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ABSTRACT. The multiplicity of a weight in a finite-dimensional irreducible representation of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ can be computed via Kostant's weight multiplicity formula. This formula consists of an alternating sum over a finite group and involves a partition function. Our main result describes the terms that contribute nonzero values to this formula, as, in practice, most terms in the sum contribute a value of zero. By taking a geometric approach, we provide concrete visualizations of these sets for all pairs of integral weights λ and μ of $\mathfrak{sl}_3(\mathbb{C})$ and show that the diagrams associated to our main result present new and surprising symmetry.

1. INTRODUCTION

In this paper, we explore the representation theory of finite-dimensional Lie algebra $\mathfrak{g}$ by studying Kostant's weight multiplicity formula, which gives the multiplicity of the weight μ in an irreducible representation of $\mathfrak{sl}_3(\mathbb{C})$ with highest weight λ. Kostant's weight multiplicity formula is defined as [5]:

$$m(\lambda,\mu) = \sum_{\sigma\in W}(-1)^{\ell(\sigma)}\wp(\sigma(\lambda+\rho)-(\mu+\rho)). \tag{1}$$

In equation (1), σ denotes the elements of the Weyl group W, $\ell(\sigma)$ is the length of σ, ρ is equal to half the sum of the positive roots, and $\wp$ denotes Kostant's partition function. The partition function is defined on weights ξ and $\wp(\xi)$ gives the number of ways of expressing ξ as a nonnegative integral sum of positive roots [4]. We note that using KWMF is very difficult. This occurs because of the fundamental properties of the formula, which cause the order of W to grow factorially in the rank of the Lie algebra considered. In addition to this, a closed formula for the partition function, $\wp$, is not known in general.

Our work focuses on reducing the computation, by describing the elements of the Weyl group that contribute nontrivially to the formula when we consider the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Hence, we are interested in cases when $\sigma(\lambda+\rho)-(\mu+\rho)$ is a linear combination of the simple roots α_1 and α_2 with nonnegative integer coefficients [4]. This leads to the following definition.

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Definition 1.1. We define the *Weyl alternation set*, denoted $\mathcal{A}(\lambda,\mu)$, as the set of Weyl group elements for which $\wp(\sigma(\lambda+\rho)-(\mu+\rho))>0$.

Let ϖ_1 and ϖ_2 be the fundamental weights of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$. Our main result computes the Weyl alternation sets $\mathcal{A}(\lambda,\mu)$ when $\lambda = c_1\varpi_1 + c_2\varpi_2$ for $c_1, c_2 \in \mathbb{Z}$ and $\mu = n\alpha_1 + m\alpha_2$ with $n, m \in \mathbb{Z}$.

Theorem 1.2. *Fix $\mu = n\alpha_1 + m\alpha_2$ with $n, m \in \mathbb{Z}$. If $\lambda = (3x+y)\varpi_1 + y\varpi_2$ for some $x, y \in \mathbb{Z}$, then*

$$\mathcal{A}(\lambda,\mu) = \begin{cases} \{1\} & x < m+1,\ x > -1-n \\ \{s_1\} & y < -2x+n,\ y > -2x-m-2 \\ \{s_2\} & y > -x-n-2,\ y < -x+m \\ \{s_1s_2\} & y > -2x-m-2,\ y < -2x+n \\ \{s_2s_1\} & y < -x+m,\ y > -x-n-2 \\ \{s_1s_2s_1\} & x > -n-1,\ x < 1+m \\ \{1, s_1\} & y \geq -2x+n,\ x \leq -1-n \\ \{s_1, s_2s_1\} & y \geq -x+m,\ y \leq -2x-m-2 \\ \{s_2s_1, s_1s_2s_1\} & x \leq -n-1,\ y \leq -x-n-2 \\ \{s_1s_2, s_1s_2s_1\} & y \leq -2x-m-2,\ x \geq 1+m \\ \{s_2, s_1s_2\} & y \leq -x-n-2,\ y \geq -2x+n \\ \{1, s_2\} & x \geq m+1,\ y \geq -x+m \\ \emptyset & otherwise \end{cases}$$

where 1, s_1, s_2, s_1s_2, s_2s_1, $s_1s_2s_1$ are the Weyl group elements of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$.

Theorem 1.2 is a generalization of [3, Theorem 2.2.1] which only computed the Weyl alternation sets $\mathcal{A}(\lambda, 0)$, where $\lambda = (3x+y)\alpha_1 + y\alpha_2$. In this paper we also generalize the idea of the μ weight *Weyl alternation diagram*. These diagrams provide a visualization of the Weyl alternation sets $\mathcal{A}(\lambda,\mu)$ by associating all fundamental weights with lattice points and encoding the Weyl elements in Weyl alternation sets via colored dots. Figure 1, first appearing in [3, Figure 2.16], presents the Weyl alternation diagram of $\mu = 0$, which depicts the 13 distinct Weyl alternation sets, $\mathcal{A}(\lambda, 0)$ for $\lambda = c_1\varpi_1 + c_2\varpi_2$ with $c_1, c_2 \in \mathbb{Z}$. For example, the Weyl alternation diagram shows that $\mathcal{A}(3\varpi_1, 0) = \{1, s_2\}$, which is indicated by the red dot at the lattice point $(3, 0)$ having Weyl alternation set $\{1, s_2\}$, as shown in the key. Likewise, any weight $\lambda = c_1\varpi_1 + c_2\varpi_2$ (or lattice point (c_1, c_2)) with no colored dot implies that $\mathcal{A}(\lambda, 0) = \emptyset$.

Our work presents the Weyl alternation diagrams in the case that $\mu = n\alpha_1 + m\alpha_2$, where $n, m \in \mathbb{N} := \{0, 1, 2, \ldots\}$. The importance of these diagrams is that they state exactly which Weyl group elements will contribute nontrivially to the multiplicity $m(\lambda,\mu)$, allowing us to reduce the computation to exactly those elements. Additionally, in the case where the diagram implies that the Weyl alternation set is empty, we know that the multiplicity will be zero. We do remark that the only portion of the fundamental weight lattice that encodes the irreducible representations of $\mathfrak{sl}_3$ is the first quadrant, but we consider the entire lattice in our analysis, as it provides surprising symmetrical patterns.

