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Abstract. We continue the study of U-factorization and τ-factorization by exploring further possible definitions of irreducible elements and associate relations. We are primarily interested in strongly associate rings where relationships between these definitions are more well behaved. We also show that all these definitions coincide for présimplifiable rings, such as any domain or quasi-local ring.

1. Introduction and Background

Anderson and Frazier introduced a concept called τ-factorization, in [2] in 2011. This provided a general theory of factorization which consolidated much of the existing literature on factorization theory in integral domains into a single overarching framework of factorization theory. Recently, the first author has looked at several different methods to extend this τ-factorization to commutative rings with zero-divisors, see [18, 19, 20, 21, 22].

The relation τ_n, in particular, has garnered considerable attention in the integers, for example, see [13, 14], where factorizations are only allowed if each factor is congruent modulo (n). A. Mahlum and the first author began the process of studying these τ_n-factorization properties in $\mathbb{Z}/m\mathbb{Z}$ in [16]. Unfortunately, there were issues which arose primarily due to the presence of idempotent elements in $\mathbb{Z}/m\mathbb{Z}$ when there were multiple prime divisors of m. These idempotent elements seemed to be the only thing preventing the rings from satisfying the nicer τ_n-finite factorization properties. The standard way to resolve factorization issues coming about from idempotent elements is the method of U-factorization. This method was introduced by Fletcher in [11] and has been studied by many authors since then as an improved way of studying factorization in commutative rings with zero-divisors. In particular, we expand on the work done in τ-U-factorization in [6, 20] by exploring several other possible definitions for the irreducible elements and looking at several associate relations. Because we were primarily interested in exploring τ_n-factorizations in $\mathbb{Z}/m\mathbb{Z}$, we devote most of our attention to strongly associate rings. We see that in strongly associate rings several diverging definitions will converge and

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nice relationships can be demonstrated between the various irreducible elements.
In Section 2, we present many definitions and background results which will be used throughout the paper to study relationships between the irreducible elements. In Section 3, we study the relationships between factorization properties of individual elements which were defined in Section 2. Finally, we show that in a présimplifiable ring, in particular, any integral domain or quasi-local ring, all of the definitions coincide. This is clearly what we desire as a generalization of τ-factorization from integral domains.

2. Preliminaries

We carry over much of the notation from [16]. We assume R is a commutative ring with $1 \neq 0$. Let $R^* = R - \{0\}$, $U(R)$ be the set of units of R, and $R^\# = R^* - U(R)$ be the non-zero, non-units of R.

Definition 2.1.

(1) Let $a \sim b$ if $(a) = (b)$ where $(a) = \{ra \mid r \in R\}$, the principal ideal generated by a. In this case, a and b are said to be *associates*.

(2) Let $a \approx b$ if there exists $\lambda \in U(R)$ such that $a = \lambda b$. In this case, a and b are said to be *strong associates*.

(3) Let $a \cong b$ if (1) $a \sim b$ and (2) $a = b = 0$ or if $a = rb$ for some $r \in R$ then $r \in U(R)$. In this case, a and b are said to be *very strong associates*.

Definition 2.2. A ring R is said to be a *strongly associate* ring if for any $a, b \in R$, $a \sim b$ implies $a \approx b$.

See [1] for further discussion of this property.

Definition 2.3. A ring R is said to be a *very strongly associate (equivalently, présimplifiable as it is often called as below)*) ring if for any $a, b \in R$, $a \sim b$ implies $a \cong b$.

As in [7, 8, 9, 10], R is présimplifiable means for any $x, y \in R$, we have $x = yx$ implies $x = 0$ or $y \in U(R)$. Examples of présimplifiable rings include integral domains and quasi-local rings which are rings with a unique maximal ideal, but not necessarily Noetherian. We briefly observe that the présimplifiable property is equivalent to what we have been calling a very strongly associate ring.

Proposition 2.4. *R is présimplifiable if and only if R is a very strongly associate ring.*

Proof. ($\Rightarrow$) Let R be présimplifiable. Suppose $a \sim b$, then we must show $a \cong b$. If $a = 0$, then $b = 0$ also since $(a) = (b)$. Thus, we have $a \cong b$ as desired. If $b = 0$, the argument is symmetric. If neither a nor b is 0, then suppose $a = rb$ for some $r \in R$. Since $(a) = (b)$, we have $b = sa$ for some $s \in R$. Thus $a = rsa$, and we can use présimplifiable to conclude that $rs \in U(R)$. This means $r \in U(R)$ which shows that $a \cong b$.
($\Leftarrow$) Let R be a very strongly associate ring. Suppose $x = yx$ for some $x, y, \in R$. We must show that $x = 0$ or $y \in U(R)$. Certainly $(x) = (x)$ so $x \sim x$. Since R is very strongly associate, it follows that $x \cong x$. If $x \neq 0$ and we have $x = yx$ for some $y \in R$, then y must be a unit as desired. □

It follows from definitions that very strong associates are strong associates and strong associates are associates. Both $\sim$ and $\approx$ are equivalence relations, while $\cong$ fails only to be reflexive. Therefore, in a strongly associate ring, $\sim$ and $\approx$ coincide and in a very strongly associate ring, $\sim$, $\approx$ and $\cong$ all coincide. We direct the reader to [3] where these relations are explored much more thoroughly and many examples are provided to show that they are distinct in rings with zero-divisors.
All very strongly associate rings are therefore strongly associate rings. We pause to give a few examples to show that these kinds of rings are not terribly exotic, but in fact quite common. We highly recommend [1] for the interested reader where these questions are studied specifically.

