In this paper we examine a more primitive equalizer. We look at the case where the second player simply takes a second turn after their first, to make up for having to go second. The turn order would continue as it would in normal play thereafter. So if the first player is 1 and the second player is 2, the order of play will be $1,2,2,1,2,1,2,\ldots$. As this 'compensates' the second player for not being able to play first, we will refer to this as the *compensation variant* going forward. In studying this, we find that rather than make a game more balanced, this compensation variant gives the second player a distinct advantage for most starting positions.

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We begin by examining the compensation variant in the general case of impartial games, which are games where both players have the same set of moves available to them [2]. As we will discuss in detail in Section 2, every position in a normal play impartial game is either an **N** position (meaning that assuming optimal play, the next player to take a turn will win) or a **P** position (meaning the previous player to take a turn will win). We demonstrate that the compensation variant gives the second player a winning strategy for many starting positions, and that there is only a very specific class of games that the first player will win under this variant. This leads to the following major result we prove in Section 2:

Theorem 2.1. *Let G be an impartial game, played with the compensation variant. If the first player can on their first turn go to either a terminal position, or to some* **N** *position with no possible moves to other* **N** *positions or terminal positions, then the first player will win. Otherwise, the second player will win.*

We demonstrate the clear advantage this gives the second player by using a few prototypical examples of impartial games. We first look at normal play Nim, as described in [3], and describe explicitly all possible starting positions the first player will win under the compensation variant. We then prove that these are the only possible games of Nim that meet the conditions of the theorem above.

We then look at Chomp, which is a misère impartial game, under this compensation variant. A misère game is a game where the player who brings the game to a terminal position loses [2]. We look at different possible starting boards, and describe exactly which starting positions allow the first player to win under the compensation variant. This also turns out to have a small number of such positions, due to the particular rules of the game. For every game that begins with a board larger than 3 by 3, the second player has a trivial winning strategy, so they win on the first player's second turn.

After examining these example games, we describe some theoretical possible situations that would allow the first player to still have an advantage in the compensation variant. We also describe other variants on turn order we are interested in studying, and conjecture on how they will affect who will win.

2. Impartial Games

2.1. **Basic Definitions.** Recall that impartial games are games where both players have the same set of moves available to them. In that way, the only difference between the two players is that one of them goes first, and the other goes second. For the sake of convenience we will call the player going first player one, and the player going second player two. Under standard rule conventions, the last player able to make a legal move wins the game. We call a position with no further moves possible a terminal position of the game. This means that assuming optimal strategy on both sides, the winner of a game can be deduced simply from the position of the game board at the start of the game. We will define the following terms as they are used in the standard alternating turn order, and then apply them to study the compensation variant described in the introduction.

We say that at every point an impartial game is in either a **P** position, meaning the **P**revious player to go will eventually win with optimal play, or an **N** position, meaning the **N**ext player to go will eventually win with optimal play. We see that **P** and **N** positions satisfy the following properties, which can be used to recursively define them as in [1].

(1) Every move from a **P** position goes to an **N** position.

(2) There exists a move from each **N** position to some **P** position.

(3) The terminal position for the game is a **P** position.

We note that under this definition there are possibly also moves which take you from one **N** position to another **N** position. These moves will be key to finding the winning strategy under the compensation variant.

With standard alternating turn order, the starting position of the game is enough to tell you which player has a winning strategy. At the beginning of a game, if the board is in a **P** position it will be a player two win, and if it is in an **N** position it will be a player one win. The winning strategy requires that a player must move so that at the end of their turn the board is always in a **P** position. The definitions of the two position types given above ensure this is always possible, as follows:

If the game begins in a **P** position, then every move the first player makes must go to a **N** position. At that point, there exists a move that player two can make that brings it to some **P** position. If player two has not won, the first player must again go to a **N** position. By continuing in this fashion, eventually the game will reach a terminal position (which is a **P** position) immediately following one of player two's moves, making them the winner.

If the game begins in a **N** position, then there exists a move that player one can take that brings it to some **P** position. At this point, every move the second player makes must go to a **N** position, and there then exists a move that player one can make that goes to another **P** position. This process eventually leads to the game reaching a terminal position immediately following one of player one's moves, making them the winner.