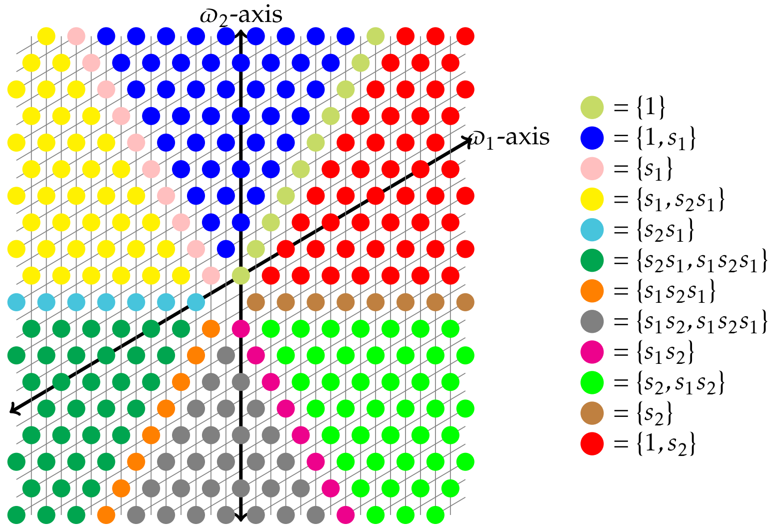


Figure 1. Weyl alternation diagram for $\mu = 0$. Reproduced with author's permission from [4].

This paper is organized as follows: Section 2 provides the necessary background and definitions needed throughout the remainder of the paper. Section 3 provides the proof of Theorem 1.2. Section 4 provides the construction of the Weyl alternation diagrams. We end with Section 5 where we provide a few open problems in this area.

2. Background

We begin by providing the necessary background and definitions as presented in [4].

Definition 2.1. A vector space $\mathfrak{g}$ over a field $\mathbb{F}$ together with a bilinear map $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ is said to be a *Lie algebra* if the map satisfies:

(1) $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$ (skew symmetry), and

(2) $[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]] = 0$ for all $X, Y, Z \in \mathfrak{g}$ (Jacobi identity).

From here on out, we specialize to the Lie algebra $\mathfrak{sl}_3 := \mathfrak{sl}_3(\mathbb{C})$, whose elements are 3×3 complex matrices with trace zero and the lie bracket is defined by the commutator bracket, i.e. if $X, Y \in \mathfrak{sl}_3$, then $[X, Y] = XY - YX$. The Cartan subalgebra, $\mathfrak{h}$, is a subset of $\mathfrak{sl}_3$, consisting of matrices that have all nondiagonal entries equal to zero. Then $\mathfrak{h}^*$, the dual of the Cartan, is the set of all linear functionals on $\mathfrak{h}$, which are maps from $\mathfrak{h}$ to the complex numbers. Associated to a Cartan subalgebra $\mathfrak{h}$ and a Lie algebra $\mathfrak{g}$, is a set of simple roots, which form a basis for $\mathfrak{h}^*$. In $\mathfrak{sl}_3$, the simple roots are denoted by α_1 and α_2, and the set of positive roots consists of α_1, α_2, and $\alpha_1 + \alpha_2$.

Another basis for $\mathfrak{h}^*$ are the fundamental weights, denoted by ϖ_1 and ϖ_2. The fundamental weights have the following relationship with the simple roots [4]:

$$\varpi_1 = \frac{2}{3}\alpha_1 + \frac{1}{3}\alpha_2, \tag{2}$$

$$\varpi_2 = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2. \tag{3}$$

The fundamental weights play an important role in the representation theory of $\mathfrak{sl}_3$. In order to state this formally, we first present the representation theory background beginning with the definition of a representation.

Definition 2.2. A *representation*, τ, is a map from a Lie algebra $\mathfrak{g}$ to the set of all invertible linear transformations from the vector space V to itself. Namely, $\tau : \mathfrak{g} \to \mathfrak{gl}(V)$. We say the representation τ is *irreducible* if the only subspaces that satisfy $(\tau(\mathfrak{g}))W \subset W$ are V or (0).

The theorem of the highest weight proves that there is a bijection between the set of dominant integral weights and the set of irreducible representations of a Lie algebra $\mathfrak{g}$ [2, Theorem 3.2.5]. Therefore, in our setting, each irreducible representation of $\mathfrak{sl}_3$ corresponds to a dominant integral weight, which is defined as

$$\lambda = c_1\varpi_1 + c_2\varpi_2 \quad \text{with } c_1, c_2 \in \mathbb{Z}.$$

The converse is also true. That is, any dominant integral weight $\lambda = n_1\varpi_1 + n_2\varpi_2$ with $n_1, n_2 \in \mathbb{N}$ corresponds to an irreducible representation of $\mathfrak{sl}_3$.

One can also study the structure of an irreducible representation by decomposing the underlying vector space V (of the representation) into the direct sum of subspaces (called weight spaces in this context) as follows:

$$V = \bigoplus_{\mu} V_\mu,$$

where the decomposition is indexed by a finite set of weights μ. The dimension of each weight space V_μ can be found from studying the associated weight rather than the overarching representation theory. In fact, Kostant's weight multiplicity formula (equation (1)) gives a way to compute the dimension of V_μ in the representation corresponding to the dominant integral weight λ.

This naturally allows us to consider the computation involved in Kostant's weight multiplicity formula in order to compute the dimension of the associated weight spaces. In particular our work is motivated by determining what Weyl group elements contribute nontrivially to Kostant's weight multiplicity formula. To do so we first need to understand the Weyl group elements and how they act on weights. This is the purpose of the next section.