Example 2.5.

- Every integral domain is very strongly associate, which includes the integers $\mathbb{Z}$, polynomial rings over an integral domain ($\mathbb{Z}[X]$, $\mathbb{Q}[X]$, $\mathbb{R}[X]$, $\mathbb{C}[X]$, etc), among many others.
- Quasi-local rings are very strongly associate.
- The modular integers $\mathbb{Z}/m\mathbb{Z}$ are strongly associate.
- Direct products of strongly associate rings are strongly associate, which means direct products of any of the above would generate strongly associate rings which are not very strongly associate.
- Principal ideal rings (rings in which every ideal is principal) or Artinian rings (rings in which every descending chain of ideals is finite) are strongly associate.
- Semi-quasilocal rings (rings with a finite number of maximal ideals) are strongly associate, but need not be very strongly associate.
- Idealization provides another technique for constructing strongly associate rings which are not very strongly associate.

Let τ be a symmetric relation on $R^\#$, that is, $\tau \subseteq R^\# \times R^\#$ and if $(a,b) \in \tau$, then $(b,a) \in \tau$ and we will write $a\tau b$. For non-units $a, a_i \in R$, and $\lambda \in U(R)$, $a = \lambda a_1 \cdots a_n$ is said to be a *τ-factorization* if $a_i \tau a_j$ for all $i \neq j$. If $n = 1$, then this is said to be a trivial *τ-factorization*. Given the above τ-factorization, we would say that *a_i is a τ-factor of a* or write $a_i \mid_\tau a$. We note that 0 cannot appear as a τ-factor, except in the trivial factorization $0 = \lambda 0$ for some $\lambda \in U(R)$.
We pause to provide some standard examples of τ-relations which have been of interest in the literature.

Example 2.6. Let R be a commutative ring with $1 \neq 0$.

(1) $\tau_d = R^\# \times R^\#$. This yields the usual factorizations in R and $\mid_{\tau_d}$ is the same as the usual divides.

(2) $\tau = \emptyset$. This is the other extreme, where for every $a \in R^\#$, there are only trivial factorizations and $a \mid_\tau b \Leftrightarrow a = \lambda b$ for $\lambda \in U(R) \Leftrightarrow a \approx b$.

(3) Let I be an ideal in R. Set $a\tau_I b$ if and only if $a - b \in I$.

   (a) Let $R = \mathbb{Z}$. Let $I = (n)$. Then this is τ_n which was studied extensively in [13, 14].

(b) In [16], we were especially interested in $R = \mathbb{Z}/m\mathbb{Z}$ and $I = (\overline{n})$ ($\mathbb{Z}/m\mathbb{Z}$ is a principal ideal ring so all ideals are principal).

(4) We obtain the co-maximal factorizations studied in [17] by $a\tau b$ if and only if $(a,b) = R$. Furthermore, for any $\star$-operation, we obtain $\star$-co-maximal factorizations studied in [15] by $a\tau_\star b$ if and only if $(a,b)^\star = R$.

Recall that in an integral domain (commutative ring with unity and no non-zero, zero-divisors), a non-unit element p is *prime* if $p \mid ab$ implies $p \mid a$ or $p \mid b$ and p is *irreducible* if $a \sim bc$ implies $a \sim b$ or $a \sim c$. These definitions are often used as the standard way to extend these concepts from integral domains into rings with zero-divisors. We will investigate more completely later, but pause to introduce an example which highlights one of many problems with factorization in rings with zero-divisors to motivate the technique of U-factorization.

Example 2.7. Let $R = \mathbb{Z}/6\mathbb{Z}$ and consider the element $\overline{3}$. Now $\overline{3}$ is prime and therefore irreducible, yet we have $\overline{3} = \overline{3}\cdot\overline{3}$, which means $\overline{3}$ is a non-trivial idempotent element. This problem is further compounded by the fact that $3 = \overline{3}^i$ for all $i \geq 1$ yielding arbitrarily long prime and irreducible factorizations of 3 in $\mathbb{Z}/6\mathbb{Z}$. The idea is then that these additional copies of the idempotent element, $\overline{3}$ are not contributing anything to the principal ideal we are trying to factor, $(\overline{3})$. In some sense, these additional factors are no different than factoring 6 in the integers as $6 = 1 \cdot 1 \cdot 1 \cdots 1 \cdot 2 \cdot 3$ and claiming the factorizations are no longer unique.

This is the motivation behind the use of what is called U-factorization which we define formally below. We use the same definitions for τ-U-factorization and τ-$\overline{U}$-factorization from [4, 5, 6, 20].