2.2. **Compensation Variant.** We now look at what happens under the compensation variant described in the introduction, where the second player is given a single extra turn. Under this turn order, the first three turns are unusual (in that they go 1,2,2), and then turns resume alternating as usual. However, this slight change means that knowing whether the starting board is a **P** or **N** position is no longer enough to determine who has a winning strategy. We examine all possible cases below:

2.2.1. *Game begins in a **P** Position.* We first examine the case where the game begins in a **P** position. In that case, player one's first move will always result in an **N** position.

(1) If there exists a move from this **N** position to another **N** position, then the second player should take it with their first turn. At this point the game is back to alternating turns in an **N** position with the second player to play. Therefore, the second player has a winning strategy.

(2) If there does not exist a move from this **N** position to another **N** position, then the second player's first turn must be to a **P** position. At this point either the second player has won, or the game is back to alternating turns in an **P** position with the second player to play. Therefore, the first player has a winning strategy.

2.2.2. *Game begins in an **N** Position.* We next examine the case where the game begins in an **N** position. There are now more possibilities for the first player that must be considered.

(1) If the first player can end the game on their first turn, by moving to a terminal position, the first player has a winning strategy.

(2) If not, the first player could move to a **P** position. If they do, the second player's first move will take them to an **N** position. At this point the game is back to an alternating turn order in an **N** position with the second player to play. Therefore, the second player has a winning strategy.

(3) The first player could possibly move to another **N** position on their first turn. If they do so, the arguments above for when the game begins in a **P** position apply. Either the second player can move to yet another **N** position on their first turn, and they have a winning strategy, or there are no further **N** positions available. In that case, either the second player ends the game on their first move and wins, or they end their first turn in a **P**-position. At this point the game is back to an alternating turn order in an **P** position with the second player to play, which means the first player has a winning strategy.

Looking at each of these cases, we see that there are really two key aspects that determine whether the first player can win that we must look at. The first is whether the first player can immediately win the game, without giving the second player a chance to go. The second is whether the first player can use their turn to move to an **N** position with no possible moves to other **N** positions nor to a terminal position. In all other situations, the second player will win. These results are summarized in the following theorem (previewed in the introduction).

Theorem 2.1. *Let G be an impartial game, played with the compensation variant. If the first player can on their first turn go to either a terminal position, or to some **N** position with no possible moves to other **N** positions or terminal positions, then the first player will win. Otherwise, the second player will win.*

The first case examined above for when a game begins in an **N** position describes a game that was a first player win under standard play and will remain a first player win under the compensation variant. However, this situation only arises for very simple starting positions, as it requires the game to be ended in a single turn, and we will refer to these positions as having *Winning Openings*. For the remainder of this section, we will restrict our focus to games that do not have winning openings.

The more interesting case is when the first player can use their turn to move to an **N** position with no possible moves to other **N** positions nor to a terminal position. These positions, where every possible move is to a non-terminal **P** position, will be very important to us. As they limit available options, we will call these positions *choke-point positions*.

2.3. **Choke-Point Positions.** We see that determining which player will win under the compensation variant depends entirely on whether the first player is able to move to one of these choke-point positions on their first turn. If so, then the second player is forced to move to a **P** position with their first move, and to an **N** position with their second move. Therefore, the first player will have a winning strategy for this game. This winning strategy results from a situation where the second player would prefer not to take the second turn they are given in the compensation variant, but they have no choice. Zugzwang is the formal term used for this situation, where a player would prefer to pass rather than take their turn, as taking their turn forces them to go to a weaker position [2].

We devote the rest of this section to studying these choke-point positions, and trying to determine when exactly they arise. To better understand these positions, we wil use Sprague-Grundy Numbers (commonly known as nimbers). The Sprague-Grundy Theorem demonstrates that every position of a normal play impartial game is equivalent to a nimber [9],[6]. Nimbers can be defined recursively in the following way [2]:

(1) The nimber of any **P** position is 0.
(2) The nimber of an **N** position is the minimum excluded value of nimbers among positions that can be moved to.

