The Minnesota Journal of Undergraduate Mathematics

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The Minnesota Journal of Undergraduate Mathematics

Volume 4 (2018-2019 Academic Year)

Sponsored by
School of Mathematics
University of Minnesota
Minneapolis, MN 55455

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Abstract. A right tetrahedron, also known as trirectangular tetrahedron, is a tetrahedron with three right angled triangles and a base triangle. By studying Pythagorean triples and quadruples, we deduce the existence of different types of right tetrahedra. Three cases of right tetrahedra are the focus of this paper: (i) Right tetrahedra with all integer edges (ii) Right tetrahedra with integer face areas, and (iii) Right tetrahedra with integer face areas and integer principal edges. Case (iii) constitutes the original work of this paper. The constructive method we use for this case allow us to classify existence of such tetrahedra as well as provide an algorithm to find those integer values from a given set of Pythagorean quadruple.

1. Introduction: Right Tetrahedra

Definition 1.1. A *tetrahedron* is a polyhedron that has four triangular faces.

Definition 1.2. A *right tetrahedron*, also known as a *trirectangular tetrahedron*, is a tetrahedron with three faces which are right-angled triangles or, equivalently, all three face angles at one vertex are right angles and a fourth triangular base face.

The base triangle (i.e. the base face) of the right tetrahedron is the opposite of the vertex where the right angles meet. In this paper, we will observe different kinds of right tetrahedra, but our primary focus is on right tetrahedra whose edges, principal edges, or face areas are integer. Before exploring different kinds of right tetrahedra, here is the definition of *principal edges*.

Definition 1.3. *Principal edges* are the edges of a right tetrahedron which are connected to each other by a same vertex and form a right angle on the vertex.

Here are the kinds of right tetrahedra that will be discussed.

I) Right tetrahedra with all integer edges (see Section 2.2). The existence of such right tetrahedron has already been covered in [1].

II) Right tetrahedra with integer face areas (see Section 2.3). The existence of such right tetrahedron is essentially covered by De Gua's Theorem and Pythagorean quadruples.

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III) Right tetrahedra with integer face areas and integer principal edges (see Section 3). This case constitutes the original work of this paper. The constructive method we use also allow us to classify all such tetrahedra.

Before we proceed with the contents of the paper, here's an interesting fact about a tetrahedron. There exists a tetrahedron whose edges, areas of the faces, and volumes are all natural numbers (see [1, Section 15.24]). This tetrahedron has two edges 896, 990, and the remaining four edges are 1073 each. Two faces of this tetrahedron have areas equal to 436800, the other two have 47120 and the volume is 62092800. But a right tetrahedron with all integer edges and areas is yet to be determined.

2. Background

The contents under this section are not part of the original proof but are essential in deducing that a right tetrahedron with integer edges and integer area for the faces exists.

2.1. **Pythagorean Triples and Quadruples.** We start by reviewing Pythagorean triples and quadruples.

2.1.1. *Pythagorean Triples.* Pythagorean triples are set of three integers (x, y, z) such that $x^2 + y^2 = z^2$. From [1, Chapter 2].

Definition 2.1. A given Pythagorean triple is said to be primitive if x, y and z have no common divisor.

Theorem 2.2. *A triple of positive integers x, y, and z, is a solution for the equation $x^2+y^2 = z^2$, if the following equations can be satisfied:*

$x = u^2 - v^2$

$y = 2uv$

$z = u^2 + v^2$

where u and v are some positive integers such that $u > v > 0$. Moreover x, y, z is a primitive triple if and only if $\gcd(u, v) = 1$ *and z is odd, that is to say u and v are not both odd nor both even, so one is even and the other is odd.*

There exist infinitely many primitive Pythagorean triples. Also, it is not possible to get the same primitive root from different pairs (u, v) that satisfy the restrictions.

2.1.2. *Pythagorean Quadruples.* The classification of Pythagorean quadruples depends on four parameters. More precisely, we have the following Theorem

Theorem 2.3. *A quadruple (x, y, z, m) of positive integers, where z is odd and* $\gcd(x, y, z) = 1$, *is a solution of the Diophantine equation*

$$x^2 + y^2 + z^2 = m^2$$

if, as stated in [2], *we can write:*

$x = 2(uw - vt)$,

$y = 2(ut + vw)$,

$z = (u^2 + v^2 - w^2 - t^2)$,

$m = (u^2 + v^2 + w^2 + t^2)$.