Definition 2.8. Let $a \in R$ be a non-unit. If $a = a_1a_2\cdots a_m b_1 b_2 \cdots b_n$, where $a_i, b_j \in R$ are non-units, then $a = a_1a_2\cdots a_m\lceil b_1b_2\cdots b_n\rceil$ is a *U-factorization* of a if

(1) $a_i(b_1b_2\cdots b_n) = (b_1b_2\cdots b_n)$ for $1 \leq i \leq m$, and

(2) $b_j(b_1\cdots\widehat{b_j}\cdots b_n) \neq (b_1b_2\cdots\widehat{b_j}\cdots b_n)$ for $1 \leq j \leq n$,

where $\widehat{b_j}$ means the element is omitted from the product and the ceiling symbols are used as part of the definition to separate essential divisors from inessential divisors.
We call the b_j's the *essential divisors* of this particular U-factorization of a, and the a_i's are the *inessential divisors* of this particular U-factorization of a.

Definition 2.9. Let a be a non-unit, $a_i, b_j \in R^\#$, and $\lambda \in U(R)$. Then $a = a_1a_2\cdots a_m\lceil b_1b_2\cdots b_n\rceil$ is said to be a *τ-U-factorization* if $a = a_1a_2\cdots a_m\lceil b_1b_2\cdots b_n\rceil$ is a U-factorization and the factorization $a = \lambda a_1\cdots a_m b_1\cdots b_n$ is a τ-factorization. If $n = 1$, then this is said to be a *trivial τ-U-factorization*.

Definition 2.10. Let a be a non-unit, $a_i, b_j \in R^\#$ and $\lambda \in U(R)$. Then we say that the factorization $a = \lambda a_1a_2\cdots a_m\lceil b_1b_2\cdots b_n\rceil$ is a *τ-$\overline{U}$-factorization* if $a = a_1a_2\cdots a_m\lceil b_1b_2\cdots b_n\rceil$ is a U-factorization and $b_i\tau b_j$ for all $i \neq j$. If $n = 1$, then this is said to be a *trivial τ-$\overline{U}$-factorization*.

Now, equipped with our U-factorization definitions, we can look at the factorizations that were giving us trouble before in the following example.

Example 2.11. We continue to look at the ring $R = \mathbb{Z}/6\mathbb{Z}$ and factorizations of $\overline{3}$. Because $(\overline{3}^2) = (\overline{3})$, we have $\overline{3} = \overline{3}^i = \overline{3}^{i-1}\lceil\overline{3}\rceil$. Thus we see that these arbitrarily long factorizations of $\overline{3}$ actually all have the same essential divisor, a single $\overline{3}$. The extra inessential factors of $\overline{3}$ are not contributing anything to the principal ideal. This type of factorization was introduced and studied extensively in [11, 12], where it is shown that in fact using U-factorization, $\mathbb{Z}/n\mathbb{Z}$ is actually a U-unique factorization ring (in the sense of Fletcher) for any $n \in \mathbb{N}$.

The next hurdle is all the different possible definitions of irreducible element that abound in rings with zero-divisors. We include those defined in [3] since those were the starting point of our research. Let $a \in R$ be a non-unit. Then a is said to be *prime* if $a \mid bc$ implies $a \mid b$ or $a \mid c$. We say a is said to be *irreducible* if $a = bc$ implies we have $a \sim b$ or $a \sim c$. We say a is *strongly irreducible*if $a = bc$ implies we have $a \approx b$ or $a \approx c$. We say that a is *m-irreducible* if (a) is maximal among principal ideals. We say a is *very strongly irreducible* if $a = bc$ implies $a \cong b$ or $a \cong c$.
We provide some of these examples from [3] below to give the reader a sense of how these definitions may differ in a ring with zero-divisors even in the traditional factorization setting (when $\tau = R^{\#} \times R^{\#}$).

Example 2.12.

- An element which is prime and irreducible, but not strongly irreducible.
  Let $R = F[X,Y,Z]/(X - XYZ)$ where F is a field. We let x, y, and z represent the image of X, Y, and Z in R and note that $x = xyz$ so $(x) = (xy)$, $x \sim xy$; however $x \not\approx xy$. Thus x is prime and therefore irreducible, but not strongly irreducible since $x = (xy)z$, but $x \not\approx xy$ and $x \not\approx z$.

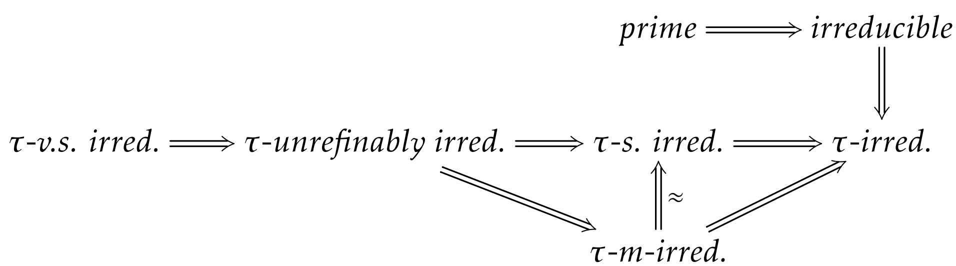
- An element which is strongly irreducible, but not m-irreducible.
  Let $R = \mathbb{Z} \times \mathbb{Z}$. Then $(0,1)$ is strongly irreducible, but not m-irreducible since $((0,1)) \subsetneq ((2,1)) \subsetneq R$.

