The Minnesota Journal of Undergraduate Mathematics

Twisted Torus Links and the General Braid

Brandy G. Doleshal[1], Eric J. Hanson[2], Taylor E. Martin[1], and Max Zhou[3]

[1]Sam Houston State University
[2]Brandeis University
[3]UCLA

The Minnesota Journal of Undergraduate Mathematics

Volume 4 (2018-2019 Academic Year)

Sponsored by
School of Mathematics
University of Minnesota
Minneapolis, MN 55455

The Minnesota Journal of Undergraduate Mathematics

Volume 4 (2018-2019 Academic Year)

Twisted Torus Links and the General Braid

Brandy G. Doleshal[1], Eric J. Hanson* [2], Taylor E. Martin[1], and Max Zhou[3]

[1]Sam Houston State University
[2]Brandeis University
[3]UCLA

Abstract. Birman and Kofman define the twisted torus links as the closure of a certain braid. We show that we can find another braid representation of some twisted torus links, which we call the general braid. Using the general braid, we are able to determine the number of components of some twisted torus links as well as classify the link type of several twisted torus links.

1. Introduction

A mathematical *knot* is a nonintersecting closed curve in the three dimensional sphere S^3; a disjoint union of knots is called a *link.* The mathematical study of knots and links is an important and well studied area of topology, with many applications to other areas of mathematics and science. There are many lenses through which to study knots and links, but one unfortunate barrier is the fact that there does not exist a group structure, or even a binary operation, to place on the category of links in S^3. One solution to this obstacle is to study *braids*, which are closely related to links. We can define a braid in several ways; the most straightforward is that an n-strand braid, for n an integer greater than 1, is a collection of n disjoint, descending strands in a cylinder $\mathbb{D}^2 \times I$. The *closure* of an n-strand braid is a *link*, a collection of disjoint knots in 3-space, which is formed by attaching the bottom of the cylinder to the top. Conversely, every knot and link can be represented as the closure of a braid. Figure 1 depicts a three strand braid and its closure, which is a knot.

One way in which braids differ from links is that the set of all n-strand braids form a group called the *braid group*, denoted $\mathcal{B}_n$. The group operation on $\mathcal{B}_n$ is concatenation, which means to stack one braid on top of another. There are $n-1$ generators for the braid group $\mathcal{B}_n$, the set $\{\sigma_i\}_{i=1}^{n-1}$, where σ_i is a single crossing of the $i+1^{st}$ strand over the i^{th} strand. The survey article [1] provides a thorough introduction to braids and the braid group.

* *Corresponding author*

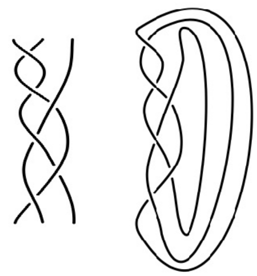


Figure 1. A three strand braid and its closure.

Torus knots are knots that can be drawn on the torus $S^1 \times S^1$. A torus knot is typically denoted by $T(p,q)$, where p and q are relatively prime integers; this notation reflects the fact that the (p,q) torus knot is the closure of the p-strand braid $(\sigma_1\sigma_2\ldots\sigma_{p-1})^q$ realized by passing the leftmost strand under the other $p-1$ strands (an action we will call a *single positive twist*) q times. The restriction that $\gcd(p,q)=1$ guarantees that the closure of this braid will be a knot; relaxing this requirement yields a torus *link*.

The work of this paper is to study a generalization of torus links, called the *twisted torus link*, which was defnted by Birman and Kofman's [2] as the closure of the braid

$$(\sigma_1\sigma_2\cdots\sigma_{r_1-1})^{s_1}(\sigma_1\sigma_2\cdots\sigma_{r_2-1})^{s_2}\cdots(\sigma_1\sigma_2\cdots\sigma_{r_k-1})^{s_k}$$

where $2 \le r_1 \le r_2 \le \ldots \le r_k$, and $0 < s_i$ for $i = 1,2,\ldots,k$. A twisted torus link can be thought of as a generalization of the more often explored twisted torus knot ([3, 4, 5, 6, 7, 8, 9, 10], just to name a few). Using our notation, twisted torus knots are the subset of twisted torus links with $k=2$, $\gcd(r_2,s_2)=1$ and $s_1 = nr_1$ for some nonzero integer n. In this work, we focus on the case $k=2$ and we allow for negative twisting as well as positive twisting, i.e. s_2 may be positive or negative. Following notation of Birman and Kofman [2], we denote the closure of $(\sigma_1\sigma_2\cdots\sigma_{r_1-1})^{s_1}(\sigma_1\sigma_2\cdots\sigma_{r_2-1})^{s_2}$ by $T((r_1,s_1),(r_2,s_2))$.

The existing literature focuses on characterizing certain families of twisted torus knots as cable knots [5], composite knots in [8], or fibered [4]. The Alexander polynomial of some twisted torus knots is calculated in [10]. However, these results cannot be easily generalized to twisted torus links because of a fundamental problem: given the parameters r_1, s_1, r_2, and s_2, it is not easy to discern the number of components in the resulting link $T((r_1,s_1),(r_2,s_2))$.

One of our goals is to find a formula for the component number of a twisted torus link. In any braid, the number of components in the braid closure is determined by the way the strands in the braid are permuted; if we number the strands, 1 to n, from left to right at the top of the braid, the position of the strands at the bottom of the braid gives a permutation in the symmetric group S_n. The number of disjoint cycles in this permutation tells us the number of components in the closure of the braid. In the case of twisted torus links, we can write a program to directly compute the twisted torus link's corresponding permutation. However, we have yet to find a simple formula that describes the result in general. This work is a step in that direction.

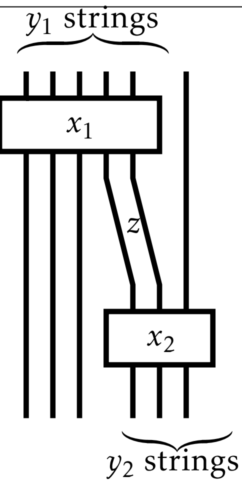


Figure 2. $GB(x_1, y_1, x_2, y_2, z)$

Our approach is to introduce a new braid presentation that we call the *general braid*, $GB(x_1, y_1, x_2, y_2, z)$, which is defined in Section 2. In Corollary 2.6 , we describe precisely which twisted torus links have general braid presentations. Our main result, Theorem 3.3 relates the number of components in the closure of a general braid to the greatest common divisor of the parameters, x_1, y_1, x_2, y_2. As a corollary, the results in Section 3 completely determine the component number of all twisted torus links which correspond to general braids of the form $GB(\pm x_1, y_1, \mp x_2, y_2, z)$ with $z \in \{0, 1, 2\}$. In Section 4, we use our results to determine the link type of some twisted torus links.

2. The General Braid

We define a single positive twist on r strands to be obtained by pulling the leftmost strand behind the remaining $r-1$ strands and a single negative twist to be obtained by pulling the leftmost strand in front of the remaining $r-1$ strands. In $\mathcal{B}_n$, with $n \geq r$, a single positive twist is represented by the braid element $\sigma_1 \sigma_2 \cdots \sigma_{r-1}$ and a single negative twist is represented by the braid element $\sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{r-1}^{-1}$. Note that, with this definition of a negative twist, a single positive twist followed by a single negative twist on the same r strands does not result in cancellation. However, a full positive twist, $(\sigma_1 \sigma_2 \cdots \sigma_{r-1})^r$, is the inverse of a full negative twist, $(\sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{r-1}^{-1})^r$.

Before introducing the general braid, we prove the following lemma regarding positive and negative twists.

Lemma 2.1. *A full twist (positive or negative) on s strings is equivalent to its reverse. In other words, $(\sigma_1 ... \sigma_{s-1})^s$ has the same closure as $(\sigma_{s-1} ... \sigma_1)^s$ and $(\sigma_1^{-1} ... \sigma_{s-1}^{-1})^s$ has the same closure as $(\sigma_{s-1}^{-1} ... \sigma_1^{-1})^s$*

Proof. We know that a full positive twist and a full negative twist are inverses. Thus, $(\sigma_1 ... \sigma_{s-1})^{-s} = (\sigma_1^{-1} ... \sigma_{s-1}^{-1})^s$. In general, $(\sigma_1 ... \sigma_{s-1})^{-1} = \sigma_{s-1}^{-1} ... \sigma_1^{-1}$. It follows that $(\sigma_1^{-1} ... \sigma_{s-1}^{-1})^s = (\sigma_1 ... \sigma_{s-1})^{-s} = (\sigma_{s-1}^{-1} ... \sigma_1^{-1})^s$. This means $(\sigma_1^{-1} ... \sigma_{s-1}^{-1})^s$ and $(\sigma_{s-1}^{-1} ... \sigma_1^{-1})^s$ have the same closure. The proof that $(\sigma_1 ... \sigma_{s-1})^s$ has the same closure as $(\sigma_{s-1} ... \sigma_1)^s$ is analogous. $\square$

Definition 2.2 (General Braid)**.** Let $x_1, y_1, x_2, y_2, z \in \mathbb{Z}$ with $y_1, y_2 \geq 2$ and $0 \leq z \leq \min(y_1, y_2)$. We define the *general braid* with parameters x_1, y_1, x_2, y_2, and z, denoted $GB(x_1, y_1, x_2, y_2, z)$, as the braid on $y_1 + y_2 - z$ strings consisting of x_1 twists on the leftmost y_1 strings followed by x_2 twists on the rightmost y_2 strings. This is shown in Figure 2.