Conditions for the above integer solution (x, y, z, m) *to be unique are (see* [4]*):*

- $uw > vt$
- $u^2 + v^2 > w^2 + t^2$
- *(1)* $u \geq 1, v \geq 0$ *(2)* $w \geq 1, t \geq 0$ *(3)* $t + v \geq 1$
- $u + v + w + 1 \equiv 1 (\text{mod } 2)$
- $(u^2 + v^2, w^2 + t^2, ut + vw) = 1$
- $t = 0 \rightarrow u \leq v, v = 0 \rightarrow w \leq t$

Proof. Since the result for Pythagorean quadruples is less known than that of Pythagorean triples, we sketch the proof given in [3]. Assume $x_1 = \frac{1}{2}x$ and $y_1 = \frac{1}{2}y$. Using the given Pythagorean equation $x^2 + y^2 + z^2 = m^2$, we have:

$$\begin{aligned} (2x_1)^2 + (2y_1)^2 + z^2 &= m^2 \\ (2x_1)^2 + (2y_1)^2 &= m^2 - z^2 \\ (2x_1)^2 + (2y_1)^2 &= (m-z)(m+z) \\ {x_1}^2 + {y_1}^2 &= \frac{(m-z)(m+z)}{4} \\ {x_1}^2 + {y_1}^2 &= \frac{(m-z)}{2}\frac{(m+z)}{2} \end{aligned}$$

Let us set $f = \gcd(x_1, y_1), f_1 = \gcd(f, \frac{1}{2}(m+z))$ and $f_2 = (f, \frac{1}{2}(m-z))$. Since, by hypothesis $x, y,$ and z are relatively prime, we have that

$$\gcd(f_1, f_2) = 1, \text{ and } f = f_1 \cdot f_2. \tag{1}$$

Now set $x_2 = \frac{x_1}{f}$, $y_2 = \frac{y_1}{f}$, $z_1 = \frac{(m+z)}{2{f_1}^2}$ and $z_2 = \frac{(m-z)}{2{f_2}^2}$. Then, from equation (1), we get ${x_2}^2 + {y_2}^2 = z_1 \cdot z_2$ where $\gcd(x_2, y_2) = 1$ and z_1 and z_2 are not necessarily relatively prime. Here, let

$$x_2 + iy_2 = \prod_{j=1}^{n} (\pi_j)$$

be a factorization into gaussian primes where $x_2 + iy_2$ cannot be divided by a rational prime p since $\gcd(x_2, y_2) = 1$. Also, none of π_j is a rational prime congruent to 3 (mod 4) as it contradicts $(x_2, y_2) = 1$.

Now, $x_2 - iy_2 = \prod_{j=1}^{n}(\overline{\pi}_j)$ and

$$z_1 \cdot z_2 = \prod_{j=1}^{n}(\pi_j\overline{\pi}_j), \quad z_1 = \prod_{j=1}^{m}(\pi_j\overline{\pi}_j), \quad z_2 = \prod_{j=m+1}^{n}(\pi_j\overline{\pi}_j).$$

Set

$$u_1 + iv_2 = \prod_{j=1}^{m}(\pi_j), \quad w_1 + il_1 = \prod_{j=m+1}^{n}(\pi_j).$$

Then

$$\begin{aligned} z_1 &= (u_1 + iv_2)(u_1 - iv_2), \\ z_2 &= (w_1 + il_2)(w_1 - il_2), \\ x_2 + iy_2 &= (u_1 + iv_2)(w_1 + il_2) \end{aligned}$$

where $u_1 + iv_2 = \gcd(z_1, x_2 + iy_2)$ and $w_1 + il_2 = \gcd(z_2, x_2 + iy_2)$.

Now, setting $u = f_1 u_1$, $v = f_1 v_1$, $w = f_2 w_1$, and $t = f_2 t_1$, we obtain the Pythagorean quadruple equation's integer solutions. □

2.2. **Right Tetrahedra with All Integer Edges.** The existence of a right tetrahedron with all integer edges has already been covered in [1] using Pythagorean triples. The following theorem obtained from [1, Section 15.17] shows that a right tetrahedron can have all integer edges.

Theorem 2.4. *Let (x, y, z) be a Pythagorean triple. Let $a = x|4y^2 - z^2|$, $b = y|4x^2 - z^2|$, $c = 4xyz$. Then a right tetrahedron having $a, b,$ and c as length of its principal edges, has all integer edges.*

Proof. Let (x, y, z) be a Pythagorean triple and take: $a = x|4y^2 - z^2|$, $b = y|4x^2 - z^2|$, $c = 4xyz$. Using $x^2 + y^2 = z^2$ and the expressions of $a, b,$ and c, we can obtain the following identities:

$$\begin{aligned} a^2 + b^2 &= (z^3)^2 \\ a^2 + c^2 &= x^2(4y^2 + z^2)^2 \\ b^2 + c^2 &= y^2(4x^2 + z^2)^2 \end{aligned}$$

For the absolute value expressions, we assessed the square of absolute value expression (e.g. $|4y^2 - z^2|^2$) as only the square of the expression (e.g. $(4y^2 - z^2)^2$). This assessment was done since the absolute value of a given expression is always either positive (e.g. $(4y^2 - z^2)$) or negative (e.g. $-(4y^2 - z^2)$) value so square of those values are the same (e.g. $(4y^2 - z^2)^2 = (-(4y^2 - z^2))^2$).