- An element which is m-irreducible, but not very strongly irreducible.
  Let $R = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and consider the element $(1,0)$. The ideal $((1,0))$ is maximal and therefore certainly maximal among principal ideals.

We now summarize several definitions given in [19] and [21] which extended the above definitions to use τ-factorization. Let $a \in R$ be a non-unit. Then a is said to be τ*-irreducible* or τ*-atomic* if for any τ-factorization $a = \lambda a_1 \cdots a_n$, we have $a \sim a_i$ for some i. We say a is τ*-strongly irreducible* or τ*-strongly atomic* if for any τ-factorization $a = \lambda a_1 \cdots a_n$, we have $a \approx a_i$ for some i. We say that a is τ*-m-irreducible* or τ*-m-atomic* if for any τ-factorization $a = \lambda a_1 \cdots a_n$, we have $a \sim a_i$ for all i. Note: the m is for "maximal" since such an a is maximal among principal ideals generated by elements which occur as τ-factors of a. As in [21], $a \in R$ is said to be a τ*-unrefinably irreducible* or τ*-unrefinably atomic* if a admits only trivial τ-factorizations. We say that a is τ*-very strongly irreducible* or τ*-very strongly*

atomic if $a \cong a$ and a has no non-trivial τ-factorizations. We refer the reader to [19] and [21] for a further discussion and more equivalent definitions of these various forms of τ-irreducibility.
We have the following relationship between the various types of τ-irreducibles which is proved in [19, Theorem 3.9] as well as [21].

Theorem 2.13. *The following diagram illustrates the relationship between the various types of τ-irreducibles a might satisfy where $\approx$ represents R being a strongly associate ring.*



Since our rings will be strongly associate rings, we have the following simplified relationship between irreducibles: τ-very strongly irreducible $\Rightarrow$ τ-unrefinably irreducible $\Rightarrow$ τ-m-irreducible $\Rightarrow$ τ-strongly irreducible $\Rightarrow$ τ-irreducible. Furthermore, examples are provided in [19, 21] which show that no further implications will hold in general. Here we introduce the definitions of *τ-U-irreducible* and *τ-$\overline{U}$-irreducible* in [6].

Definition 2.14. An element $a \in R - U(R)$ is said to be *τ-U-irreducible* if whenever the factorization $a = a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-U-factorization, $a \sim b_i$ for some i. An element $a \in R - U(R)$ is said to be *τ-$\overline{U}$-irreducible* if whenever $a = a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-$\overline{U}$-factorization, $a \sim b_i$ for some i.

We now can naturally extend these definitions to use the other associate relations and types of irreducible.

Definition 2.15. Let $a \in R - U(R)$.

(1) We say that a is *τ-U-strongly irreducible* (resp. *τ-$\overline{U}$-strongly irreducible*) if whenever $a = a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-U-factorization (resp. τ-$\overline{\text{U}}$-factorization), $a \approx b_i$ for some i.

(2) We say that $a \in R - U(R)$ is *τ-U-m-irreducible* (resp. *τ-$\overline{U}$-m-irreducible*) if whenever $a = a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-U-factorization (resp. τ-$\overline{\text{U}}$-factorization), $a \sim b_i$ for all i.

(3) We say that $a \in R - U(R)$ is *τ-U-unrefinably irreducible* (resp. *τ-$\overline{U}$-unrefinably irreducible*) if a admits only trivial τ-U-factorizations (resp. τ-$\overline{\text{U}}$-factorizations).

(4) We say that $a \in R - U(R)$ is *τ-U-very strongly irreducible* (resp. *τ-$\overline{U}$-very strongly irreducible*) if $a \cong a$ and a has only trivial τ-U-factorizations (resp. τ-$\overline{\text{U}}$-factorizations).

3. Results about elements in a strongly associate ring.

In this section, we begin by studying relationships between the various irreducible elements defined above within a strongly associate ring.

In [6], M. Axtell and C. Mooney proved that in any strongly associate ring R, if we let $a \in R - U(R)$ and for any τ a symmetric relation on $R^\#$, the following are equivalent.

(1) a is τ-irreducible.

(2) a is τ-U-irreducible.

(3) a is τ-$\overline{\text{U}}$-irreducible.

We will look to expand upon these types of results for the other types of irreducible elements defined in Section 2. To do so, we will find the following lemma quite useful. Many of these facts are clear from definitions and were proved in [6], but we prove (3) because it helps to demonstrate to the reader why R being strongly associate is so important for our purposes.