Theorem 2.3. *Let $r_1, r_2, s_1, s_2 \in \mathbb{N}$ with $r_2 \geq r_1 \geq s_1 + s_2$. If $r_1 \bmod (s_1 + s_2)$ is not between s_1 and s_2, then $T((r_1, s_1), (r_2, -s_2))$ is isotopic to the closure of $GB(x_1, y_1, x_2, y_2, z)$ with parameters:*

$$x_1 = \begin{cases} (r_1 - r_2) - s_2\lfloor \frac{r_1}{s_1+s_2} \rfloor - (r_1 \bmod (s_1 + s_2)) & \text{if } r_1 \bmod (s_1+s_2) \leq \min(s_1, s_2) \\ (r_1 - r_2) - s_2\lfloor \frac{r_1}{s_1+s_2} \rfloor + (r_1 \bmod (s_1 + s_2)) - s_1 - 2s_2 & \text{if } r_1 \bmod (s_1+s_2) \geq \max(s_1, s_2) \end{cases}$$

$$y_1 = s_2$$

$$x_2 = \begin{cases} s_1\lfloor \frac{r_1}{s_1+s_2} \rfloor + (r_1 \bmod (s_1 + s_2)) & \text{if } r_1 \bmod (s_1+s_2) \leq \min(s_1, s_2) \\ s_1\lfloor \frac{r_1}{s_1+s_2} \rfloor + (r_1 \bmod (s_1 + s_2)) - s_2 & \text{if } r_1 \bmod (s_1+s_2) \geq \max(s_1, s_2) \end{cases}$$

$$y_2 = s_1$$

$$z = \begin{cases} r_1 \bmod (s_1 + s_2) & \text{if } r_1 \bmod (s_1+s_2) \leq \min(s_1, s_2) \\ s_1 + s_2 - (r_1 \bmod (s_1 + s_2)) & \text{if } r_1 \bmod (s_1+s_2) \geq \max(s_1, s_2) \end{cases}$$

Proof. Let $r_1, r_2, s_1, s_2 \in \mathbb{N}$ such that $r_2 \geq r_1 \geq s_1 + s_2$. We begin with the standard braid representation of $T((r_1, s_1), (r_2, -s_2))$, show in Figure 3. From here, we can see the s_1 positive twists on the first r_1 strands as a full twist on the first s_1 strands followed by the s_1 strands passing behind the next $r_1 - s_1$ strands. We can also view the s_2 negative twists on all r_2 strands as a full twist on the first s_2 strands followed by the s_2 strands passing in front of the next $r_2 - s_2$ strands. After isotopy to simplify the the braid, we obtain the second braid in Figure 3.

In order to make the braid easier to manipulate, we redraw the picture, looking at the braid instead from the back, which yields the braid shown in Figure 4. Since we are interested in the closure of the braid, we notice that by closing the first strand of the first braid shown in Figure 5, we can pull this strand off over the twist box so that is passes over the the strands below the twist box instead. This operation reduces the number of strands of the braid by 1 and increases the number of negative twists in the leftmost twist box by 1 so that the new number of twists is $-s_2 - 1$. We can use this move $r_2 - r_1$ times. For ease of notation, we use x_1 and x_2 to represent the number of twists on each set of strings. We obtain the braid shown in Figure 6, with $x_1 = r_1 - r_2 - s_2$ and $x_2 = s_1$.

If there are at least $2(s_1 + s_2)$ remaining strings, we can reduce the total number of strings by performing belt tricks. This move is named from a party trick: lay a long belt in a loop, flat on a surface. Then, grab the buckle and end of the belt and pull taut. The belt will no longer lay flat on the surface, but will contain one full twist. Similarly, if some number k of leftmost strands in a braid are closed up by attaching bottom to top, and then the k adjacent closed strands are pulled downwards in the braid, the result will be a braid of k fewer strands, where the strands that were closed off now contain a full twist. We use this process frequently in simplifying a twisted torus link. The overall process is shown in Figure 7. We can perform $\lfloor \frac{r_1 - (s_1+s_2)}{(s_1+s_2)} \rfloor$ pairs of belt tricks, each of which

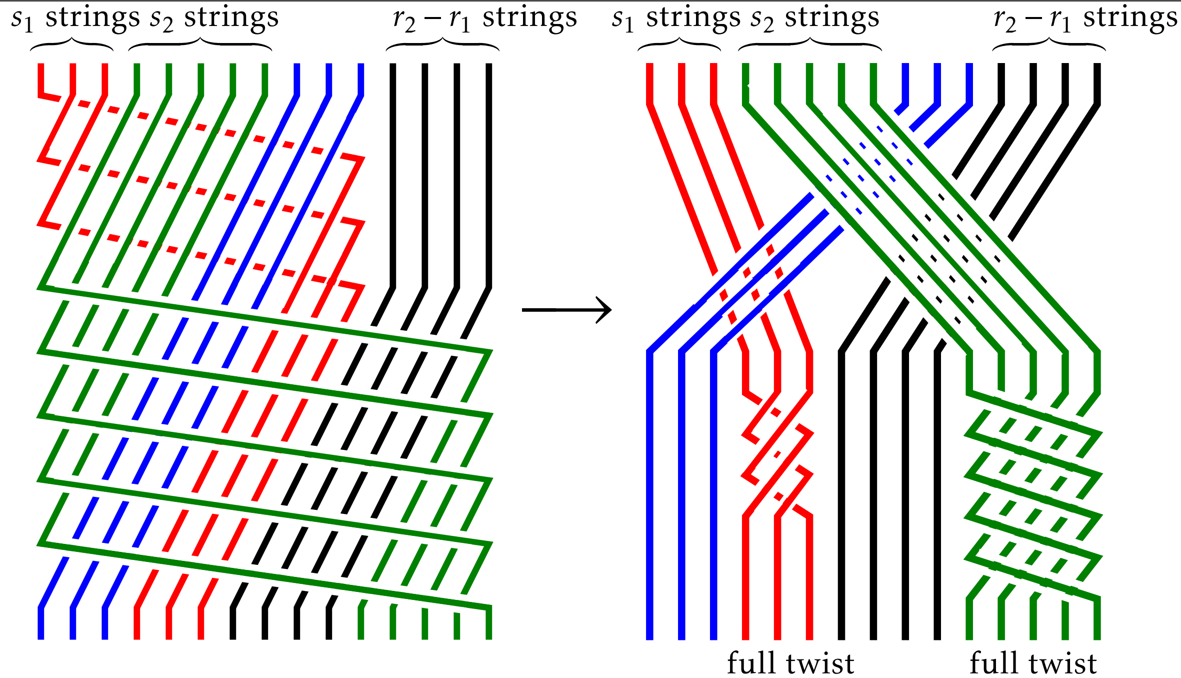


FIGURE 3. Collecting twists from the standard braid representation of $T((r_1,s_1),(r_2,s_2))$

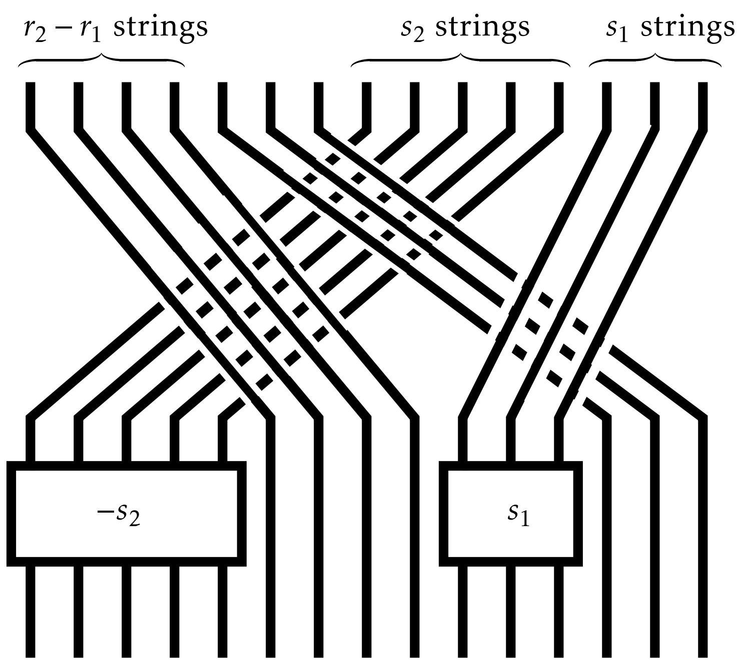


FIGURE 4. The back of $T((r_1,s_1),(r_2,s_2))$

adds s_2 negative twists to the set of negative twists and s_1 positive twists to the set of positive twists. The result is the braid shown in Figure 8 with $x_1 = r_1 - r_2 - s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor$ and $x_2 = s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor$ and $y_1 = s_2$, $y_2 = s_1$. We consider two cases: $r_1 \bmod (s_1+s_2) \le \min(s_1,s_2)$ and