The minimum excluded value of a set is defined to be the smallest element of the universe not included in that set. Within the context above, this means it is the smallest value that none of the game positions that can be moved to in a single turn take [4].

Since there exists a move from every **N** position to a **P** position, we see that no **N** positions can have a nimber of 0. This leads to the following result for nimbers of the choke-point positions.

Proposition 2.2. *All choke-point positions must have a nimber of 1.*

Proof. Being a choke-point position meant all possible moves from that position must be to non-terminal **P** positions, which each have nimber 0. Therefore the minimum excluded value will be 1. □

However, we also note that being a choke-point position is not equivalent to having a nimber of 1 since it is possible a game position could move to a position with nimber 0 (a **P** position) or 2 (an **N** position) and still have a nimber of 1. It is also possible the position with nimber 0 could be a terminal position.

We can now use this proposition to classify a category of games that will all be second player wins under the compensation variant.

Theorem 2.3. *If a game starts in a position with nimber 1, and cannot be won in a single move, then it will be a second player win.*

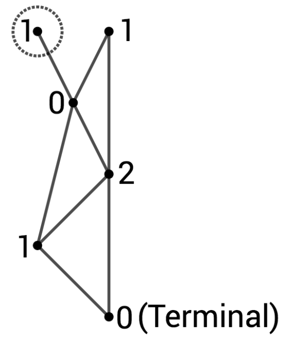

Figure 1. Game tree featuring three positions with nimber 1, only one of which (circled) is a choke-point position.

Proof. This follows immediately from proposition 1. If a position has nimber 1, that means none of the moves from that position will have nimber 1. Therefore they cannot be choke-point positions, and by the analysis of section 2.2 above, the second player will have a winning strategy. □

In the following sections, we will analyze the winning strategy for various positions of Nim and Chomp under the compensation variant. In the course of doing so, we will attempt to analyze exactly when choke-point positions arise in these games, and to make predictions about when they arise in general.

3. Nim

3.1. **Game Description.** The first example we will study under this turn order variant is Nim. Nim was first analyzed in detail by Charles Bouton in 1901, and is one of the fundamental examples of a combinatorial game [3]. While many variants of Nim exist, we will focus on normal play Nim. This is a game played with n heaps of objects, where each heap consists of an arbitrary number k_i of objects. On a player's turn, they may remove any number of objects they want from any single heap. The winner is the player who removes the final object.

One key numerical value that is associated with Nim is called the Nim sum [3]. To find the Nim sum of a given position, you first write the number of objects in each heap in base 2, and take the sum of the values in each place in $\mathbb{Z}_2$. This means if there are an even number of 1s, you get a 0 for that place, and if there are an odd number of 1s you get a 1 for that place (we denote this operation with $\oplus$). The Nim sum is the decimal number equal to that result thought of in base 2. For example, if there are heaps of size 4, 5, and 7, you would have 100, 101, and 111. Following this procedure gives 110 which is equal to 6 as a decimal number. We can write this as $4 \oplus 5 \oplus 7 = 6$.

The Nim sum, as described above, and the nimber, as defined in section 1.3, are in fact equal [2]. Therefore, we see that under standard turn order, the first player will win if the starting Nim sum is non-zero (an **N** position), and the second player will win if the Nim sum is 0 (a **P** position). As must be the case based on the definitions of **P** and **N** positions given in section 1, if the Nim sum is 0 then any move will result in a Nim sum which is non-zero, whereas if the Nim sum is non-zero then there exists some move that will result in a Nim sum of 0.

3.2. **Nim Under the Compensation Variant.** We now analyze a game of Nim played under our compensation variant, in order to determine which player has a winning strategy and what games have possible choke-point positions. Aside from a limited number of very specific starting positions (with either very small number of heaps, or small-sized heaps), we will see that the second player always has a winning strategy. We begin by examining the few special cases where our general argument will not apply, and discuss how they relate to the analysis of impartial games in general we saw in the previous section.