2.1. **Weyl group construction.** The Weyl group of the Lie algebra $\mathfrak{sl}_3$ is isomorphic to the symmetric group on 3 letters and hence has 6 elements. Harris in [4] provides a geometric construction of the elements of the Weyl group of $\mathfrak{sl}_3$ as being generated by reflections which are perpendicular to the simple roots α_1 and α_2. This is depicted in Figure 2. If we let s_1 be the root reflection associated with α_1, and s_2 be the root reflection associated with α_2, then the remaining Weyl group elements are generated by concatenations of s_1 and s_2. This can be visualized by observing that we can take any root $(\alpha_1, \alpha_2, \alpha_1 + \alpha_2, -\alpha_1, -\alpha_2, -\alpha_1 - \alpha_2)$ to any of the other roots by reflecting across s_1 and s_2.

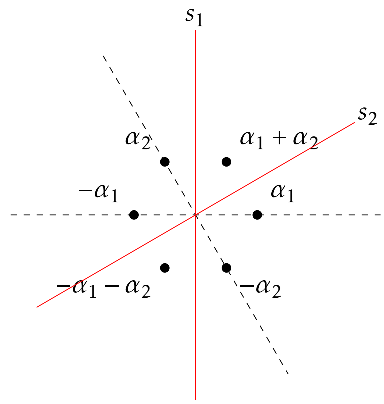


Figure 2. Roots of $\mathfrak{sl}_3(\mathbb{C})$ and the root reflections s_1 and s_2 which generate all Weyl group elements. Reproduced with author's permission from [4].

The unique Weyl group elements extrapolated from this construction are $1, s_1, s_2, s_1s_2, s_2s_1$, and $s_1s_2s_1$ and these account for all of the elements of the Weyl group. We note that as expressed, all elements are of minimal length; any other combination (such as $s_2s_1s_1s_2$) will reduce to one of the elements already given ($s_2s_1s_1s_2 = 1$). There is much work done in the area of minimal representations of Coxeter group elements (Weyl groups are Coxeter groups), and we point the interested readers to [1] for further reading.

Lastly we will need to consider how the elements of the Weyl group act on the simple roots and the fundamental weights of $\mathfrak{sl}_3$. For $1 \le i, j \le 2$

$$s_i(\alpha_j) = \begin{cases} -\alpha_j & \text{if } i = j \\ \alpha_i + \alpha_j & \text{if } i \neq j \end{cases}$$

and

$$s_i(\varpi_j) = \begin{cases} \varpi_j - \alpha_j & \text{if } i = j \\ \varpi_j & \text{if } i \neq j. \end{cases}$$

In order to determine how a longer element of W acts, we use the fact that the action of W is linear. Hence we can distribute over addition and pull out scalars. For example, $s_2s_1(3\alpha_1) = s_2(3s_1(\alpha_1)) = s_2(-3\alpha_1) = -3s_2(\alpha_1) = -3(\alpha_1+\alpha_2) = -3\alpha_1-3\alpha_2$. The computations will be used throughout this paper to evaluate the expression $\sigma(\lambda+\rho)-(\mu+\rho)$ which is the input of the partition function as stated in equation (1).

3. Proof of main result

We aim to find the Weyl alternation sets $\mathcal{A}(\lambda, \mu)$ for any weight $\lambda = c_1\varpi_1 + c_2\varpi_2$ with $c_1, c_2 \in \mathbb{Z}$ and $\mu = n\alpha_1 + m\alpha_2$ with $n, m \in \mathbb{Z}$. Thus, we must compute the partition function value on the expression $\sigma(\lambda+\rho)-(\mu+\rho)$ for all $\sigma \in W$. To do so we must convert the expressions $\xi = \sigma(\lambda+\rho)-(\mu+\rho)$ into a sum of simple roots, as the Weyl alternation set (Definition 1.1) consists of the Weyl group elements σ for which ξ can be written as a nonnegative integral sum of positive roots. As an example, we let $\mu = 0$ and choose $\sigma = 1$.

In this case we have

$$\sigma(\lambda+\rho)-(\mu+\rho) = 1(c_1\varpi_1+c_2\varpi_2+\rho)-(0+\rho) = c_1\varpi_1+c_2\varpi_2. \tag{4}$$

Using the relationship between the fundamental weights and the simple roots (given in equations (2)-(3)), we simplify equation (4) as:

$$c_1\varpi_1+c_2\varpi_2 = \left(\frac{2c_1+c_2}{3}\right)\alpha_1+\left(\frac{c_1+2c_2}{3}\right)\alpha_2. \tag{5}$$

equation (5) can be evaluated for any $c_1, c_2 \in \mathbb{Z}$ to determine if 1 is an element of the Weyl alternation set $\mathcal{A}(c_1\varpi_1+c_2\varpi_2, 0)$. Again, this only happens when both coefficients are nonnegative integers, so we are interested in the cases when $\frac{2c_1+c_2}{3}, \frac{c_1+2c_2}{3} \in \mathbb{N}$ since these weights satisfy $\wp(1(\lambda+\rho)-\rho)>0$.

While equation (4) was specific to the case $\mu=0$, we generalize the equation to hold for any $\mu = n\alpha_1+m\alpha_2$, where $n, m \in \mathbb{Z}$. This is accomplished by simply subtracting n and m from the coefficients of α_1 and α_2 in equation (5), respectively. Repeating this process for the remaining equations arising from $\sigma(\lambda+\rho)-(\mu+\rho)$, where $\sigma \in \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$, yields the following equations:

$$\begin{aligned}
1(c_1\varpi_1+c_2\varpi_2+\rho)-\rho-\mu &= \left(\frac{2c_1+c_2-3n}{3}\right)\alpha_1+\left(\frac{c_1+2c_2-3m}{3}\right)\alpha_2,\\
s_1(c_1\varpi_1+c_2\varpi_2+\rho)-\rho-\mu &= \left(\frac{-c_1+c_2-3n-3}{3}\right)\alpha_1+\left(\frac{c_1+2c_2-3m}{3}\right)\alpha_2,\\
s_2(c_1\varpi_1+c_2\varpi_2+\rho)-\rho-\mu &= \left(\frac{2c_1+c_2-3n}{3}\right)\alpha_1+\left(\frac{c_1-c_2-3m-3}{3}\right)\alpha_2,\\
s_1s_2(c_1\varpi_1+c_2\varpi_2+\rho)-\rho-\mu &= \left(\frac{-c_1-2c_2-3n-6}{3}\right)\alpha_1+\left(\frac{c_1-c_2-3m-3}{3}\right)\alpha_2,\\
s_2s_1(c_1\varpi_1+c_2\varpi_2+\rho)-\rho-\mu &= \left(\frac{-c_1+c_2-3n-3}{3}\right)\alpha_1+\left(\frac{-2c_1-c_2-3m-6}{3}\right)\alpha_2,\\
s_1s_2s_1(c_1\varpi_1+c_2\varpi_2+\rho)-\rho-\mu &= \left(\frac{-c_1-2c_2-3n-6}{3}\right)\alpha_1+\left(\frac{-2c_1-c_2-3m-6}{3}\right)\alpha_2.
\end{aligned} \tag{6}$$

Since we are trying to determine when the coefficients of the simple roots (in the above equations) are nonnegative integers, we first deal with the divisibility condition via a substitution. Since the equations in (3) must have that the coefficients of the simple roots are divisible by 3 we note that if we let $c_1 = 3x+y$ and $c_2 = y$ where $x, y \in \mathbb{Z}$, then

$$\lambda = (3x+y)\varpi_1+y\varpi_2 = (2x+y)\alpha_1+(x+y)\alpha_2.$$

In this case every increment of x and y will guarantee that the coefficients of the simple roots in the equations in (3) are divisible by 3. This substitution allows us to eliminate the denominator in the equations in (3). We can now prove our main result.

Proof of Theorem 1.2. Let $\lambda = (3x+y)\varpi_1 + y\varpi_2$ for some $x, y \in \mathbb{Z}$ and let $\mu = n\alpha_1 + m\alpha_2$ with $n, m \in \mathbb{Z}$. By the equations in (3) we have that

$$\begin{aligned}
1(\lambda+\rho)-\rho-\mu &= (2x+y-n)\alpha_1 + (x+y-m)\alpha_2,\\
s_1(\lambda+\rho)-\rho-\mu &= (-x-1-n)\alpha_1 + (x+y-m)\alpha_2,\\
s_2(\lambda+\rho)-\rho-\mu &= (2x+y-n)\alpha_1 + (x-1-m)\alpha_2,\\
s_1s_2(\lambda+\rho)-\rho-\mu &= (-x-y-2-n)\alpha_1 + (x-1-m)\alpha_2,\\
s_2s_1(\lambda+\rho)-\rho-\mu &= (-x-n-1)\alpha_1 + (-2x-y-m-2)\alpha_2,\\
s_1s_2s_1(\lambda+\rho)-\rho-\mu &= (-x-y-n-2)\alpha_1 + (-2x-y-m-2)\alpha_2.
\end{aligned}$$

By Definition 1.4 and from each of the above equations we have that

$$\begin{aligned}
1 \in \mathcal{A}(\lambda,\mu) &\Leftrightarrow 2x+y-n \geq 0 \text{ and } x+y-m \geq 0,\\
s_1 \in \mathcal{A}(\lambda,\mu) &\Leftrightarrow -x-1-n \geq 0 \text{ and } x+y-m \geq 0,\\
s_2 \in \mathcal{A}(\lambda,\mu) &\Leftrightarrow 2x+y-n \geq 0 \text{ and } x-1-m \geq 0,\\
s_1s_2 \in \mathcal{A}(\lambda,\mu) &\Leftrightarrow -x-y-2-n \geq 0 \text{ and } x-1-m \geq 0,\\
s_2s_1 \in \mathcal{A}(\lambda,\mu) &\Leftrightarrow -x-n-1 \geq 0 \text{ and } -2x-y-m-2 \geq 0,\\
s_1s_2s_1 \in \mathcal{A}(\lambda,\mu) &\Leftrightarrow -x-y-n-2 \geq 0 \text{ and } -2x-y-m-2 \geq 0.
\end{aligned}$$

Intersecting these inequalities along with the selection of points on the lattices whose coefficients of α_1 and α_2 are divisible by 3 yields the desired result. □

4. Weyl alternation set diagrams

We showed in the proof of Theorem 1.2 that each element of the Weyl group belongs to the Weyl alternation set $\mathcal{A}(\lambda,\mu)$ based on the solutions to a system of inequalities. Namely, we are finding regions of a plane on which the coefficients of the simple roots in the expressions $\sigma(\lambda+\rho)-(\mu+\rho)$ are nonnegative integers.

The Weyl alternation diagrams will involve graphing the solution sets to each pair of linear inequalities arising from the equations in (3) on the fundamental weight lattice of $\mathfrak{sl}_3$, which is given by $\mathbb{Z}\varpi_1 \oplus \mathbb{Z}\varpi_2$, which we superimpose on a plane defined by the $\mathbb{R}$-span of the fundamental weights ϖ_1 and ϖ_2. This plane and weight lattice is depicted in Figure 3, where the axes of this plane correspond to the $\mathbb{R}$-span of the fundamental weights with placement of simple roots as in Figure 2, i.e. $\alpha_1 = 2\varpi_1 - \varpi_2$ and $\alpha_2 = -\varpi_1 + 2\varpi_2$.

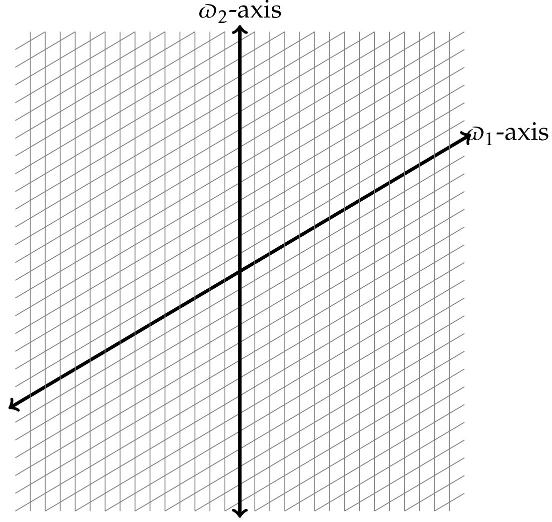


Figure 3. Weight lattice $\mathbb{Z}\varpi_1 \oplus \mathbb{Z}\varpi_2$.