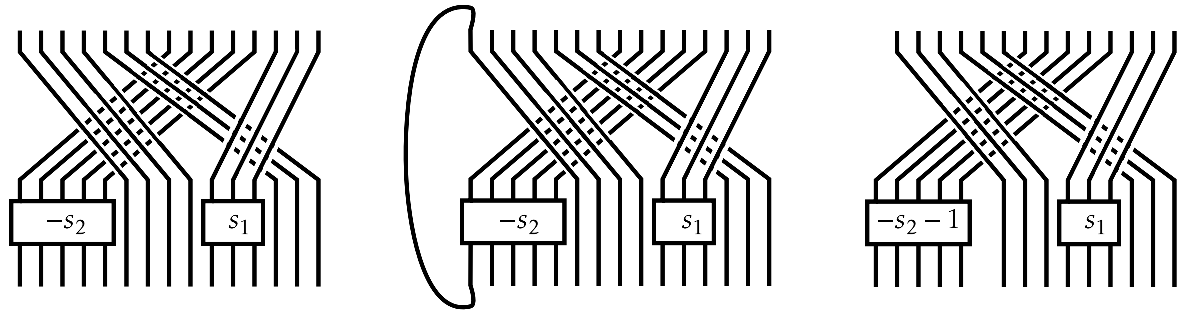


Figure 5. The links in these three images are equivalent

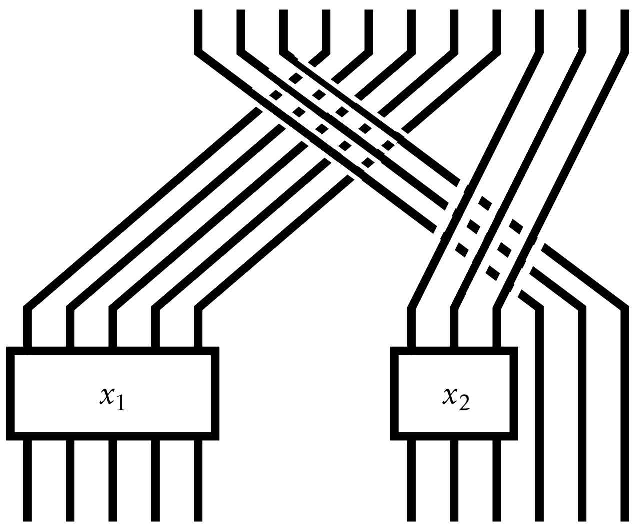


Figure 6. Our braid after performing $r_2 - r_1$ of the moves shown in Figure 5

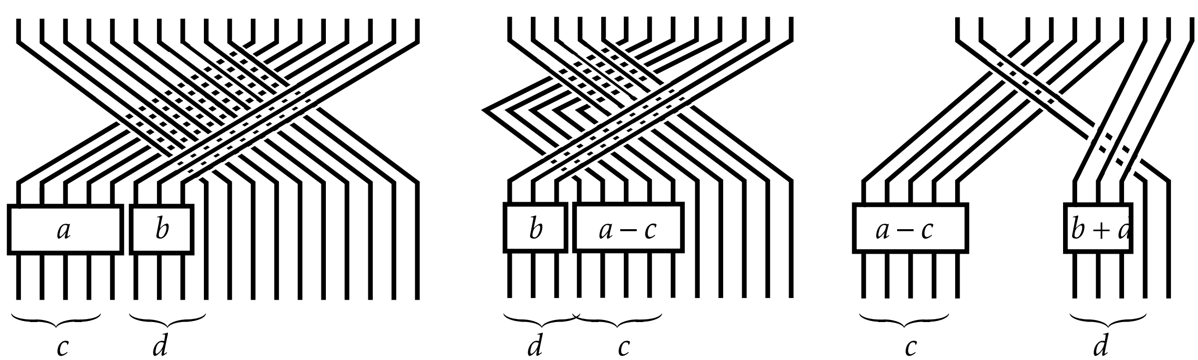


Figure 7. We move between these braids by performing belt tricks.

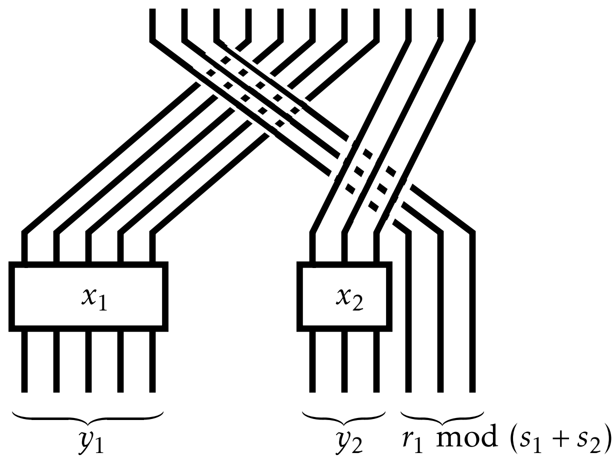


FIGURE 8. Our braid after performing $\left\lfloor \dfrac{r_1-(s_1+s_2)}{(s_1+s_2)} \right\rfloor$ pairs of belt tricks

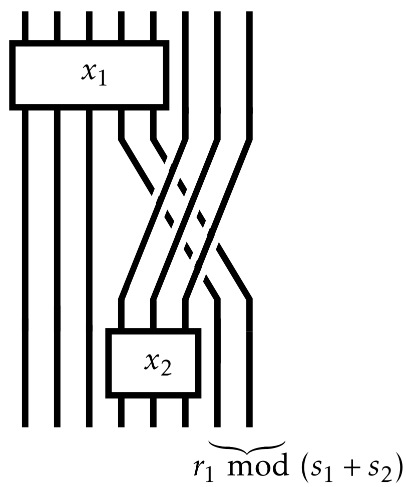


FIGURE 9. Case I: $r_1 \bmod s_1+s_2 \leq s_2$

$r_1 \bmod (s_1+s_2) \geq \max(s_1, s_2)$.

Case I: $r_1 \bmod (s_1+s_2) \leq \min(s_1, s_2)$:

As $r_1 \bmod (s_1+s_2) \leq s_2$, we no longer perform more belt tricks. We close the braid partially to slide strands across tangles and reduce the number of strings in our braid to s_1+s_2. The resulting braid is shown in Figure 9 with $x_1 = r_1 - r_2 - s_2\lfloor \frac{r_1}{s_1+s_2} \rfloor - (r_1 \bmod (s_1+s_2))$ and $x_2 = s_1 \lfloor \frac{r_1}{s_1+s_2} \rfloor$. From here, we partially close $r_1 \bmod (s_1+s_2)$ more strands to obtain the general braid, where the parameters are $x_1 = (r_1-r_2) - s_2\lfloor \frac{r_1}{s_1+s_2} \rfloor - (r_1 \bmod (s_1+s_2)), y_1 = s_2, x_2 = s_1\lfloor \frac{r_1}{s_1+s_2} \rfloor + (r_1 \bmod (s_1+s_2)), y_2 = s_1$, and $z = r_1 \bmod (s_1+s_2)$, which was our goal.

Case II: $r_1 \bmod (s_1+s_2) \geq \max(s_1, s_2)$:

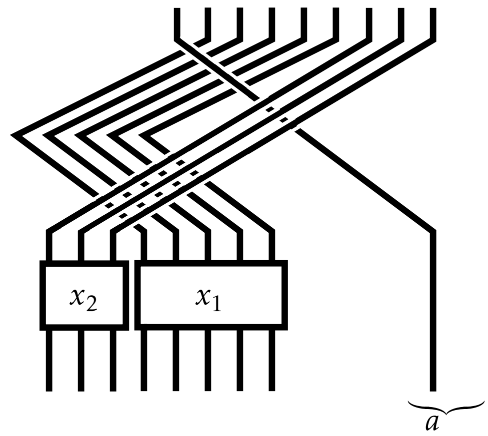


FIGURE 10. Our braid after performing an additional belt trick

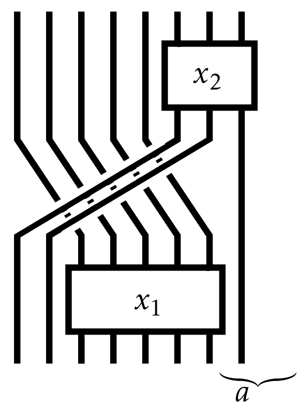


FIGURE 11. Our braid after closing the $(r_1 \bmod (s_1+s_2))-s_2$ strands

When $r_1 \bmod (s_1+s_2) \geq s_2$ we perform one additional belt trick, leaving us with the braid shown in Figure 10 with $x_1 = (r_1-r_2)-s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor - s_2, x_2 = s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor$, and $a = (r_1 \bmod (s_1+s_2))-s_2$. At this point, we can perform $(r_1 \bmod (s_1+s_2))-s_2$ partial closure moves, as above, to obtain the braid shown in Figure 11 with $x_1 = (r_1-r_2)-s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor - s_2, x_2 = s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor + (r_1 \bmod (s_1+s_2))-s_2$, and $a = (r_1 \bmod (s_1+s_2))-s_2$.

From here, we cancel $s_1 - a = s_2 + s_1 - (r_1 \bmod (s_1+s_2))$ of the $x_1 = (r_1-r_2)-s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor - s_2$ twists by closing the leftmost $s_1 - a$ of the strings. The result can be conjugated to $GB(x_1,y_1,x_2,y_2,z)$ with $x_1 = (r_1-r_2)-s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor + (r_1 \bmod (s_1+s_2)) - s_1 - 2s_2, y_1 = s_2, x_2 = s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor + (r_1 \bmod (s_1+s_2))-s_2, y_2 = s_1$, and $z = s_1+s_2-(r_1 \bmod (s_1+s_2))$ (shown in Figure 2), which was our goal. □

Remark. We note that if $s_1 = 1$ or $s_2 = 1$, Theorem 2.3 still holds; however, this case is not interesting, as a twist on a single strand has no effect.

Corollary 2.4. *Let $r_1, r_2, s_1, s_2 \in \mathbb{N}$ with $r_2 \geq r_1 \geq s_1 + s_2$. If $r_1 \bmod (s_1+s_2)$ is not between s_1 and s_2, then $T((r_1,-s_1),(r_2,s_2))$ is isotopic to the closure of $GB(x_1,y_1,x_2,y_2,z)$ with parameters:*

$$x_1 = \begin{cases} (r_2-r_1)+s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor+(r_1 \bmod (s_1+s_2)) & \text{if } r_1 \bmod (s_1+s_2) \leq \min(s_1,s_2) \\ (r_2-r_1)+s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor-(r_1 \bmod (s_1+s_2))+s_1+2s_2 & \text{if } r_1 \bmod (s_1+s_2) \geq \max(s_1,s_2) \end{cases}$$

$$y_1 = s_2$$

$$x_2 = \begin{cases} -s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor-(r_1 \bmod (s_1+s_2)) & \text{if } r_1 \bmod (s_1+s_2) \leq \min(s_1,s_2) \\ -s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor-(r_1 \bmod (s_1+s_2))+s_2 & \text{if } r_1 \bmod (s_1+s_2) \geq \max(s_1,s_2) \end{cases}$$

$$y_2 = s_1$$

$$z = \begin{cases} r_1 \bmod (s_1+s_2) & \text{if } r_1 \bmod (s_1+s_2) \leq \min(s_1,s_2) \\ s_1+s_2-(r_1 \bmod (s_1+s_2)) & \text{if } r_1 \bmod (s_1+s_2) \geq \max(s_1,s_2) \end{cases}$$

Proof. We note that the mirror image of $T((r_1,-s_1),(r_2,s_2))$ is $T((r_1,s_1),(r_2,-s_2))$ and the mirror image of $GB(x_1,y_1,x_2,y_2,z)$ is $GB(-x_1,y_1,-x_2,y_2,z)$. Thus, we apply Theorem 2.3 to $T((r_1,s_1),(r_2,-s_2))$ then take the mirror image, which gives us our result. □

Definition 2.5 (Inverse Image)**.** We say that a general braid, $GB(x_1,y_1,x_2,y_2,z)$ has an *inverse image* if there is a twisted torus link which is mapped to $GB(x_1,y_1,x_2,y_2,z)$ by Theorem 2.3 or Corollary 2.4.