3.2.1. *Special Cases.*

Case 1: If there's just one heap, then the first player wins, as in the standard game. This is an example of a game with a Winning Opening. The game has an available opening move that immediately brings the game to a terminal position, giving the first player the victory. This case is no different than it would be under standard play.

Case 2: If there are two heaps, with any number of objects in each, then the second player will win. If the first player takes a complete heap, the second player wins on their first turn by taking the other heap. If the first player does not take a complete heap, the second player wins on their second turn (they take first one whole heap, and then the other). (Note: In standard play the second player would only win if both heaps had the same number of objects in them, otherwise the first player would win).

Case 3: If there are $n \geq 3$ heaps which each contain exactly one object, then the first player wins if n is even, and the second player wins if n is odd. There is no choice in the moves either player can make in this game. (Note: In standard play the first player would win if n is odd, and the second player would win if n is even).

Case 4: If there are $n \geq 3$ heaps of objects, where exactly one heap has $k > 1$ objects in it and the rest have one object, then the first player wins. Their first move should remove $k-1$ objects from the large heap if n is odd, and all of the large heap if n is even. This first action reduces the game to an odd number of heaps of size one for the second player's first turn, and as in Case 3 above, there is no further choice in either player's moves. This eventually leads to a first player win, as in the standard game. (Note: In standard play, the first player's strategy is to remove $k-1$ objects from the large heap if n is even, and all of the large heap if n is odd).

The possible first player wins in Cases 3 and 4 come from their ability to move to a choke-point position on their first turn. We now explicitly prove that these are in fact choke-point positions, as defined previously.

Proposition 3.1. *A game with an odd number of heaps greater than one, where each heap has exactly 1 object in it, is in a choke-point position.*

Proof. Assume there are an odd number of heaps greater than one, where each heap has exactly 1 object in it. This means there are at least 3 heaps, an odd number of heaps, and every heap has exactly 1 object in it. Therefore the Nim sum looks like $1 \oplus 1 \oplus \cdots \oplus 1 = 1$, with an odd number of 1's, and every possible move removes exactly one of these 1's. This gives an even number of 1's, and therefore Nim sum 0. Since any game with Nim sum 0 is in a **P** position, we see that the only available moves are to **P** positions. In addition, since there are at least 2 heaps remaining these must be non-terminal **P** positions. By definition, choke-point positions are positions where all possible moves lead to non-terminal **P** positions. Therefore, game positions of this type must be choke-point positions. □

The strategy described in case 4 above always leads to one of these choke-point positions, and is therefore a winning strategy for the first player.

3.2.2. *General Case.* Having looked at the exceptional cases above, we now turn our attention to all other possible initial positions. We find that they will in fact all be second player wins.

Theorem 3.2. *Under the compensation variant, for every starting position of Nim with $n \geq 3$ heaps where at least two heaps have more than one object, the second player will win.*

The key to this result, as we saw in section 2.2, is the existence of moves that take you from one N-position to another. We will describe exactly when these moves exist, by means of the following corollary:

Corollary 3.3. *For $n \geq 3$ heaps containing $k_1, k_2, \ldots k_n$ objects respectively, with $k_1 > 1$ and non-zero Nim sum, there is always an available move to another position that also has non-zero Nim sum.*

Proof. We will show this by looking at four possible cases, depending on whether the original Nim sum was even or odd, and whether k_1 is even or odd.

(1) If $k_1 \oplus k_2 \oplus \cdots k_n$ gives an odd nimber (i.e. the binary representation ends in 1) and k_1 is even, then we can remove all k_1 objects from heap 1 and still have non-zero Nim sum.

(2) If $k_1 \oplus k_2 \oplus \cdots k_n$ gives an odd nimber (i.e. the binary representation ends in 1) and k_1 is odd, then we can remove $k_1 - 1$ objects from heap 1 and still have non-zero Nim sum.

(3) If $k_1 \oplus k_2 \oplus \cdots k_n$ gives an even nimber (i.e. the binary representation ends in 0) and k_1 is even, then we can remove $k_1 - 1$ objects from heap 1 and be sure to have non-zero Nim sum.