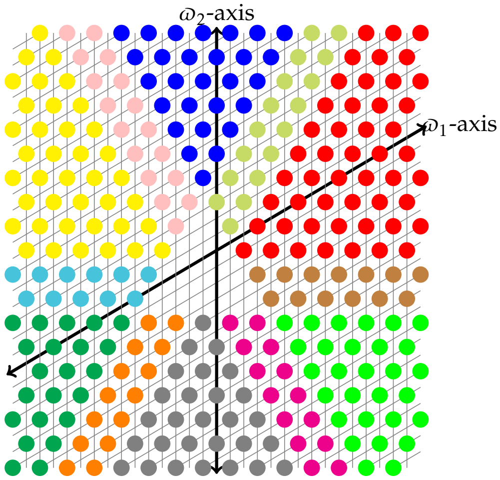
To create the Weyl alternation diagrams we graph the inequalities (of the equations in (3)) as we would on $\mathbb{R}^2$, but rather than shading the solution set (since not all weights can be written as a sum of simple roots using nonegative integers), we place a colored solid circle only on the integral weights for which $\sigma(\lambda+\rho)-(\mu+\rho) \in \mathbb{N}\alpha_1 \oplus \mathbb{N}\alpha_2$ since these are the integral weights for which the solution conditions hold. Then the Weyl alternation diagram is a multi-colored diagram on the fundamental weight lattice of $\mathfrak{sl}_3$, where the number of colors is given by the number of non-empty Weyl alternatiuon sets $\mathcal{A}(\lambda,\mu)$ where $\mu = n\alpha_1 + m\alpha_2$ is fixed (with $n, m \in \mathbb{N}$) and as we vary $\lambda = c_1\varpi_1 + c_2\varpi_2$ with $c_1, c_2 \in \mathbb{Z}$. Once we choose the appropriate number of colors, we assign a distinct color to each of the non-empty Weyl alternation sets. Then each integral weight with the same non-empty Weyl alternation set gets a small solid circle of the color assigned to that particular Weyl alternation set.

Before we present some Weyl alternation diagrams for $\mu = n\alpha_1 + m\alpha_2$ where $n, m \in \mathbb{N}$ we provide the following definition.

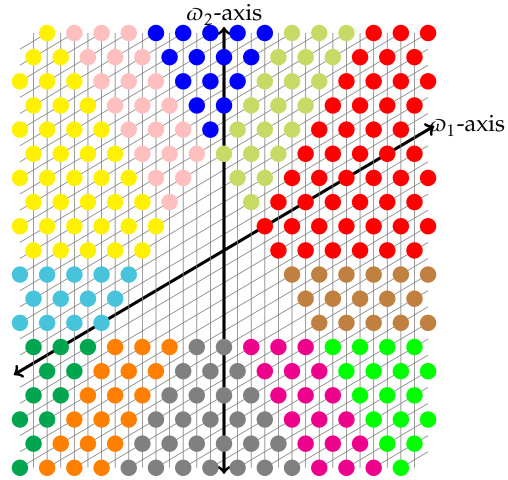
Definition 4.1. An *empty region* on the lattice $\mathbb{Z}\varpi_1 \oplus \mathbb{Z}\varpi_2$ is a set of lattice points such that every point (λ,μ) satisfies $\mathcal{A}(\lambda,\mu) = \emptyset$.

Hence for any pair of weights (λ,μ) in an empty region, $m(\lambda,\mu) = 0$. As observed in Figure 1, the center shape of the empty region is a collection of a few lattice points such that $\mathcal{A}(\lambda,\mu) = \emptyset$ located around $-\rho$. We will shortly observe that the center shape of the empty region changes for varying values of μ. We now apply the above mentioned procedure to create some Weyl alternation diagrams.

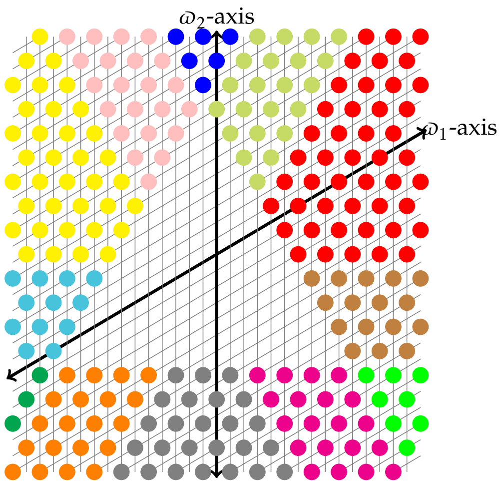
4.1. **Case of $\mu = \mathbf{n}\alpha_1$.** Figures 4a-4d depict the Weyl alternation diagrams for $\mu = n\alpha_1$, where $n = 1$, 2, 3, and 4, and Figure 4e provides the list of Weyl alternation sets, which is a key for the Weyl alternation diagram. Observe that changing from $\mu = 0$ (Figure 1) to $\mu = \alpha_1$ (Figure 4a) results in a graph with an upward facing triangle as the center shape of the empty region and increasing the coefficient of α_1 in $\mu = n\alpha_1$ from $n = 1$ to