By inverting the parameter formulas in Theorem 2.3 and Corollary 2.4, we obtain the following corollary.

Corollary 2.6. *$GB(x_1,y_1,x_2,y_2,z)$ has an inverse image if and only if one of the following holds:*

(1) $GB(x_1,y_1,x_2,y_2,z)$ corresponds to $T((\frac{y_1}{y_2}(x_2-z)+x_2,y_2),(x_2-x_1-z,-y_1))$ when

(a) y_2 divides $y_1(x_2-z)$,

(b) $x_1 < 0 < x_2$,

(c) $x_2-x_1-z \geq \frac{y_1}{y_2}(x_2-z)+x_2 \geq y_1+y_2$, and

(d) $(\frac{y_1}{y_2}(x_2-z)+x_2) \bmod (y_1+y_2) \leq \min(y_1,y_2)$.

(2) $GB(x_1,y_1,x_2,y_2,z)$ corresponds to $T((\frac{y_1}{y_2}(x_2+z)+x_2,y_2),(x_2-x_1-z,-y_1))$ when

(a) y_2 divides $y_1(x_2+z)$,

(b) $x_1 < 0 < x_2$,

(c) $x_2-x_1-z \geq \frac{y_1}{y_2}(x_2+z)+x_2 \geq y_1+y_2$, and

(d) $(\frac{y_1}{y_2}(x_2+z)+x_2) \bmod (y_1+y_2) \geq \max(y_1,y_2)$.

(3) $GB(x_1,y_1,x_2,y_2,z)$ corresponds to $T((\frac{-y_1}{y_2}(x_2+z)-x_2,-y_2),(x_1-x_2-z,y_1))$ when

(a) y_2 divides $y_1(x_2+z)$,

(b) $x_2 < 0 < x_1$,

(c) $x_1 - x_2 - z \geq \frac{-y_1}{y_2}(x_2 + z) - x_2 \geq y_1 + y_2$, *and*

(d) $(\frac{-y_1}{y_2}(x_2 + z) - x_2) \bmod (y_1 + y_2) \geq \max(y_1, y_2)$.

(4) $GB(x_1, y_1, x_2, y_2, z)$ *corresponds to* $T((\frac{-y_1}{y_2}(x_2 - z) - x_2, -y_2), (x_1 - x_2 - z, y_1))$ *when*

(a) y_2 *divides* $y_1(x_2 - z)$,

(b) $x_2 < 0 < x_1$,

(c) $x_1 - x_2 - z \geq \frac{-y_1}{y_2}(x_2 - z) - x_2 \geq y_1 + y_2$, *and*

(d) $(\frac{-y_1}{y_2}(x_2 - z) - x_2) \bmod (y_1 + y_2) \leq \min(y_1, y_2)$.

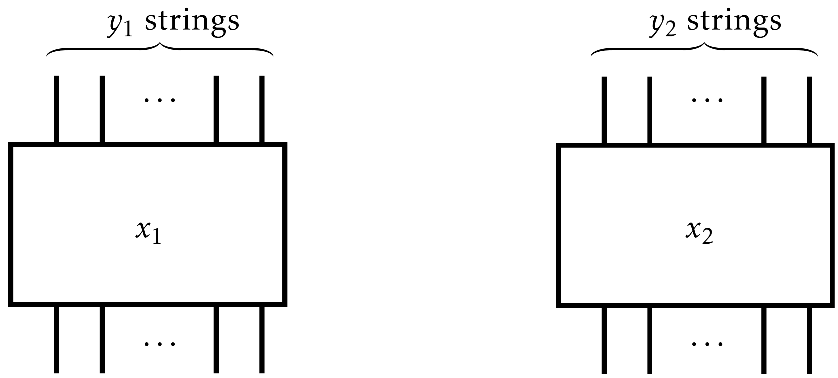
3. Restricted Permutations and Component Number

In this section, we describe a tool that is useful in describing the number of components in the closure of a general braid, which we call the braid's *restricted permutation*. We begin by noting that the permutation which corresponds to $GB(x_1, y_1, x_2, y_2, z)$ is $(y_1 + y_2 - z, y_1 + y_2 - z - 1, \ldots, y_1 - z + 1)^{x_2} \circ (y_1, y_1 - 1, \ldots, 1)^{x_1} \in S_{y_1+y_2-z}$. The number of cycles in the cycle decomposition of this permutation is the same as the number of components in the closure of the general braid. We will refer to $(y_1, y_1 - 1, \ldots, 1)^{x_1} \in S_{y_1+y_2-z}$ as the *top permutation* of the general braid and $(y_1 + y_2 - z, y_1 + y_2 - z - 1, \ldots, y_1 - z + 1)^{x_2} \in S_{y_1+y_2-z}$ as the *bottom permutation* of the general braid throughout this section. In general, we use the symbol $\#\alpha$ to represent the number of components in the closure of a braid α. Likewise, $\overline{\alpha}$ will refer to the normal braid closure of the braid α. We will denote by $\#\sigma$ the number of cycles in the cyclic decomposition of a permutation σ. We now begin with the definition of a restricted permutation.

Definition 3.1. Let $\sigma \in S_n$ and $S \subseteq \{1, 2, \ldots, n\}$. Let a_i be the i^{th} smallest element of S for $i \in \{1, \ldots, |S|\}$. For $a_i \in S$, let m_i be the smallest positive integer such that $\sigma^{m_i}(a_i) \in S$. Say $\sigma^{m_i}(a_i) = a_{j_i}$. We define the *restricted permutation*, σ restricted to S (denoted $\sigma\big|_S$), as the permutation $\sigma_S \in S_{|S|}$ given by $\sigma_S(i) = j_i$

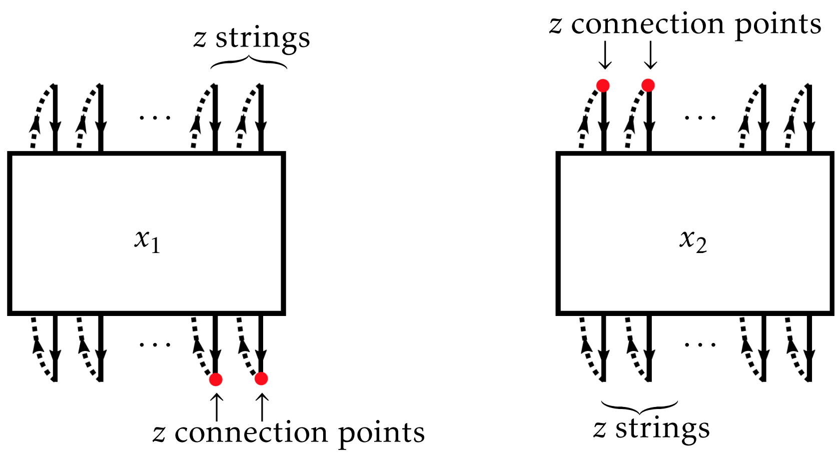
In a general braid's permutation, the only elements which do not stay on the same side of the braid after either the top permutation or bottom permutation are elements of the set $S = \{y_1 - z + 1, \ldots, y_1\}$ (unless the the top permutation or the bottom permutation is the identity). As a result, we will be restricting permutations of general braids to the set S throughout this section. Thus, we will refer to the top permutation of a general braid restricted to S as the *top restricted permutation* and the bottom permutation of a general braid restricted to S as the *bottom restricted permutation* of $GB(x_1, y_1, x_2, y_2, z)$. This allows us to propose the following Lemma:

Lemma 3.2. *Let* $GB(x_1, y_1, x_2, y_2, z)$ *be a general braid. Let* c_1 *be the number of cycles in the top permutation which are properly contained in* $\{1, \ldots, y_1 - z\}$ *and let* c_2 *be the number of cycles in the bottom permutation which are properly contained in* $\{y_1 + 1, \ldots, y_1 + y_2 - z\}$.



α β

FIGURE 12. Tangles α and β



$\overline{\alpha}$ $\overline{\beta}$

FIGURE 13. Links $\overline{\alpha}$ and $\overline{\beta}$

Let σ' be the top restricted permutation and let ρ' be the bottom restricted permutation, then $\#GB(x_1, y_1, x_2, y_2, z) = c_1 + c_2 + \#(\rho' \circ \sigma')$.

Proof. Let $g = GB(x_1, y_1, x_2, y_2, z)$ be a general braid. Let σ be the top permutation of g and ρ be the bottom permutation of g. We consider g as a $(y_1 + y_2 - z)$-tangle. Let α be the y_1-tangle which consists of x_1 twists (so the permutation associated with α is σ restricted to $\{1, \ldots, y_1\}$) and let β be the y_2-tangle which consists of x_2 twists (so the permutation associated with β is ρ restricted to $\{y_1 + 1 - z, \ldots, y_1 + y_2 - z\}$). These tangles are shown in Figure 12.