(4) If $k_1 \oplus k_2 \oplus \cdots k_n$ gives an even nimber (i.e. the binary representation ends in 0) and k_1 is odd, then we can remove all k_1 objects from heap 1 and be sure to have non-zero Nim sum.

□

This corollary, together with cases 1 and 2 of the previous section, also proves that the choke-point position seen in cases 3 and 4 is the only possible choke-point position in Nim. This lack of choke-point positions is what leads to the second player's dominance in the compensation variant version of Nim.

For completeness, we explicitly describe the winning strategy for the second player in the general case described above. However, we already know such a strategy must exist by the analysis of section 2.2. As a reminder, the starting position is $n \geq 3$ heaps, where at least two heaps have more than one object in them. We begin by analyzing the possible first moves for the first player:

(1) Take an entire heap. This either leaves 2 heaps (which is a trivial win for the second player) or it leaves at least 3 heaps, of which at least one has more than one object in it.

(2) Take some other number of objects from a single heap. This leaves at least 3 heaps, of which at least one has more than one object in it.

So at the start of the second player's turn, we see they either trivially win, or there are $m \geq 3$ heaps where at least one has more than one object in it. Either the Nim sum is 0, or it is not. If the Nim sum is 0, then their first move will always lead to a non-zero Nim sum. They can then always find some move on their second turn to return to Nim sum 0. On the other hand, if the Nim sum begins non-zero then by Corollary 1 there is some first move which will lead to another non-zero Nim sum. They can then always find some move on their second turn which gives Nim sum 0. So in either situation, they have put the game in a **P** position with the first player to play and alternating turns. At this point, if the second player plays optimally they can always end their turn with Nim sum 0, and eventually win the game.

We therefore see that the second player wins almost every possible starting position under the compensation variant of Nim, with a few classes of exceptions as given in section 3.2.1. In those classes, the loss was either due to the first player immediately winning the game, or being able to move to a position with non-zero Nim sum, where every possible move was to Nim sum 0. We also have proven the following result on the choke-point positions of Nim:

Proposition 3.4. *The only choke-point positions in Nim are those which have an odd number of heaps, more than 1 heap, and exactly one object in each heap.*

4. Chomp

4.1. **Game Description.** Chomp is an impartial combinatorial game in which two players alternate removing squares from a given starting arrangement of squares on a grid. If a square is present, we require all squares below and to the left of it be present as well. When one player chooses a space, that player "then removes all squares above it in that column and all squares to the right of it in that row, together with all squares above and to the right of these" [10]. The game continues until someone is forced to take the space at the bottom left of the board. Chomp is a misère game, thus the player who takes this space is declared the loser. One story illustrating a possible context for the game is that the rectangular grid that the game typically takes place on represents a chocolate bar and the bottom left piece is poisoned, which is why neither player wants to remove or "eat" that piece [10]. One method used to analyze this game is to consider it a non-misère game where the terminal position is a board with only the poisoned square remaining.

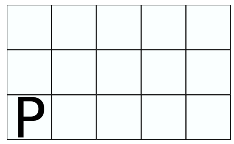

Figure 2. A 3×5 Chomp board. The "P" represents the poisoned piece.

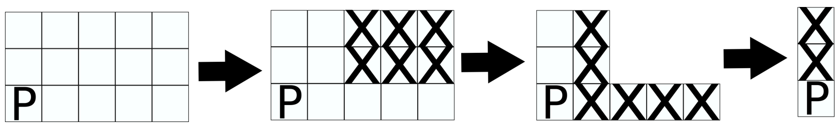

Figure 3. A possible game of Chomp played on the above board. Each "X" is a space that is removed on that turn, where the bottom left "X" is the space selected by the player. The person who played last in the above game is the winner, since in the next move the poisoned square is taken.

Under standard turn order, we have the following well-known results about Chomp:

Proposition 4.1. *When Chomp is played on a rectangular grid under standard turn order, the first player has a winning strategy.*

We remind the reader why this is through the method used by Stewart in [10].