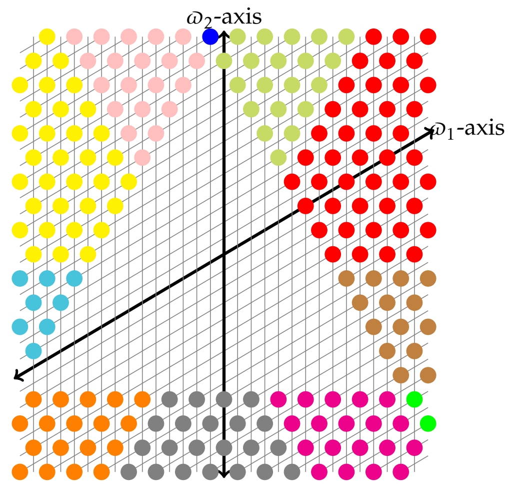
(a) $\mu = \alpha_1$



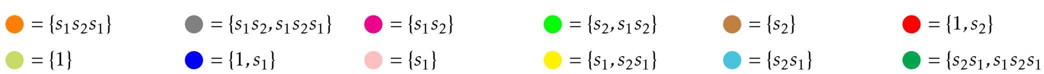
(b) $\mu = 2\alpha_1$



(c) $\mu = 3\alpha_1$



(d) $\mu = 4\alpha_1$

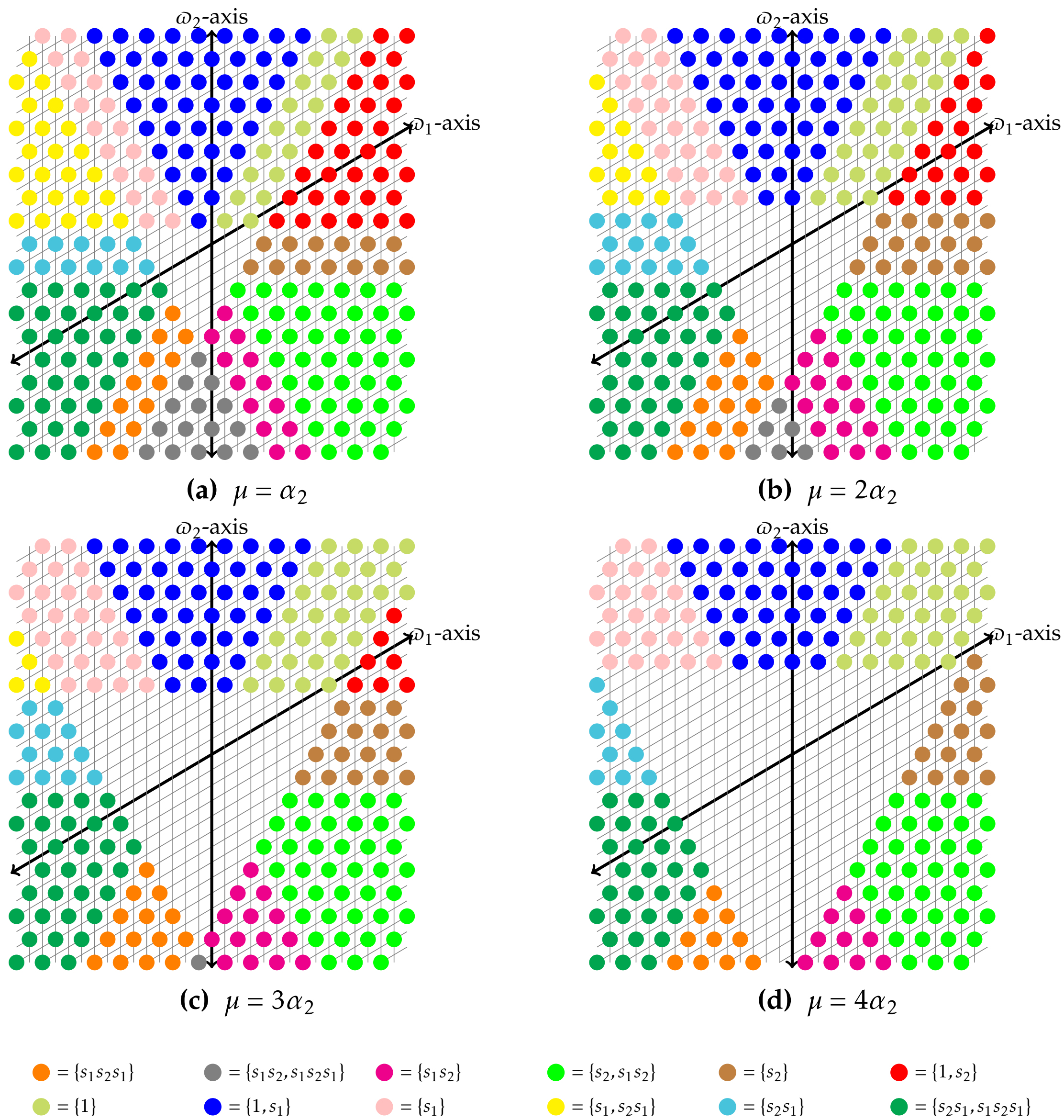


(e) Key

Figure 4. Weyl alternation diagrams for $\mu = \alpha_1,\ 2\alpha_1,\ 3\alpha_1,\ 4\alpha_1$.

$n = 2$, increases the size of the triangle's base. This continues to occur when $\mu = n\alpha_1$ as n increases.

4.2. **Case of $\mu = \mathbf{m}\alpha_2$.** Figures 5a-5d depict the Weyl alternation diagrams for $\mu = n\alpha_2$, where $n = 1$, 2, and 3, and Figure 5e provides the list of Weyl alternation sets, which is a key for the Weyl alternation diagram. Changing from $\mu = 0$ to $\mu = \alpha_2$ results in a similar graph to the one in the $\mu = \alpha_1$ case, except the triangle is a horizontal reflection of the one seen in Figure 4a, i.e. the center shape of the empty region is a downwards facing triangle. Notice that increasing the coefficient of α_2 in $\mu = m\alpha_2$ from $m = 1$ to $m = 2$ increases the size of the triangle's base. This continues to occur when $\mu = m\alpha_2$ as m increases.