As we can see, the sum of squares of any two principal edges' length gives a perfect square solution which determines that the length of the remaining edge is integer as well. Therefore, we can associate a right tetrahedron with all integer edges to any Pythagorean triple. □

Since we know that there exists infinitely many primitive Pythagorean triples from Section 2.1.1, we can also say that there exists infinitely many right tetrahedra with all integer edges.

2.3. **Right Tetrahedra with Integer Areas for the Faces.** We will also summarize the prior results of De Gua that address the case of when a tetrahedron has faces with all integer areas. With reference to Figure 1, A, B, C are the areas of respective labeled right triangle faces of a right tetrahedron and D is the area of the respective labeled base face of the tetrahedron. In this section we will determine under what circumstances areas A, B, C, and D are all integers. In order to do so, we apply the De Gua's Theorem which provides an algebraic relation among A, B, C, and D. For reader's sake, we provide a detailed proof to the De Gua's Theorem which is based on the De Gua's Theorem's original proof.

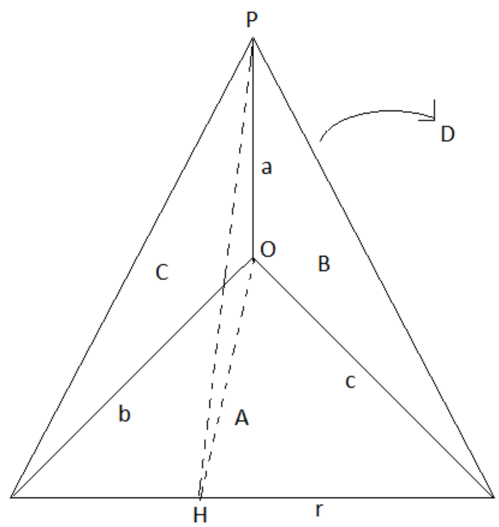

Figure 1.

Theorem 2.5. *Let A, B, C, D be the face areas of a right tetrahedron where A, B and C represents the areas of right angled triangle faces meeting at vertex o and D is the area of the base face. Then*

$$A^2 + B^2 + C^2 = D^2. \tag{2}$$

Proof. Here, let A, B, C, D be the face areas of a right tetrahedron and a, b, c be the length of its *principal edges*. With reference to Figure 1, we have:

$$A = \frac{bc}{2}, \quad B = \frac{ac}{2}, \quad C = \frac{ab}{2}, \quad r = \sqrt{b^2 + c^2}, \quad D = \frac{\overline{PH} \cdot r}{2},$$

where $\overline{PH}$ is the height of the face with area D and $\overline{PH} = \sqrt{a^2 + \overline{OH}^2} = \sqrt{a^2 + \dfrac{b^2c^2}{b^2+c^2}}$. So we get:

$$\begin{aligned}
\overline{PH} &= \sqrt{\frac{a^2b^2 + a^2c^2 + b^2c^2}{b^2+c^2}},\\
\overline{PH}^2 &= \frac{a^2b^2 + a^2c^2 + b^2c^2}{r^2},\\
\overline{PH}^2 r^2 &= a^2b^2 + a^2c^2 + b^2c^2,\\
\frac{\overline{PH}^2 r^2}{4} &= \frac{a^2b^2}{2^2} + \frac{a^2c^2}{2^2} + \frac{c^2b^2}{2^2}.
\end{aligned}$$

The last equation is exactly:

$$D^2 = A^2 + B^2 + C^2.$$

□

An integer solution of equation (2) is a Pythagorean quadruple. However, so far, we did not provide a complete proof of the existence of a right tetrahedron with all integer faces. This will be done in the next section, where we will construct explicit examples.

3. Right Tetrahedra with Integer Faces and Integer Principal Edges

In this section, we present the original result of this paper. We establish the existence of right tetrahedra with integer faces and integer edges. We also provide a constructive algorithm to determine the principal edges of such tetrahedra.