We now consider the links $\overline{\alpha}$ and $\overline{\beta}$, shown in Figure 13. We observe that $\overline{g}$ can be obtained from $\overline{\alpha}$ and $\overline{\beta}$ by attaching these braids at each of the corresponding z connection points of each link (highlighted in Figure 13). The result is shown in Figure 14

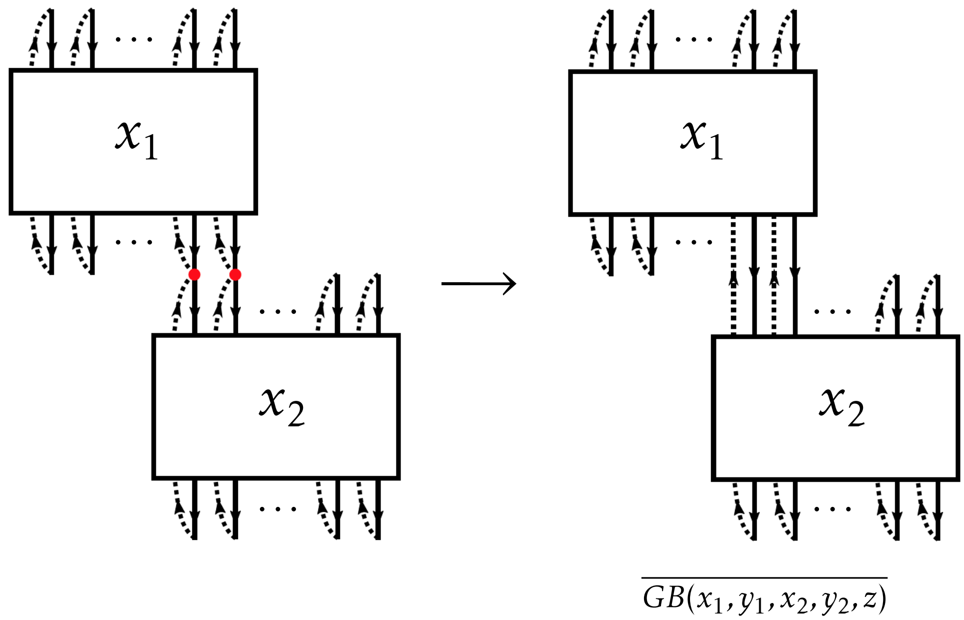


Figure 14. Attaching at corresponding connection points of $\overline{\alpha}$ and $\overline{\beta}$.

With the relationship between $\overline{\alpha}, \overline{\beta}$, and $\overline{g}$ established, we proceed by removing components from $\overline{\alpha}$ and $\overline{\beta}$. Specifically, we remove components which do not contain any of the z strings whose closures we connected to form $\overline{g}$. Let c_1 be the number of components removed from $\overline{\alpha}$ and let c_2 be the number of components removed from $\overline{\beta}$. We note that c_1 is the number of cycles of σ restricted to $\{1,\ldots,y_1\}$ which do not contain elements of $\{y_1-z+1,\ldots,y_1\}$ and c_2 is the number of cycles of ρ restricted to $\{y_1+1,\ldots,y_1+y_2-z\}$ which do not contain elements of $\{1,\ldots,z\}$. We will refer to the link obtained from removing c_1 components of $\overline{\alpha}$ as L_1, the link obtained from removing c_2 components of $\overline{\beta}$ as L_2, and the link obtained from removing the corresponding c_1+c_2 components of g as L_3. These steps are shown in Figure 15. We conclude that $\#g = c_1 + c_2 + \#L_3$.

Having removed the components of $\overline{\alpha}$ which do not include any of the z shared strings, we can isotope L_1 to the closure of a z-tangle, α', on the strings numbered $y_1-z+1,\ldots,y_1$ such that the z connection points remain fixed. Similarly, having removed the components of $\overline{\beta}$ which do not include any of the z shared strings, we can isotope L_2 to the closure of a z tangle, β', on the strings numbered $1,\ldots,z$ such that the z connection points remain fixed. Performing these isotopies shows that L_3 is isotopic to the closure of the concatenation of α' and β', denoted $\alpha' * \beta'$. These steps are shown in Figure 16 and indicate $\#g = c_1 + c_2 + \#(\alpha' * \beta')$.

Finally, we note that the permutation associated with α' is the top restricted permutation of g and the permutation associated with β' is the bottom restricted permutation of g, as we have now reduced each permutation to be an element of S_z. Let σ' and ρ' be the top

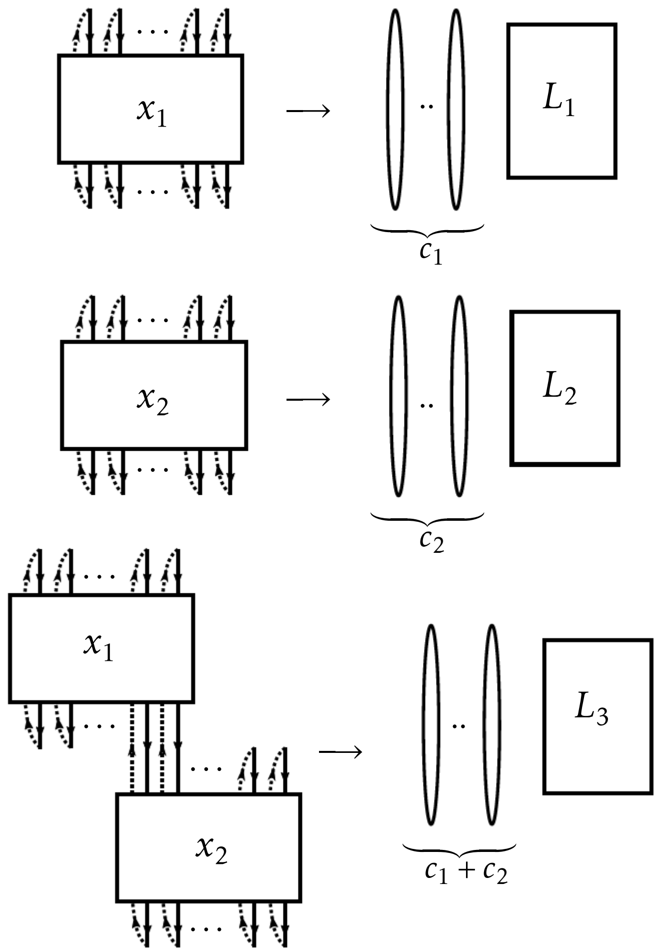


FIGURE 15. Removing components from $\overline{\alpha}$, $\overline{\beta}$, and $\overline{g}$

and bottom restricted permutations of g respectively, then $\#(\alpha' * \beta') = \#(\rho' \circ \sigma')$. It follows that $\#g = c_1 + c_2 + \#(\rho' \circ \sigma')$. □

Having proved this lemma, we are ready to prove some results regarding the component number of general braids.

Now we focus only on general braids which have inverse images, as in Corollary 2.6. However, it is possible that many of the results proven in this section apply to general braids which do not have inverse images. Since, in general, the closure of $GB(\pm x_1, y_1, \pm x_2, y_2, z)$ has the same number of components as the closure of $GB(x_1, y_1, x_2, y_2, z)$, our proofs consider the parameters of the general braid in absolute value. We use $\#GB(x_1, y_1, x_2, y_2, z)$ to denote the number of components in the closure of $GB(x_1, y_1, x_2, y_2, z)$

Our goal is to prove the following theorem regarding the component number of general braids:

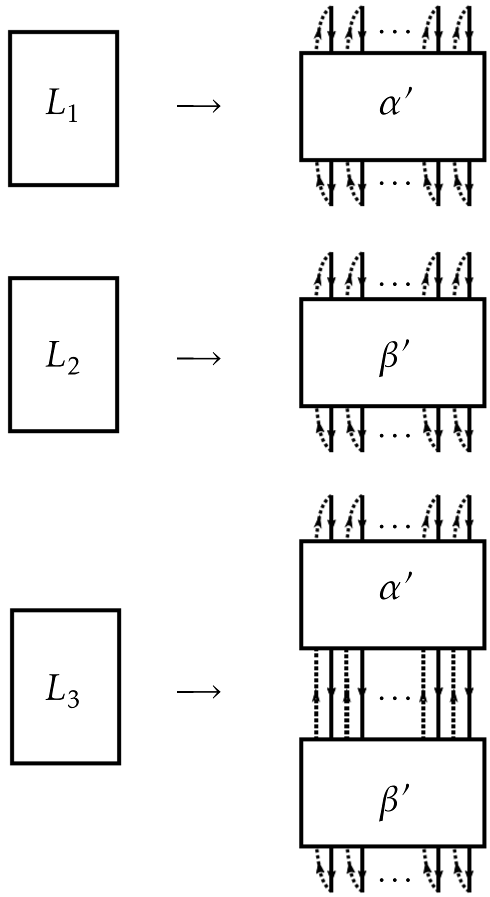


FIGURE 16. Isotoping L_1, L_2, and L_3 to be closures of z-tangles.

Theorem 3.3. *Let $x_1, y_1, x_2, y_2, z \in \mathbb{N}$ such that $GB(\pm x_1, y_1, \mp x_2, y_2, z)$ has an inverse image. Then, the following statements hold:*

(1) For a positive integer d, if d divides $\gcd(x_1, y_1, x_2, y_2)$, then

$$\#GB(x_1, y_1, x_2, y_2, z) = d\ \#GB\left(\frac{x_1}{d}, \frac{y_1}{d}, \frac{x_2}{d}, \frac{y_2}{d}, \frac{z}{d}\right)$$

.

(2) If $\gcd(x_1, y_1) + \gcd(x_2, y_2) = z$, then $\#GB(x_1, y_1, x_2, y_2, z) = 2\gcd(x_1, y_1, x_2, y_2)$.

(3) If $\gcd(x_1, y_1) + \gcd(x_2, y_2) > z$, then $\#GB(x_1, y_1, x_2, y_2, z) = \gcd(x_1, y_1) + \gcd(x_2, y_2) - z$.