Lemma 3.1. *Let R be a commutative ring with identity and τ be a symmetric relation on $R^\#$.*

(1) *Any τ-U-factorization is a τ-$\overline{U}$-factorization.*

(2) *Any τ-U-factorization is a τ-factorization.*

(3) *If R is strongly associate, then any non-trivial τ-$\overline{U}$-factorization can be transformed into a non-trivial τ-U-factorization with the same essential divisors.*

(4) *Any τ-factorization can be rearranged into a factorization which is a τ-U-factorization, hence also a τ-$\overline{U}$-factorization.*

(5) *All τ-factorizations, τ-U-factorizations, and τ-$\overline{U}$-factorizations are factorizations in the usual sense.*

Proof. As discussed above, we prove only (3) as the rest is immediate or proved in [6]. Let R be a strongly associate ring and let $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ be a τ-$\overline{\text{U}}$-factorization. Then by definition of U-factorization, $(a) = (b_1 \cdots b_n)$ and therefore since R is strongly associate, $a \approx b_1 \cdots b_n$, say $a = \mu b_1 \cdots b_n$ for some unit μ. But then $a = \mu \lceil b_1 \cdots b_n \rceil$ is a τ-U-factorization because each of the b_i was essential and therefore is still essential. Thus we have found a τ-U-factorization with the same essential divisors. □

We see the relationship between various irreducible elements which use the strongly associate relation. When the ring is strongly associate, we see that these actually coincide with the τ-irreducible, τ-U-irreducible, and τ-$\overline{\text{U}}$-irreducible elements from [6].

Theorem 3.2. *Let R be a commutative and strongly associate ring with identity, let τ be a symmetric relation on $R^\#$, and let $a \in R - U(R)$. Then the following statements are equivalent.*

(1) *a is τ-strongly irreducible.*

(2) *a is τ-U-strongly irreducible.*

(3) *a is τ-$\overline{\text{U}}$-strongly irreducible.*

(4) *a is τ-irreducible.*

(5) *a is τ-U-irreducible.*

(6) *a is τ-$\overline{\text{U}}$-irreducible.*

Proof. (2) $\Rightarrow$ (1) Let a be τ-U-strongly irreducible and suppose that $a = \lambda a_1 \cdots a_n$ is a τ-factorization. Then by Lemma 3.1, this factorization can be rearranged into a τ-U-factorization. Since a is τ-U-strongly irreducible, a is strongly associate to one of the essential divisors in this τ-U-factorization, which is a τ-factor. Thus, we have shown that a is τ-strongly irreducible.
(3) $\Rightarrow$ (2) Let a be τ-$\overline{\text{U}}$-strongly irreducible and suppose that $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-U-factorization. Then by Lemma 3.1, this is also a τ-$\overline{\text{U}}$-factorization. Since a is τ-$\overline{\text{U}}$-strongly irreducible, $a \approx b_i$ for some i. Thus, a is τ-U-strongly irreducible.
(1) $\Rightarrow$ (3) Let a be τ-strongly irreducible and suppose that $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-$\overline{\text{U}}$-factorization. Then we can get $(a) = (b_1 \cdots b_n)$. In a strongly associate ring, it implies that $a = \mu b_1 \cdots b_n$ for some $\mu \in U(R)$. And we have $b_i \tau b_j$ for any $i \neq j$ since $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-$\overline{\text{U}}$-factorization. So $a = \mu b_1 \cdots b_n$ is a τ-factorization. And hence, a is τ-strongly irreducible implies that $a \approx b_i$ for some i, proving a is τ-$\overline{\text{U}}$-strongly irreducible as desired.
(1) $\Leftrightarrow$ (4) through (6) It suffices to show that (1) $\Leftrightarrow$ (4) since (4) through (6) were shown to be equivalent in [6]. In a strongly associate ring $\sim$ and $\approx$ coincide, so if $a = \lambda a_1 \cdots a_n$ is a τ-factorization, then $a \sim a_i$ if and only if $a \approx a_i$. □

The relations between m-irreducible elements are a bit different. We summarize these relations in the following theorem.

Theorem 3.3. *Let R be a commutative and strongly associate ring with identity, let τ be a symmetric relation on $R^{\#}$, and let $a \in R - U(R)$. Then we consider the following statements.*

(1) *a is τ-m-irreducible.*

(2) *a is τ-U-m-irreducible.*

(3) *a is τ-$\overline{\text{U}}$-m-irreducible.*

Then (1) $\Rightarrow$ (2) and (2) $\Leftrightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Let a be τ-m-irreducible and $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ a τ-U-factorization. Then we have $a \sim b_1 \cdots b_n$ since $(a) = (b_1 \cdots b_n)$. In a strongly associate ring, it implies that $a = \mu b_1 \cdots b_n$ for some $\mu \in U(R)$. We have $b_i \tau b_j$ for any $i \neq j$ since $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-U-factorization. So $a = \mu b_1 \cdots b_n$ is a τ-factorization. And hence, a is τ-m-irreducible implies that $a \sim b_i$ for all i, proving a is τ-U-m-irreducible.
(2) $\Rightarrow$ (3) Let a be τ-U-m-irreducible. Let $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ be a τ-$\overline{\text{U}}$-factorization. Then we can get $(a) = (b_1 \cdots b_n)$. In a strongly associate ring, it implies that $a = \mu b_1 \cdots b_n$ for some $\mu \in U(R)$. And we have $b_i \tau b_j$ for any $i \neq j$ since $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-$\overline{\text{U}}$-factorization. So $a = \mu \lceil b_1 \cdots b_n \rceil$ is a τ-U-factorization. And hence, a is τ-U-m-irreducible

implies that $a \sim b_i$ for all i, proving a is τ-$\overline{\text{U}}$-m-irreducible as desired.
(3) ⇒ (2) Let a be τ-$\overline{\text{U}}$-m-irreducible. Let $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ be a τ-U-factorization. Then by Lemma 3.1, this is also a τ-$\overline{\text{U}}$-factorization. Since a is τ-$\overline{\text{U}}$-m-irreducible, $a \sim b_i$ for all i. Thus, a is τ-U-m-irreducible. □