Proof. On a $1 \times n$ board, the first player clearly has a winning strategy by taking the piece adjacent to the poisoned square. On a 2×2 board, the first player has a winning strategy by taking the piece diagonal to the poisoned square. Now, suppose the second player has a winning strategy on a rectangular board larger than 2×2. This means that for any move the first player makes on their turn, the second player will be able to win. Say the first player takes the square in the top right corner. Then the second player takes their first turn, and they begin to play their winning strategy. Regardless of where the second player goes on this turn, the first player could have made that move on their first turn (See figure 4).

We see that the board at the end of the second player's turn is identical to a move that the first player could have made on their first turn. This means the first player could 'steal' this winning strategy, and win themselves. ☐

We find that the first player can win any rectangular chomp board other than the (trivial) 1×1 board, though we do not give a winning strategy explicitly. Therefore every rectangular board is an **N**-position.

We now begin looking at non-rectangular boards. The simplest example is the Chomp board in figure 5.

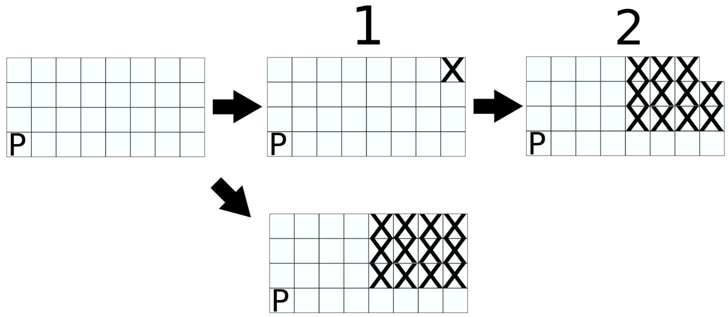

Figure 4. A 4×8 Chomp board. The 1's and 2's above the boards note who's turn it is.

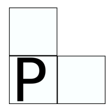

Figure 5. The simplest non-rectangular Chomp board.

This is clearly a second player win, as the first player must take one adjacent square and the second player must take the other. The order that these squares are taken does not matter, as one square does not eliminate the other. The first player must then take the poison square and lose. Thus, this board is a second player win. In fact, we can show that this leads to an entire class of **P** positions, through strong induction.

Proposition 4.2. *Any* 2 *row board where the bottom row has n squares and the top row has* $n-1$ *squares will be a second player win (i.e. a* **P** *position).*

Proof. We've already proven the base case above. Now, we assume that any board of this type with less than k squares in the bottom row is a **P** position, and use this to prove that a board of this type with k squares in the bottom row is also a **P** position.

On a board with k squares in the bottom row and $k-1$ squares in the top, the first player can make a move on either the bottom row or the top row. If they move on the bottom row, the resulting board will be a perfectly rectangular board, no matter where the first player moved to. The second player can then move in the upper right corner to bring the game to what the induction hypothesis implies is a **P** position. On the other hand, if the first player moves on the top row then they reduce it to having m squares, where $m < k-1$. The second player can then move on the bottom row to reduce it to $m+1$ squares, again bringing the game to a **P** position by the induction hypothesis. Thus, by the definitions of **P** and **N** positions, we see that the 2 row board with k squares in the bottom row and $k-1$ squares in the top row is also a **P** position. □

We extend these arguments one step further, to obtain the following result:

Proposition 4.3. *Any* 2 *row board where the bottom row does not have exactly one more square than the top row is an* **N** *position.*

Proof. In the case of a rectangular board, we're already finished by proposition 4.1. By the rules of Chomp given earlier, the bottom row must have at least as many squares in it as the top row, so if this is not the case then it must have at least two more squares. The first player can take enough to reduce it to the case of proposition 4.2, which is a **P** position. Therefore, the initial board must be an **N** position. □

We end our discussion of winning strategies for standard turn order Chomp here, with the caveat that determining the winner on larger non-rectangular boards can become quite difficult. We will see that while strategies for standard Chomp can become rather complex, especially for non-rectangular boards, the compensation variant makes Chomp much simpler.