(a) $\mu = \alpha_2$

(b) $\mu = 2\alpha_2$

(c) $\mu = 3\alpha_2$

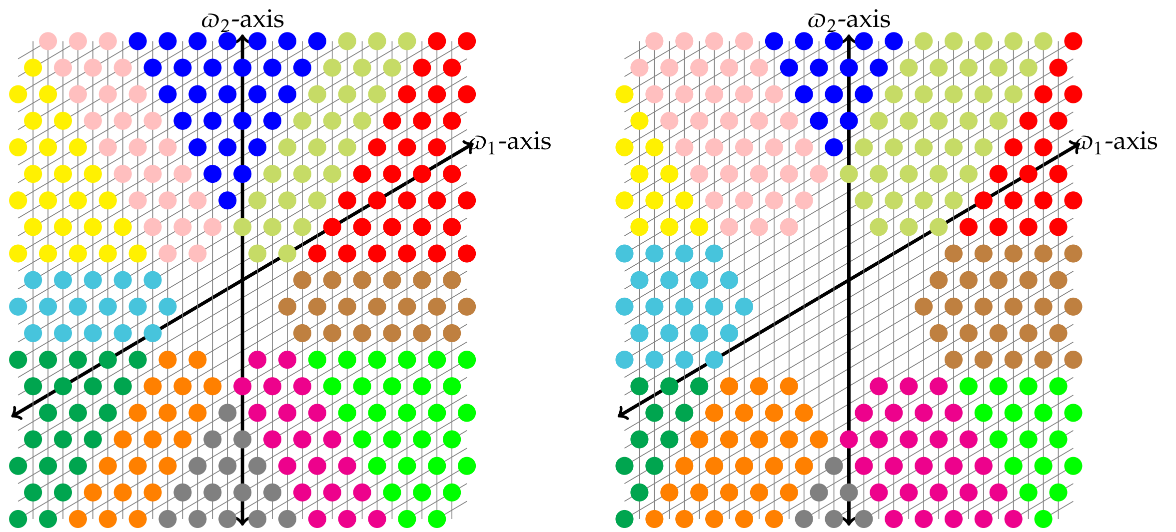
(d) $\mu = 4\alpha_2$

(e) Key

Figure 5. Weyl alternation diagrams for $\mu = \alpha_2,\ 2\alpha_2,\ 3\alpha_2,\ 4\alpha_2$.

4.3. **Case of $\mu = \mathbf{n}\alpha_1 + \mathbf{m}\alpha_2$.** The case $\mu = \alpha_1 + \alpha_2$ yields a more interesting pattern (Figure 6): the two triangles arising in the cases $\mu = \alpha_1$ and $\mu = \alpha_2$ overlap each other to create a *star* as the center shape of the empty region.

This happens because the values $n = 1$ and $m = 1$ prevent the corresponding linear inequalities of the equations in (3) from shading within the six triangular regions depicted in Figure 7. Thus, the resulting Weyl alternation diagram has a center empty region that is shaped like a six pointed star.



(a) $\mu = \alpha_1 + \alpha_2$ **(b)** $\mu = 2\alpha_1 + 2\alpha_2$

● = $\{s_1s_2s_1\}$ ● = $\{s_1s_2, s_1s_2s_1\}$ ● = $\{s_1s_2\}$ ● = $\{s_2, s_1s_2\}$ ● = $\{s_2\}$ ● = $\{1, s_2\}$
● = $\{1\}$ ● = $\{1, s_1\}$ ● = $\{s_1\}$ ● = $\{s_1, s_2s_1\}$ ● = $\{s_2s_1\}$ ● = $\{s_2s_1, s_1s_2s_1\}$

(c) Key

Figure 6. Weyl alternation diagram for $\mu = \alpha_1 + \alpha_2$ and $\mu = 2\alpha_1 + 2\alpha_2$.

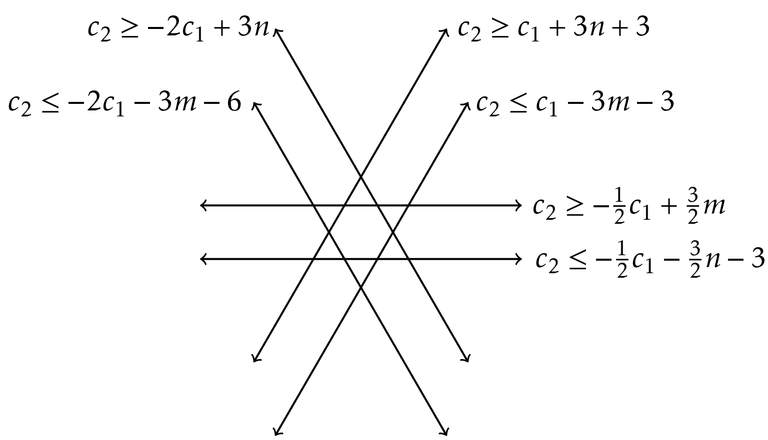
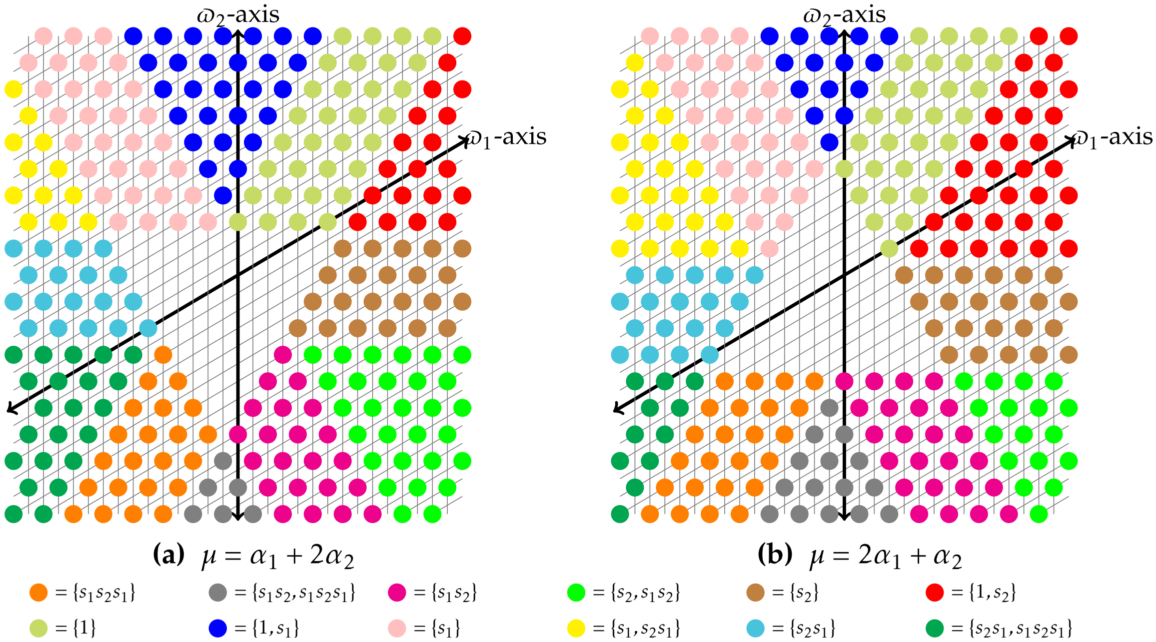