We pause to provide an example which shows that (1) and (2) from above are not equivalent. That is τ-m-irreducible is stronger than τ-U-m-irreducible (equivalently τ-$\overline{\text{U}}$-m-irreducible).

Example 3.4. Let $R = \mathbb{Z} \times \mathbb{Z}$. Let $\tau = \{[(1,0),(2,0)],[(2,0),(1,0)]\}$. The element we consider is $(2,0)$. The only non-trivial τ-factorization of $(2,0)$ is $(2,0) = (1,0)(2,0)$. But $(2,0)$ is not associate to $(1,0)$, so it is not τ-m-irreducible. On the other hand, $(1,0)$ will never be an essential divisor in any τ-U-factorization (the only nontrivial one is above $(1,0)(2,0)$ and $(1,0)$ is inessential). Thus, the only essential divisor in τ-U-factorizations must be $(2,0)$ (or unit multiples of $(2,0)$ from the trivial τ-U-factorizations). And hence, $(2,0)$ is τ-U-m-irreducible and τ-$\overline{\text{U}}$-m-irreducible but not τ-m-irreducible.

We see that the unrefinably irreducible elements behave similarly to m-irreducible elements. We summarize these results formally in the following theorem.

Theorem 3.5. *Let R be a commutative and strongly associate ring with identity, let τ be a symmetric relation on $R^{\#}$, and let $a \in R - U(R)$. Then we consider the following statements.*

(1) *a is τ-unrefinably irreducible.*

(2) *a is τ-U-unrefinably irreducible.*

(3) *a is τ-$\overline{U}$-unrefinably irreducible.*

Then (1) ⇒ (2) and (2) ⇔ (3).

Proof. (1) ⇒ (2) Let a be τ-unrefinably irreducible, so a admits only trivial τ-factorization. We can prove that a is τ-U-unrefinably irreducible as well by contradiction. Suppose a is not τ-U-unrefinably irreducible and that there exists a non-trivial τ-U-factorization, say $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ for $n > 1$. By Lemma 3.1, $a = \lambda a_1 \cdots a_m b_1 \cdots b_n$ is a τ-factorization. Since $n > 1$, it is a non-trivial τ-U-factorization, which contradicts that a is τ-unrefinably irreducible.
(2) ⇒ (3) Let a be τ-U-unrefinably irreducible, so a admits only trivial τ-U-factorizations. By Lemma 3.1, there is no non-trivial τ-$\overline{\text{U}}$-factorization, or it can be transformed into a non-trivial τ-U-factorization. Thus, a admits only trivial τ-$\overline{\text{U}}$-factorization, and hence, a is τ-$\overline{\text{U}}$-unrefinably irreducible as desired.
(3) ⇒ (2) Let a be τ-$\overline{\text{U}}$-unrefinably irreducible, so a admits only trivial τ-$\overline{\text{U}}$-factorizations. If there exists a non-trivial τ-U-factorization of a with more than one essential factor, then by Lemma 3.1, it is a τ-$\overline{\text{U}}$-factorization as well. Having more than one essential factor contradicts our assumption, thus a admits only trivial τ-U-factorization and is τ-U-unrefinably irreducible as desired. □

We again show that (1) and (2) are not equivalent by providing an example which shows that τ-unrefinably irreducible is strictly stronger than τ-U-unrefinably irreducible (equivalently τ-$\overline{\text{U}}$-unrefinably irreducible).

Example 3.6. We let $R = \mathbb{Z}\times\mathbb{Z}$ and $\tau = \{[(1,0),(1,0)]\}$. We consider the element $(1,0)$. This element is not τ-unrefinably irreducible, since the factorization $(1,0) = (1,0)(1,0)\cdots(1,0)$ is a non-trivial τ-factorization. Furthermore, all of the τ-U-factorizations have the form $(1,0) = \lambda(1,0)\cdots(1,0)\lceil(1,0)\rceil$, where $\lambda \in U(R)$. Thus, the element $(1,0)$ admits only trivial τ-U-factorizations and hence it is τ-U-unrefinably irreducible and τ-$\overline{\text{U}}$-unrefinably irreducible but not τ-unrefinably irreducible.

We now see that very strongly irreducible elements are quite nicely behaved in a strongly associate ring. These results are presented in the following Theorem.