In order to prove Theorem 3.3, we first prove the following lemmas:

Lemma 3.4. *If $x_1, y_1, x_2, y_2, z \in \mathbb{N}$ such that $GB(\pm x_1, y_1, \mp x_2, y_2, z)$ has an inverse image, then $\gcd(x_2, y_2)$ divides z.*

Proof. Since the braid $GB(\pm x_1, y_1, \mp x_2, y_2, z)$ has an inverse image, its closure is isotopic to $T((r_1, \mp s_1),(r_2, \pm s_2))$ for some $r_1, s_1, r_2, s_2 \in \mathbb{N}$. We consider four cases:

(1) $GB(-x_1, y_1, x_2, y_2, z)$ has inverse image $T((r_1, s_1),(r_2, -s_2))$ and $r_1 \mod (s_1+s_2) \leq \min(s_1, s_2)$

(2) $GB(-x_1, y_1, x_2, y_2, z)$ has inverse image $T((r_1, s_1),(r_2, -s_2))$ and $r_1 \mod (s_1+s_2) \geq \max(s_1, s_2)$

(3) $GB(x_1, y_1, -x_2, y_2, z)$ has inverse image $T((r_1, -s_1),(r_2, s_2))$ with $r_1 \mod (s_1+s_2) \leq \min(s_1, s_2)$

(4) $GB(x_1, y_1, -x_2, y_2, z)$ has inverse image $T((r_1, -s_1),(r_2, s_2))$ with $r_1 \mod (s_1+s_2) \geq \max(s_1, s_2)$

Case (1): Consider the case where the closure of the braid $GB(-x_1, y_1, x_2, y_2, z)$ is isotopic to $T((r_1, s_1),(r_2, -s_2))$ with $r_1 \mod (s_1+s_2) \leq \min(s_1, s_2)$. We then have $x_2 = s_1\lfloor \frac{r_1}{s_1+s_2} \rfloor + (r_1 \mod (s_1+s_2)), y_2 = s_1$, and $z = r_1 \mod (s_1+s_2)$ by Theorem 2.3. It follows that $\gcd(x_2, y_2) = \gcd(z, s_1)$, which divides z.

Case (2): Consider the case where the closure of the braid $GB(-x_1, y_1, x_2, y_2, z)$ is isotopic to $T((r_1, s_1),(r_2, -s_2))$ with $r_1 \mod (s_1+s_2) \geq \max(s_1, s_2)$. We then have $x_2 = s_1\lfloor \frac{r_1}{s_1+s_2} \rfloor + (r_1 \mod (s_1+s_2)) - s_2, y_2 = s_1$, and $z = s_1 + s_2 - (r_1 \mod (s_1+s_2))$ by Theorem 2.3. It follows that $\gcd(x_2, y_2) = \gcd(z, s_1)$, which divides z.

Case (3): Consider the case where the closure of the braid $GB(x_1, y_1, -x_2, y_2, z)$ is isotopic to $T((r_1, -s_1),(r_2, s_2))$ with $r_1 \mod (s_1+s_2) \leq \min(s_1, s_2)$. We then have $x_2 = -s_1\lfloor \frac{r_1}{s_1+s_2} \rfloor - (r_1 \mod (s_1+s_2)), y_2 = s_1$, and $z = (r_1 \mod (s_1+s_2))$ by Corollary 2.4. It follows that $\gcd(x_2, y_2) = \gcd(z, s_1)$, which divides z.

Case (4): Consider the case where the closure of the braid $GB(x_1, y_1, -x_2, y_2, z)$ is isotopic to $T((r_1, -s_1),(r_2, s_2))$ with $r_1 \mod (s_1+s_2) \geq \max(s_1, s_2)$. We then have $x_2 = -s_1\lfloor \frac{r_1}{s_1+s_2} \rfloor - (r_1 \mod (s_1+s_2)) + s_2, y_2 = s_1$, and $z = s_1 + s_2 - (r_1 \mod (s_1+s_2))$ by Corollary 2.4. It follows that $\gcd(x_2, y_2) = \gcd(z, s_1)$, which divides z. □

Lemma 3.5. *Let $x_1, y_1, x_2, y_2, z \in \mathbb{N}$ and suppose that $T((r_1, \mp s_1),(r_2, \pm s_2))$ is an inverse image of the braid $GB(\pm x_1, y_1, \mp x_2, y_2, z)$. Then,* $\gcd(x_1, y_1, x_2, y_2, z) = \gcd(x_1, y_1, x_2, y_2) = \gcd(r_1, s_1, r_2, s_2)$.

Proof. As in the proof of Lemma 3.4, there are four cases to consider:

(1) $GB(-x_1, y_1, x_2, y_2, z)$ has inverse image $T((r_1, s_1),(r_2, -s_2))$ and $r_1 \mod (s_1+s_2) \leq \min(s_1, s_2)$

(2) $GB(-x_1, y_1, x_2, y_2, z)$ has inverse image $T((r_1, s_1),(r_2, -s_2))$ and $r_1 \mod (s_1+s_2) \geq \max(s_1, s_2)$

(3) $GB(x_1, y_1, -x_2, y_2, z)$ has inverse image $T((r_1, -s_1),(r_2, s_2))$ with $r_1 \mod (s_1+s_2) \leq \min(s_1, s_2)$

(4) $GB(x_1, y_1, -x_2, y_2, z)$ has inverse image $T((r_1, -s_1),(r_2, s_2))$ with $r_1 \mod (s_1+s_2) \geq \max(s_1, s_2)$

We will prove the first case of this lemma. The other cases can be proved similarly.

Let $x_1, y_1, x_2, y_2, z \in \mathbb{N}$ be chosen so that the braid $GB(-x_1, y_1, x_2, y_2, z)$ has inverse image $T((r_1, s_1),(r_2, -s_2))$ and $r_1 \mod (s_1+s_2) \leq \min(s_1, s_2)$. From Lemma 3.4, $\gcd(x_1, y_1, x_2, y_2, z) = \gcd(x_1, y_1, x_2, y_2)$. Also, as $y_1 = s_2$ and $y_2 = s_1$, we know $\gcd(x_1, y_1, x_2, y_2) = \gcd(x_1, x_2, s_1, s_2)$. Let $\overline{r_1} = r_1 \mod (s_1+s_2)$, so then $r_1 = k(s_1+s_2) + \overline{r_1}$ for some $k \in \mathbb{Z}$. From Theorem 2.3, $x_1 = (r_1 - r_2) - s_2\lfloor \frac{r_1}{s_1+s_2} \rfloor - (r_1 \mod (s_1+s_2)) = k(s_1+s_2) - r_2 - s_2\lfloor \frac{r_1}{s_1+s_2} \rfloor = ks_1 - r_2$. Thus, $\gcd(x_1, x_2, s_1, s_2) = \gcd(r_2, x_2, s_1, s_2)$. From Theorem 2.3, we also know that $x_2 = s_1\lfloor \frac{r_1}{s_1+s_2} \rfloor +$

$(r_1 \bmod (s_1+s_2)) = ks_1 + \overline{r_1}$. Thus, $\gcd(r_2, x_2, s_1, s_2) = \gcd(r_2, \overline{r_1}, s_1, s_2) = \gcd(r_2, r_1, s_1, s_2)$. Hence, we have shown that $\gcd(x_1, y_1, x_2, y_2, z) = \gcd(x_1, y_1, x_2, y_2) = \gcd(r_2, r_1, s_1, s_2)$. $\square$

Lemma 3.6. *Let $x_1, y_1, x_2, y_2, z \in \mathbb{N}$ so that $GB(\pm x_1, y_1, \mp x_2, y_2, z)$ has an inverse image. If the inverse image of $GB(\pm x_1, y_1, \mp x_2, y_2, z)$ is $T((r_1, \mp s_1), (r_2, \pm s_2))$, then for a positive integer d, if d divides $\gcd(x_1, y_1, x_2, y_2, z)$, then $T((\frac{r_1}{d}, \mp\frac{s_1}{d}), (\frac{r_2}{d}, \pm\frac{s_2}{d}))$ is an inverse image of $GB(\pm\frac{x_1}{d}, \frac{y_1}{d}, \mp\frac{x_2}{d}, \frac{y_2}{d}, \frac{z}{d})$.*

Proof. As in the proofs of Lemmas 3.4 and 3.5, there are four cases to consider:

(1) $GB(-x_1, y_1, x_2, y_2, z)$ has inverse image $T((r_1, s_1), (r_2, -s_2))$ and $r_1 \bmod (s_1+s_2) \le \min(s_1, s_2)$

(2) $GB(-x_1, y_1, x_2, y_2, z)$ has inverse image $T((r_1, s_1), (r_2, -s_2))$ and $r_1 \bmod (s_1+s_2) \ge \max(s_1, s_2)$

(3) $GB(x_1, y_1, -x_2, y_2, z)$ has inverse image $T((r_1, -s_1), (r_2, s_2))$ with $r_1 \bmod (s_1+s_2) \le \min(s_1, s_2)$

(4) $GB(x_1, y_1, -x_2, y_2, z)$ has inverse image $T((r_1, -s_1), (r_2, s_2))$ with $r_1 \bmod (s_1+s_2) \ge \max(s_1, s_2)$

We will prove the first case when $GB(-x_1, y_1, x_2, y_2, z)$ has inverse image $T((r_1, s_1), (r_2, -s_2))$ with $r_1 \bmod (s_1+s_2) \le \min(s_1, s_2)$. The other cases can be proved similarly.