4.2. **Chomp under the Compensation Variant.** Recall that the compensation variant being used is a change in the original rules so that the second player makes two moves on their first turn. This seems like it might address the initial advantage the first player had, at least in the board varieties studied in the previous section. However, we find that once boards become large enough this variant makes Chomp an extremely straightforward second player win.

We also note that our previous method of analyzing an equivalent non-misère game where the terminal position is a board with only the poisoned square remaining will no longer work with the possibility of extra turns. In this case, if the second player reduces the board to just the poison square on their first turn, they will in fact lose since they must play their second turn. As such, we carefully describe the winning strategies on a variety of game boards. As with Nim, we will see that the second player frequently has a winning strategy. We begin by examining the few special cases where the general argument showing this will not apply, before proving the general result.

4.2.1. *Special Cases.* We again begin by looking at a few special cases, with small board size.

Case 1: The simplest case, a 1×1 square, is a second player win because the first player is forced to take the poisoned square on the first turn. This is unchanged from standard play.

Case 2: A $1 \times n$ board with $n > 1$ the first player should begin the game by taking the square adjacent to the poisoned square. This will result in a first player win because on the second player's first turn, they must take the last square and lose. This case is also unchanged from standard play.

Case 3: In any board with 2 rows (or equivalently 2 columns), the first player has a winning strategy. They can begin the game by taking a piece directly adjacent to the poison square on their first turn, leaving a 2×1 board (See figure 6). At this point, the second player must take two turns, so on one of those two turns they must take the poison square, leading to a first player win. Note: In standard play we saw this could be either a first player win or a second player win, depending on the specific number of squares in each row. See propositions 4.1, 4.2, and 4.3.

4.2.2. *General Case.* We now look at all boards that have at least 3 rows and 3 columns. In this case, we found the game became surprisingly simple in the compensation variant.

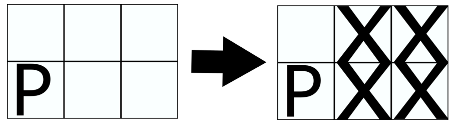

Figure 6. A 2×3 Chomp board (left) and the resulting board after the first player moves to the right of the poisoned piece (right).

Theorem 4.4. *Under the compensation variant, for every starting position of Chomp with at least 3 rows and 3 columns, the second player will win.*

Proof. There are two options for the first player's turn in boards of this type; either they initially move adjacent to the poisoned square, or they do not.

If the first player moves to a square adjacent to the poisoned square. This turns the board into a $n \times 1$ board or a $1 \times m$ board. It is now the second player's turn and they may make two moves in this variant of the game. The second player should first take the square two away from the poisoned square, leaving a 1×2 board, and on their second move, they should take the square adjacent to the poison. It is now the first player's move, and they must take the poison square, so they lose (see figure 7).

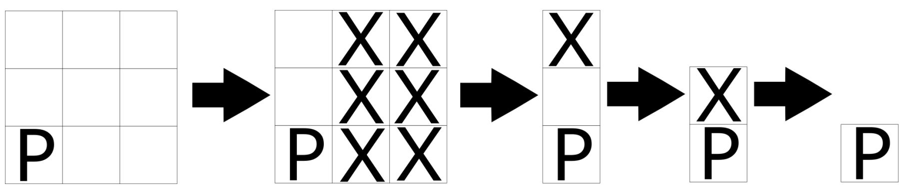

Figure 7. A 3×3 Chomp board and the turns that will play out if the first player moves adjacent to the poison under the compensation variant

The other option is for the first player to move in any space other than one of the two squares adjacent to the poison. This could change the board to a variety of possible configurations, but the important thing to note is that both of the squares adjacent to the poison will still be available. On the second player's two turns, they should take both of these squares in either order. This will reduce the board to just the poisoned square, which the first player must take on their turn (see figure 8).