Theorem 3.7. *Let R be a commutative and strongly associate ring with identity, let τ be a symmetric relation on $R^{\#}$, and let $a \in R - U(R)$. Then the following statements are equivalent.*

(1) *a is τ-very strongly irreducible.*

(2) *a is τ-U-very strongly irreducible.*

(3) *a is τ-$\overline{\text{U}}$-very strongly irreducible.*

Proof. (2) $\Rightarrow$ (1) Let a be τ-U-very strongly irreducible. Then we have $a \cong a$. Suppose that $a = \lambda a_1 \cdots a_n$ is a τ-factorization. Then by Lemma 3.1, this factorization can be rearranged into a τ-U-factorization. We claim that there can be no inessential divisors which are non-units. Suppose that after rearrangement if necessary $a = \lambda a_1 \cdots a_s \lceil a_{s+1} \cdots a_n \rceil$ with $s > 1$ is a τ-U-factorization. Then $(a) = (a_{s+1} \cdots a_n)$ which implies $ra = a_{s+1} \cdots a_n$. But then $a = \lambda a_1 \cdots a_s ra$ with $a \cong a$ which implies that $\lambda a_1 \cdots a_s r \in U(R)$ and so we have $a_1, a_2, \ldots, a_s \in U(R)$. This means $a = \lambda \lceil a_1 \cdots a_n \rceil$ is a τ-U-factorization since every non-unit factor must be essential as above. Since a is τ-U-very strongly irreducible, a admits only trivial τ-U-factorizations. So in fact $n = 1$ and we have $a = \lambda a_1$. Thus, we have shown that any τ-factorization of a is trivial, and hence, a is τ-very strongly irreducible as desired.
(3) $\Rightarrow$ (2) Let a be τ-$\overline{\text{U}}$-very strongly irreducible, so that $a \cong a$. Suppose that we have a τ-U-factorization $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$. Then by Lemma 3.1, this is also a τ-$\overline{\text{U}}$-factorization. Since a is τ-$\overline{\text{U}}$-very strongly irreducible, a admits no non-trivial τ-$\overline{\text{U}}$-factorizations. So $n = 1$. Thus, any τ-U-factorization of a is trivial, and hence, a is τ-U-very strongly irreducible.
(1) $\Rightarrow$ (3) Let a be τ-very strongly irreducible, so that $a \cong a$. Suppose that we have a τ-$\overline{\text{U}}$-factorization $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$. Then we can get $(a) = (b_1 \cdots b_n)$. In a strongly associate ring, it implies that $a = \mu b_1 \cdots b_n$ for some $\mu \in U(R)$. And we have $b_i \tau b_j$ for any $i \neq j$ since $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-$\overline{\text{U}}$-factorization. So $a = \mu b_1 \cdots b_n$ is a τ-factorization. a is τ-very strongly irreducible implies that a admits only trivial τ-factorization, which means that $n = 1$. Hence, any τ-$\overline{\text{U}}$-factorization of a is trivial, and therefore, a is τ-$\overline{\text{U}}$-very strongly irreducible. □

Now that we have seen the relationship within each level of irreducible between τ, τ-U, and τ-$\overline{\text{U}}$ factorizations, we study the relationship between the various levels of irreducible in hopes of generalizing the diagram in Theorem 2.13. The following Theorem collects these relationships.

Theorem 3.8. *Let R be a commutative and strongly associate ring with identity, let τ be a symmetric relation on $R^{\#}$, and let $a \in R - U(R)$. Then we consider the following statements.*

(1) *a is τ-U-very strongly irreducible.*

(2) *a is τ-U-unrefinably irreducible.*

(3) *a is τ-U-m-irreducible.*

(4) *a is τ-U-strongly irreducible.*

(5) *a is τ-U-irreducible.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5).

Proof. (1) $\Rightarrow$ (2) By our previous proof, τ-U-very strongly irreducible is equivalent to τ-very strongly irreducible, and τ-unrefinably irreducible yields τ-U-unrefinably irreducible. Combined with Theorem 2.13, we have a is τ-U-very strongly irreducible is equivalent to it is τ-very strongly irreducible, which implies that it is τ-unrefinably irreducible, which yields that it is τ-U-unrefinably irreducible.
(2) $\Rightarrow$ (3) Let a be τ-U-unrefinably irreducible, so that a admits no non-trivial τ-U-factorizations. The form of the τ-U-factorizations of a must be $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 \rceil$. So $(a) = (b_1)$, which means $a \sim b_1$. Since b_1 is the only essential τ-U-factor of a trival τ-U-factorization, we know a is τ-U-m-irreducible.
(3) $\Rightarrow$ (4) Let a be τ-U-m-irreducible. That is, if $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ is a τ-U-factorization, we have $a \sim b_i$ for all i. So we have $a \sim b_1$, which implies $a \approx b_1$ since the ring R is strongly associate. Hence, we get that a is τ-U-strongly irreducible.
(4) $\Leftrightarrow$ (5) Combined our previous results and Theorem 2.13, in a strongly associate ring, we have that τ-strongly irreducible and τ-U-strongly irreducible are equivalent; that τ-irreducible and τ-U-irreducible are equivalent; and that τ-irreducible is equivalent to τ-strongly irreducible. Thus, all these four are equivalent.

□

We now collect all of the previous results from this section into a single diagram which should help the reader to visualize the relationships between the various types of irreducible elements defined throughout the paper.