We consider $T((\frac{r_1}{d}, \frac{s_1}{d}), (\frac{r_2}{d}, -\frac{s_2}{d}))$. First, note that this twisted torus link can be written as a general braid. The condition $r_1 \bmod (s_1+s_2) \le \min(s_1, s_2)$ implies the condition $\frac{r_1}{d} \bmod (\frac{s_1}{d} + \frac{s_2}{d}) \le \min(\frac{s_1}{d}, \frac{s_2}{d})$, which satisfies the hypothesis of Theorem 2.3. This holds given that $\frac{r_1 \bmod (s_1+s_2)}{d} = \frac{r_1}{d} \bmod (\frac{s_1}{d} + \frac{s_2}{d})$ and $\frac{\min(s_1,s_2)}{d} = \min(\frac{s_1}{d}, \frac{s_2}{d})$. The first condition follows because $r_1 \bmod (s_1+s_2) = r_1 - k(s_1+s_2)$ for some $k \in \mathbb{Z}$, so $\frac{r_1 \bmod (s_1+s_2)}{d} = \frac{r_1}{d} - k(\frac{s_1}{d} + \frac{s_2}{d})$ is certainly congruent to $\frac{r_1}{d} \bmod (\frac{s_1}{d} + \frac{s_2}{d})$. However, since $0 < r_1 - k(s_1+s_2) \le s_1 + s_2$, then $0 < \frac{r_1}{d} - k(\frac{s_1}{d} + \frac{s_2}{d}) \le \frac{s_1}{d} + \frac{s_2}{d}$, so $\frac{r_1 \bmod (s_1+s_2)}{d}$ is also the desired coset representative.

Now, suppose that $GB(-x_1', y_1', x_2', y_2', z')$ corresponds to $T((\frac{r_1}{d}, \frac{s_1}{d}), (\frac{r_2}{d}, -\frac{s_2}{d}))$ under the correspondence given in Theorem 2.3. We claim that $x_1' = \frac{x_1}{d}$, $y_1' = \frac{y_1}{d}$, $x_2' = \frac{x_2}{d}$, $y_2' = \frac{y_2}{d}$, and $z' = \frac{z}{d}$. We only need to show that $\frac{r_1 \bmod (s_1+s_2)}{d} = \frac{r_1}{d} \bmod (\frac{s_1}{d} + \frac{s_2}{d})$, which we have already shown. Thus, $T((\frac{r_1}{d}, \frac{s_1}{d}), (\frac{r_2}{d}, -\frac{s_2}{d}))$ is an inverse image of $GB(-\frac{x_1}{d}, \frac{y_1}{d}, \frac{x_2}{d}, \frac{y_2}{d}, \frac{z}{d})$. $\square$

Lemma 3.7. *Let $r_1, s_1, r_2, s_2 \in \mathbb{Z}$ with $r_2 \ge r_1 > 0$. For a positive integer d, if d divides $\gcd(r_1, s_1, r_2, s_2)$, then $\#T((r_1, s_1), (r_2, s_2)) = d\#T((\frac{r_1}{d}, \frac{s_1}{d}), (\frac{r_2}{d}, \frac{s_2}{d}))$.*

Proof. Consider the modulo d equivalence classes of $\{1, \ldots, r_2\}$. For $x \in \{1, \ldots, d\}$, let the equivalence class of x be $S_x = \{y \in \{1, \ldots, r_2\} : y = x + kd \text{ for } k \in \{0, 1, \ldots, \frac{r_2}{d} - 1\}\}$. Let $\phi_x : S_x \to [\frac{r_2}{d}]$ be the bijection given by $\phi_x(x + kd) = 1 + k$. Let σ_1 be the top permutation of $T((r_1, s_1), (r_2, s_2))$ and σ_2 be the bottom permutation of $T((r_1, s_1), (r_2, s_2))$.

First, we show that the nontrivial cycles of σ_1 and σ_2 are contained in the modulo d equivalence classes of $\{1, \ldots, r_2\}$. For $j \in \{1, 2\}$, and $i \in \{1, \ldots, r_2\}$, we know $\sigma_j(i) \equiv i - s_j \bmod r_j$. Reducing this equation modulo d and using the hypothesis that $d \mid s_j$ and $d \mid r_j$, we see that $\sigma_j(i) \equiv i \mod d$. It follows that cycles of σ_j are contained in the modulo d equivalence classes of $\{1, \ldots, r_2\}$.

As the cycles of σ_1 and σ_2 are contained in the modulo d equivalence classes of $\{1, \ldots, r_2\}$, we see that the cycles of $\sigma = \sigma_2 \circ \sigma_1$ are also contained in these equivalence classes. In particular, $\sigma_1\big|_{S_x}$, $\sigma_2\big|_{S_x}$, and $\sigma\big|_{S_x}$ are all well defined and $\sigma\big|_{S_x} = (\sigma_2 \circ \sigma_1)\big|_{S_x} = \sigma_2\big|_{S_x} \circ \sigma_1\big|_{S_x}$.

The permutation $\sigma\big|_{S_x}$ induces a permutation on $\{1,\ldots,r_2/d\}$ given by $\sigma_x = \phi_x \circ \sigma\big|_{S_x} \circ \phi_x^{-1}$. Similarly, we let $(\sigma_1)_x = \phi_x \circ \sigma_1\big|_{S_x} \circ \phi_x^{-1}$ and $(\sigma_2)_x = \phi_x \circ \sigma\big|_{S_x} \circ \phi_x^{-1}$. From the equation $\sigma\big|_{S_x} = \sigma_2\big|_{S_x} \circ \sigma_1\big|_{S_x}$ it follows that $\sigma_x = (\sigma_2)_x \circ (\sigma_1)_x$.

Now, we will show that σ_x is the permutation associated with $T((\frac{r_1}{d},\frac{s_1}{d}),(\frac{r_2}{d},\frac{s_2}{d}))$. In light of the equation $\sigma_x = (\sigma_2)_x \circ (\sigma_1)_x$, it suffices to show that $(\sigma_i)_x$ is the permutation associated with $T(\frac{r_i}{d},\frac{s_i}{d})$. Let $k_1 \in \{1,\ldots,\frac{r_2}{d}-1\}$ and $j = x + k_1 d$. We note that $\sigma_i(j) = x + k_2 d$ for some $k_2 \in \{1,\ldots,\frac{r_2}{d}-1\}$. We now have $\sigma_i(j) \equiv j - s_i \mod r_i$ which means $\sigma_i(j) - j \equiv -s_i \mod r_i$. This implies that $(k_2 - k_1)d \equiv -s_i \mod r_i$. It follows that $k_2 - k_1 \equiv -\frac{s_i}{d} \mod \frac{r_i}{d}$. Thus, $(\sigma_i)_x(\phi_x(j)) - \phi_x(j) = \phi_x(\sigma_i(j)) - \phi_x(j) = (1+k_2) - (1+k_1) = k_2 - k_1 \equiv -\frac{s_i}{d} \mod \frac{r_i}{d}$. As ϕ_x is surjective, we observe $(\sigma_i)_x$ is the permutation associated with $T(\frac{r_i}{d},\frac{s_i}{d})$.

Since the cycle $(x+k_1d, x+k_2d,\ldots,x+k_nd)$ in $\sigma\big|_{S_x}$ maps to the cycle $(1+k_1,1+k_2,\ldots,1+k_n)$ in σ_x, we know $\sigma\big|_{S_x}$ and σ_x have the same number of cycles. It follows that $\#\sigma\big|_{S_x} = \#T((\frac{r_1}{d},\frac{s_1}{d}),(\frac{r_2}{d},\frac{s_2}{d}))$. As the total number of disjoint cycles in σ is the sum of the number of disjoint cycles of each equivalence class, we conclude $\#\sigma = \#\sigma\big|_{S_1} + \#\sigma\big|_{S_2} + \cdots + \#\sigma\big|_{S_d} = d\#T((\frac{r_1}{d},\frac{s_1}{d}),(\frac{r_2}{d},\frac{s_2}{d}))$. □

Now, we prove each part of Theorem 3.3.

Proof of Theorem 3.3(1). Suppose $GB(x_1,y_1,x_2,y_2,z)$ has an inverse image and for a positive integer d, d divides $\gcd(x_1,y_1,x_2,y_2)$. We want to show

$$\#GB(x_1,y_1,x_2,y_2,z) = d\ \#GB\left(\frac{x_1}{d},\frac{y_1}{d},\frac{x_2}{d},\frac{y_2}{d},\frac{z}{d}\right).$$

Let $d = \gcd(x_1,y_1,x_2,y_2)$ and let $GB(x_1,y_1,x_2,y_2,z)$ have inverse image $T((r_1,s_1),(r_2,s_2))$. From Lemma 3.5, $d = \gcd(x_1,y_1,x_2,y_2) = \gcd(r_1,s_1,r_2,s_2)$. The following chain of equivalences proves the result:

$$\begin{aligned}
\#GB(x_1,y_1,x_2,y_2,z) &= \#T((r_1,s_1),(r_2,s_2)) && \text{by hypothesis}\\
&= d\ \#T\left(\left(\frac{r_1}{d},\frac{s_1}{d}\right),\left(\frac{r_2}{d},\frac{s_2}{d}\right)\right) && \text{from Lemma 3.7}\\
&= d\ \#GB\left(\frac{r_1}{d},\frac{s_1}{d},\frac{r_2}{d},\frac{s_2}{d},\frac{z}{d}\right) && \text{from Lemma 3.6}
\end{aligned}$$

□

Proof of Theorem 3.3(2). Suppose $GB(x_1,y_1,x_2,y_2,z)$ has an image and that $\gcd(x_1,y_1) + \gcd(x_2,y_2) = z$. We want to show $\#GB(x_1,y_1,x_2,y_2,z) = 2\gcd(x_1,y_1,x_2,y_2)$. We must consider two cases: $\gcd(x_1,y_1,x_2,y_2) = 1$ and $\gcd(x_1,y_1,x_2,y_2) \neq 1$.

First, we consider the case $\gcd(x_1,y_1,x_2,y_2,z) = 1$. From Lemma 3.4, $\gcd(x_2,y_2) \mid z$, so from the equation $\gcd(x_1,y_1)+\gcd(x_2,y_2) = z$, $\gcd(x_2,y_2) \mid \gcd(x_1,y_1)$. Hence, $\gcd(x_2,y_2) = 1$ and $\gcd(x_1,y_1) = z-1$.