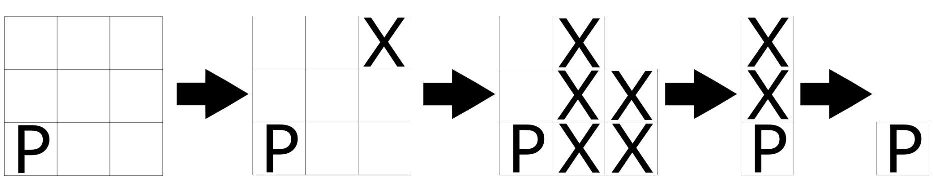

Figure 8. A 3×3 Chomp board and the turns that will play out if the first player moves somewhere not adjacent to the poison under the compensation variant

□

In the special cases looked at previously, the boards examined were too small for the second player to benefit from their advantage. However, we now see that once Chomp boards become large enough, the second player will win every time if they play under the compensation variant. Unlike some other games, this change in the rules of Chomp totally trivializes the game. Not only does the second player have a winning strategy, but they can effectively win before the first player even has a chance to take their second turn.

4.2.3. *Non-misère Alteration to Compensation Variant.* Another possibility that was considered in studying the compensation variant was how the game would change if the win condition really was to reduce the board to a single square. As mentioned at the start of section 4.2, while equivalent to the misère version under standard play, being brought to a single square is not equivalent under the variant since the second player could go to this position on their first turn, and then be forced to take the last square on their second turn.

However, this alteration only makes the second player even more dominant, as the game described in Special Case 3 (illustrated by figure 6) now also becomes a second player win. Using the strategy described for the first player leads to a win on the second player's first turn, and any alternate first player move leads to a second player win by the method described in the general case. It is the specific mechanics of a turn of Chomp, rather than the misère nature of the game, that gives the second player such an overwhelming advantage under the compensation variant.

5. Other Games

In the above analyses, we see that it requires a very specific set of conditions for these choke-point positions to occur. This is because as game trees expand, it became more and more difficult to find an **N** position that can only move to **P** positions. We already know that any position with a nimber other than one cannot be such a position, and these were very common in both Nim and Chomp.

Looking at other impartial games we would be interested in seeing if this trend continues. We conjecture this will continue to be the case, and that outside of a small class of special cases other impartial games will also become second player wins under the compensation variant. As seen in Chomp, taking two turns back to back allows for strategies that would otherwise be impractical and which quickly end the game in the second player's favor. On the other hand, while the second player could end their second turn in a winning position within Nim, they could typically not win the game outright in those two turns. We are also interested in this fundamental difference between the two games, and seeing if there is a pattern to the types of games that can be ended immediately in this way.

We are also interested in understanding the special cases where the first player still has a winning strategy, and feel there is more to learn from the specific games of this form. The choke-point positions described in section 2 are unique due to the lack of choice permitted the player, despite being in a beneficial **N** position, and it would be interesting to learn if and when such positions exist within other games as well.

6. Future Directions

Clearly this variant to turn order fails to make games more fair. Rather, it simply swings the imbalance to predominantly favor the second player, across most starting positions of impartial games. An alternative variant, that we expect to be more fair, is a 'snake-style' game. In this order, after the first player has their first turn, players would effectively alternate taking two turns in a row in a two-player game. Put another way, this would change the turn order to 1,2,2,1,1,2,2,1,.. In many ways, this can be thought of as a new standard combinatorial game, where players have more options for what they can do on their turn. However, we note that these are not quite standard games, since the first player's first turn has slightly different options available (and standard impartial combinatorial games require that all turns have the same possible moves available).

Under this variant, Chomp would continue to be a second player win for all games with at least 3 rows or columns. However, 'snake-style' Nim becomes much more interesting. After the first turn, it effectively becomes a variant of Nim where players take pieces from two different piles on each turn (Or a single pile, if they take at least 2). It would not be enough to go to a position with a Nim sum of zero, and would require a deeper analysis to determine what **P** and **N** positions would entail in this new game.

Another area we are very interested in continuing to study is the effect of change in turn order on partizan games. We have looked at some small boards of Domineering under the compensation variant focused on in this paper, and found that in general the games where the winner depended on the types of moves allowed (either 'Left' player wins or 'Right' player wins) remained wins for that type of player. This again did not hold for exceptionally small boards, but was seen to be the case for $2 \times n$ boards with n larger than 4. Looking at the results of the compensation variant for other board sizes within domineering, as well as other partizan games, would continue to be of interest.

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