If R is a strongly associate ring, then for a non-unit $a \in R$ we have the following relationship between irreducible elements.

$$\begin{array}{ccccc}
\tau\text{-very strongly irred.} & \Longleftrightarrow & \tau\text{-U-very strongly irred.} & \Longleftrightarrow & \tau\text{-}\overline{\text{U}}\text{-very strongly irred.} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\tau\text{-unrefinably irred.} & \Longrightarrow & \tau\text{-U-unrefinably irred.} & \Longleftrightarrow & \tau\text{-}\overline{\text{U}}\text{-unrefinably irred.} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\tau\text{-m-irred.} & \Longrightarrow & \tau\text{-U-m-irred.} & \Longleftrightarrow & \tau\text{-}\overline{\text{U}}\text{-m-irred.} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\tau\text{-strongly irred.} & \Longleftrightarrow & \tau\text{-U-strongly irred.} & \Longleftrightarrow & \tau\text{-}\overline{\text{U}}\text{-strongly irred.} \\
\Updownarrow & & \Updownarrow & & \Updownarrow \\
\tau\text{-irred.} & \Longleftrightarrow & \tau\text{-U-irred.} & \Longleftrightarrow & \tau - \overline{\text{U}}\text{-irred.}
\end{array}$$

We have set out to extend the notion of τ-irreducible elements in integral domains from [2]. Thus we would like these generalizations to still coincide to the original definitions when we are working in an integral domain. We see that, for non-zero elements, this is indeed the case, and in fact they all coincide in a more general class of well behaved rings with zero-divisors called présimplifiable rings. Of course, in an integral domain there are no zero-divisors, so there are only trivial factorizations of 0 anyway, so some of the definitions diverging on 0 is not terribly problematic.

Corollary 3.9. *Let R be a présimplifiable (equivalently very strongly associate) ring and let τ be a symmetric relation on $R^{\#}$. Let $a \in R$ be a non-zero, non-unit. Then the following are equivalent*

(1) *a is τ-very strongly irreducible.*

(2) *a is τ-U-very strongly irreducible.*

(3) *a is τ-$\overline{U}$-very strongly irreducible.*

(4) *a is τ-unrefinably irreducible*

(5) *a is τ-U-unrefinably irreducible.*

(6) *a is τ-$\overline{U}$-unrefinably irreducible.*

(7) *a is τ-m-irreducible.*

(8) *a is τ-U-m-irreducible.*

(9) *a is τ-$\overline{U}$-m-irreducible.*

(10) *a is τ-strongly irreducible.*

(11) *a is τ-U-strongly irreducible.*

(12) *a is τ-$\overline{U}$-strongly irreducible.*

(13) *a is τ-irreducible.*

(14) *a is τ-U-irreducible.*

(15) *a is τ-$\overline{U}$-irreducible.*

Proof. By the above diagram, it suffices to show that when R is présimplifiable or equivalently when R is a very strongly associate ring, τ-U-irreducible implies τ-very strongly irreducible. The first thing to observe is that in a présimplifiable ring, every inessential divisor must be a unit. Let $a = \lambda a_1 a_2 \cdots a_m \lceil b_1 b_2 \cdots b_n \rceil$ be a U-factorization. Since $(a) = (b_1 \cdots b_n)$, we have $ra = b_1 \cdots b_n$. This means $a = \lambda a_1 \cdots a_m b_1 \cdots b_n = (\lambda a_1 \cdots a_m r)a$ but since R is présimplifiable $a = 0$ or $\lambda a_1 \cdots a_m r \in U(R)$. Since a is non-zero, we have $\lambda a_1 \cdots a_m r \in U(R)$ and therefore $a_i \in U(R)$ for all $1 \leq i \leq m$.
With this in mind, let a be a τ-U-irreducible element. Certainly $(a) = (a)$, so $a \sim a$ which implies $a \cong a$ since R is very strongly associate by hypothesis. Let $a = \lambda c_1 \cdots c_s$ be a τ-factorization. Then $a = \lambda \lceil c_1 \cdots c_s \rceil$ is a τ-U-factorization itself since the only inessential divisors are units by our above observation. Thus, since a is τ-irreducible, $a \sim c_i$ for some i. Since R is very strongly associate, this means $a \cong c_i$ for some i. But then $a = (\lambda c_1 \cdots c_{i-1} c_{i+1} \cdots c_s) c_i$ which means $\lambda c_1 \cdots c_{i-1} c_{i+1} \cdots c_s \in U(R)$, proving a has only trivial τ-factorizations since every other factor was actually a unit and therefore is τ-very strongly irreducible as desired. □

We now have developed an understanding of the relationship between various irreducible properties in strongly associate rings. While they are still fairly complicated, especially in comparison with présimplifiable rings, we see that they are significantly more well behaved with this assumption to make studying factorization manageable. In the future, we would like to turn our attention to studying various generalizations of unique factorization properties such as half factorial rings, weak finite factorization rings, finite factorization rings, bounded factorization rings, etc. using these types of irreducible elements in strongly associate rings. This could potentially be useful to help to extend the work done in [16] and [20].

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