Theorem 3.1. *Let A, B, C and D be a Pythagorean quadruple, then there exists a right tetrahedron with areas kA, kB, kC and kD and integer principal edges for some positive integer k.*

Proof. Let A, B, C, D be a Pythagorean quadruple where A, B and C represents areas of right angled triangles and D is an area of a triangle. Then, we have a, b, c such that:

$$a = \sqrt{\frac{2BC}{A}}, \quad b = \sqrt{\frac{2AC}{B}}, \quad c = \sqrt{\frac{2AB}{C}}. \tag{3}$$

Here, we chose a, b, c such that for right triangle with area A, b and c are it's base and height and $A = \frac{bc}{2}$, for right triangle with area B, a and c are it's base and height and $B = \frac{ac}{2}$ and for right triangle with area C, b and a are it's base and height and $C = \frac{ba}{2}$.

Without loss of generality, we may assume the Pythagorean quadruple (A, B, C, D) is primitive, i.e. $\gcd(A, B, C) = 1$ and we have:

$$A = \frac{bc}{2}, \quad B = \frac{ac}{2}, \quad C = \frac{ab}{2}.$$

Hence, a, b, c are side lengths for right triangles with areas A, B, C.

As we know, (A, B, C, D) is a primitive Pythagorean quadruple. Therefore, for every positive integer k, (kA, kB, kC, kD) is still a Pythagorean quadruple. We replace our (A, B, C)

with (kA, kB, kC) respectively in equation (3) to get the following new edges for new triangles with area (kA, kB, kC):

$$a_2 = \sqrt{\frac{2kBkC}{kA}}, \quad b_2 = \sqrt{\frac{2kAkC}{kB}}, \quad c_2 = \sqrt{\frac{2kAkB}{kC}}.$$

$$a_2 = \sqrt{\frac{2kBC}{A}}, \quad b_2 = \sqrt{\frac{2kAC}{B}}, \quad c_2 = \sqrt{\frac{2kAB}{C}}. \tag{4}$$

Our goal is to determine k such that the argument of each square root is a perfect square. Initially, we factor each A, B, C such that

$$A = F_A S_A, \quad B = F_B S_B, \quad C = F_C S_C,$$

where S_A, S_B, S_C are the maximal perfect squares dividing A, B, and C respectively, and F_A, F_B, F_C are the remaining square free divisors. Notice that such factorization is unique.

We now focus on the remaining factors F_A, F_B, F_C in order to determine k so that we get a rational perfect square output. We claim that, if $k = 2F_AF_BF_C$, then a_2, b_2, c_2 are integers. As a matter of fact, by substituting $k = 2F_AF_BF_C$ in equation (4), we get:

$$a_2 = 2F_BF_C\sqrt{\frac{S_BS_C}{S_A}}, \quad b_2 = 2F_AF_C\sqrt{\frac{S_AS_C}{S_B}}, \quad c_2 = 2F_AF_B\sqrt{\frac{S_AS_B}{S_C}}.$$

Note that we have perfect square rational values for a_2, b_2, and c_2 in our above equation, so in order to achieve integer values for a_2, b_2, and c_2, we can find a new k by multiplying our $k = 2F_AF_BF_C$ with a suitable integer.

Now we form a right tetrahedron with right angled triangle faces, whose areas are integer values kA, kB and kC, meeting at vertex o and integer principal edges a_2, b_2 and c_2 similar to Figure. 1. We have constructed a tetrahedron where kA, kB, and kC are the areas of its right triangle faces. Thus by Theorem 2.5, the area of the 4th face is

$$\sqrt{(kA)^2 + (kB)^2 + (kC)^2} = \sqrt{k^2(A^2 + B^2 + C^2)} = \sqrt{(kD)^2} = kD.$$

Therefore, for every primitive Pythagorean quadruple (A, B, C, D), there exists a positive integer k such that there is a right tetrahedron with faces areas kA, kB, kC, kD and integer principal edges. □

Remark. In Theorem 3.1, along with the proof, we have constructed a map from the set of primitive Pythagorean quadruples to the set of right tetrahedra with integer faces and principal edges by choosing the minimal k . The existence of a minimal element is guaranteed by the Well Ordering Principle. With this, we characterized all possible right tetrahedra with integer faces and principal edges.

In the following table, we provide some examples of k values associated to a Pythagorean quadruple along with values for principal edges and areas of the triangle faces of the new right tetrahedron. We determine Pythagorean quadruples by using Theorem 2.3.

A	B	C	D	k	a	b	c	kA	kB	kC	kD
7	4	4	9	14	8	14	14	98	56	56	126
4	8	1	9	4	4	2	16	16	32	4	36
7	6	6	11	504	84	84	72	3024	3024	3528	5544
31	8	8	33	248	124	124	32	1984	1984	7688	8184

4. Acknowledgements

I would like to thank my academic and research advisor, Dr. Damiano Fulghesu, who guided me throughout this research. He also gave me the opportunity to participate in an Independent Study Course on Number Theory, during Spring Semester 2015 in Minnesota State University Moorhead, which aided this research.

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