Figure 7. Set of linear inequalities for determining the boundaries of the Weyl alternation sets.

We note that the star shape in the center of the empty region happens in exactly two scenarios. The first scenario we have already shown and is when $\mu = n\alpha_1 + n\alpha_2$, with $n \in \mathbb{N}$. The second scenario occurs when $\mu = n\varpi_1 + m\varpi_2 \notin \mathbb{N}\alpha_1 \oplus \mathbb{N}\alpha_2$, namely when the coefficients of ϖ_1 and ϖ_2 do not translate into integer coefficients in the simple root basis. For example, using the relationship defined in equations (2) and (3), we compute that $\mu = \varpi_1 + 2\varpi_2 = \frac{4}{3}\alpha_1 + \frac{5}{3}\alpha_2 \notin \mathbb{N}\alpha_1 \oplus \mathbb{N}\alpha_2$. In such cases the 6 boundary inequalities are present, as depicted in Figure 7, and the center shape of the empty region is a star. However, if $\mu = n\varpi_1 + m\varpi_2 = k\alpha_1 + \ell\alpha_2$ with $k, \ell \in \mathbb{N}$ but $k \neq \ell$, then the behavior of the center shape of the empty region is a triangle whose orientation depends on $\max(k, \ell)$. If



(c) Key

Figure 8. Weyl alternation diagrams for $\mu = \alpha_1 + 2\alpha_2$ and $\mu = 2\alpha_1 + \alpha_2$.

$\max(k, \ell) = k$, then the triangle faces upward, and if $\max(k, \ell) = \ell$, then the triangle faces downward. Figure 8 depicts this behavior when considering $\mu = \alpha_1 + 2\alpha_2$ and $\mu = 2\alpha_1 + \alpha_2$.

5. Open problems

Now that we have a thorough understanding of the Weyl alternation sets $\mathcal{A}(\lambda, \mu)$ in the cases where $\mu = n\alpha_1 + m\alpha_2$ is fixed with $n, m \in \mathbb{N}$ and λ varies among the integral weights of $\mathfrak{sl}_3$ one could explore the case where λ is fixed and where μ varies. Figure 9 provides a Weyl alternation diagram when we fix $\lambda = 0$, and let μ vary. We see a new configuration of Weyl alternation sets, with a large region whose color represents the fact that all six Weyl group elements are in the corresponding Weyl alternation set.

An open area of study would be to concretely characterize the Weyl alternation diagrams in this new setting. Additionally, the process presented in this paper can also be applied to other Lie algebras. In particular those of rank two: $\mathfrak{so}_5(\mathbb{C})$, $\mathfrak{sp}_4(\mathbb{C})$, $\mathfrak{so}_4(\mathbb{C})$, and the exceptional Lie algebra of type G_2. We remark that the Weyl alternation diagrams corresponding to the weight $\mu = 0$ for all rank 2 Lie algebras have appeared in [3], but the more general case of $\mu = c_1\alpha_1 + c_2\alpha_2$ or $\mu = d_1\varpi_1 + d_2\varpi_2$ with $c_1, c_2, d_1, d_2 \in \mathbb{Z}$ have not appeared elsewhere in the literature.

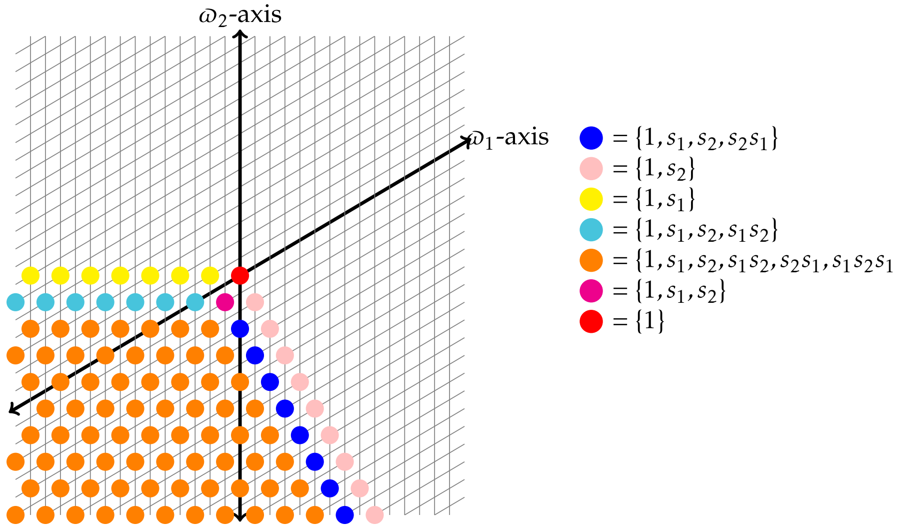


Figure 9. Weyl alternation diagram for $\mathcal{A}(0,\mu)$ with $\mu \in \mathbb{N}\varpi_1 \oplus \mathbb{N}\varpi_2$.

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