As $\gcd(x_1,y_1) = z-1$, then the top restricted permutation is the transposition $(1,z)$. As $\gcd(x_2,y_2) = 1$, then the bottom restricted permutation is a z-cycle. Composing $(1,z)$ with a z-cycle in S_z always leads to a permutation with 2 cycles. In the statement of Lemma 3.2, $c_1 = c_2 = 0$, because $\gcd(x_1,y_1) \le z$ and $\gcd(x_2,y_2) \le z$. Hence, applying Lemma 3.2, $\#GB(x_1,y_1,x_2,y_2,z) = 2$.

In the case where $\gcd(x_1,y_1,x_2,y_2,z) \neq 1$, let $d = \gcd(x_1,y_1,x_2,y_2,z)$, and suppose that $T((r_1,s_1),(r_2,s_2))$ is the inverse image of $GB(x_1,y_1,x_2,y_2,z)$. Because $\gcd(\frac{x_1}{d},\frac{y_1}{d},\frac{x_2}{d},\frac{y_2}{d}) = 1$, $\gcd(\frac{x_1}{d},\frac{y_1}{d}) + \gcd(\frac{x_2}{d},\frac{y_2}{d}) = \frac{\gcd(x_1,y_1)}{d} + \frac{\gcd(x_2,y_2)}{d} = \frac{z}{d}$, and from Lemma 3.6, $GB(\frac{x_1}{d},\frac{y_1}{d},\frac{x_2}{d},\frac{y_2}{d},\frac{z}{d})$ has inverse image $T((\frac{r_1}{d},\frac{s_1}{d}),(\frac{r_2}{d},\frac{s_2}{d}))$. When the parameters of the general braid are relatively prime, we see that $\#GB(\frac{x_1}{d},\frac{y_1}{d},\frac{x_2}{d},\frac{y_2}{d},\frac{z}{d}) = 2$. Using Theorem 3.3(1), we can say that $\#GB(x_1,y_1,x_2,y_2,z) = 2\gcd(x_1,y_1,x_2,y_2)$. □

Proof of Theorem 3.3(3). Consider the case where $GB(x_1,y_1,x_2,y_2,z)$ has an inverse image and the inequality $\gcd(x_1,y_1) + \gcd(x_2,y_2) > z$ holds. We show that $\#GB(x_1,y_1,x_2,y_2,z) = \gcd(x_1,y_1) + \gcd(x_2,y_2) - z$.

First, we consider the case when $d = \gcd(x_1,y_1,x_2,y_2,z) > 1$. Then $\gcd(x_1,y_1)+\gcd(x_2,y_2) > z$ implies that $\gcd(\frac{x_1}{d},\frac{y_1}{d}) + \gcd(\frac{x_2}{d},\frac{y_2}{d}) = \frac{\gcd(x_1,y_1)}{d} + \frac{\gcd(x_2,y_2)}{d} > \frac{z}{d}$. Further, $\gcd(\frac{x_1}{d},\frac{y_1}{d},\frac{x_2}{d},\frac{y_2}{d}) = 1$. Hence, applying the result for when the parameters of the general braid are relatively prime, we see that $\#GB(\frac{x_1}{d},\frac{y_1}{d},\frac{x_2}{d},\frac{y_2}{d},\frac{z}{d}) = \gcd(\frac{x_1}{d},\frac{y_1}{d})+\gcd(\frac{x_2}{d},\frac{y_2}{d})-\frac{z}{d} = \frac{\gcd(x_1,y_1)}{d}+\frac{\gcd(x_2,y_2)}{d}-\frac{z}{d}$. Using Theorem 3.3(1), we can say

$$\#GB(x_1,y_1,x_2,y_2,z) = d\ \#GB(\frac{x_1}{d},\frac{y_1}{d},\frac{x_2}{d},\frac{y_2}{d},\frac{z}{d}) = d\left(\frac{\gcd(x_1,y_1)}{d} + \frac{\gcd(x_2,y_2)}{d} - \frac{z}{d}\right)$$

which can be rewritten as $\gcd(x_1,y_1) + \gcd(x_2,y_2) - z$. Thus it is sufficient to show our result in the case that $\gcd(x_1,y_1,x_2,y_2,z) = 1$.

Assuming $\gcd(x_1,y_1,x_2,y_2,z) = 1$, there are three subcases to consider: $\gcd(x_1,y_1) \ge z$, $\gcd(x_2,y_2) \ge z$, and $\gcd(x_1,y_1) < z$ and $\gcd(x_2,y_2) < z$.

For the first case, the top restricted permutation is the identity, and $\gcd(x_1,y_1) - z$ components of the x_1 twisting section do not intersect the shared z strands. The bottom restricted permutation has $\gcd(x_2,y_2)$ cycles. Thus, in the statement of Lemma 3.2, $c_1 = \gcd(x_1,y_1) - z$, $c_2 = 0$, and $\#(\rho' \circ \sigma') = \gcd(x_2,y_2)$. Hence, from Lemma 3.2,

$$\#GB(x_1,y_1,x_2,y_2,z) = \gcd(x_1,y_1) + \gcd(x_2,y_2) - z.$$

For the second case, from Lemma 3.4, $\gcd(x_2,y_2) \mid z$, so $\gcd(x_2,y_2) = z$. In this case, the bottom restricted permutation is the identity. We have already discussed the case when $\gcd(x_1,y_1) \ge z$, so we may assume $\gcd(x_1,y_1) < z$. Then, there are $\gcd(x_1,y_1)$ cycles in the top restricted permutation. Thus, in the statement of Lemma 3.2, $c_1 = c_2 = 0$, and $\#(\rho' \circ \sigma') = \gcd(x_1,y_1)$. Hence, from Lemma 3.2,

$$\#GB(x_1,y_1,x_2,y_2,z) = \gcd(x_1,y_1) = \gcd(x_1,y_1) + z - z = \gcd(x_1,y_1) + \gcd(x_2,y_2) - z.$$

Now, we consider the final case, where $\gcd(x_1,y_1) < z$ and $\gcd(x_2,y_2) < z$. Because $\gcd(x_2,y_2)$ divides z, $\gcd(x_2,y_2) \le \frac{z}{2}$. Using the hypothesis $\gcd(x_1,y_1) + \gcd(x_2,y_2) \ge z+1$, it follows that $\gcd(x_1,y_1) \ge \frac{z}{2} + 1$, or equivalently, $2\gcd(x_1,y_1) \ge z+2$. In particular, $2\gcd(x_1,y_1) > z$.

By hypothesis, $\gcd(x_1, y_1) < z$. Hence, the cycle decomposition of the top restricted permutation consists of transpositions of the form $(i, i+\gcd(x_1, y_1))$ for $i = 1, \ldots, z-\gcd(x_1, y_1)$, and all other elements are fixed. In particular, the elements that are in the transpositions are $\{1, 2, \ldots, z-\gcd(x_1, y_1)\} \cup \{1+\gcd(x_1, y_1), 2+\gcd(x_1, y_1), \ldots, z\}$. Let the cycle decomposition of the bottom restricted permutation be $c_1 \circ c_2 \circ \cdots \circ c_{\gcd(x_2,y_2)}$, where c_i is the cycle containing i. Note that c_i contains all elements of $\{1, \ldots, z\}$ equivalent to $i \mod \gcd(x_2, y_2)$.

Next, we prove the following essential fact: for each cycle c_i of the bottom restricted permutation, c_i contains at most two elements from $\{1, 2, \ldots, z-\gcd(x_1, y_1)\} \cup \{1+\gcd(x_1, y_1), 2+\gcd(x_1, y_1), \ldots, z\}$. Recall that these elements are precisely those that are in the transpositions of the cycle decomposition of the top restricted permutation. From the hypothesis that $\gcd(x_1, y_1) + \gcd(x_2, y_2) \geq z + 1$, it follows that $z - \gcd(x_1, y_1) < \gcd(x_2, y_2)$. This means that in each set of $z - \gcd(x_1, y_1)$ consecutive integers, every element is distinct $\pmod{\gcd(x_2, y_2)}$. Thus, if there were some c_i which contained more than two elements of $\{1, 2, \ldots, z - \gcd(x_1, y_1)\} \cup \{1 + \gcd(x_1, y_1), 2 + \gcd(x_1, y_1), \ldots, z\}$, then c_i would have to contain more than one element from one of the two sets. This would imply that two integers from that set are congruent $\pmod{\gcd(x_2, y_2)}$, a contradiction. Hence, each c_i contains at most two elements from $\{1, 2, \ldots, z - \gcd(x_1, y_1)\} \cup \{1 + \gcd(x_1, y_1), 2 + \gcd(x_1, y_1), \ldots, z\}$.

We define a *chain* of the cycles c_i in the cycle decomposition of the bottom restricted permutation to be sequence of cycles $c_{i_1}, c_{i_2}, \ldots, c_{i_k}$ such that for $j = 1, \ldots, k-1$, there exists a transposition $(i_j, i_j + \gcd(x_1, y_1))$, $i_j \leq z - \gcd(x_1, y_1)$, so that $i_j \in c_{i_j}$ and $i_j + \gcd(x_1, y_1) \in c_{i_{j+1}}$. Note that these transpositions are exactly those in the cycle decomposition of the top restricted permutation. It follows that if $c_{i_1}, c_{i_2}, \ldots, c_{i_k}$ is a chain, then for every $j = 1, \ldots, k-1$, we have $i_{j+1} \equiv i_j + \gcd(x_1, y_1) \pmod{\gcd(x_2, y_2)}$.

We consider the equivalence classes on the cycles c_i given by the equivalence relation $c_i \sim c_j$ if and only if there is a chain containing both c_i and c_j. It is straightforward to check that this is an equivalence relation and that the equivalence classes under this equivalence relation themselves form chains.

Let $c_{i_1}, c_{i_2}, \ldots, c_{i_k}$ be a chain of k, where $k > 1$, which form an equivalence class. We claim that c_{i_1} and c_{i_k} each must contain only one element of $\{1, 2, \ldots, z - \gcd(x_1, y_1)\} \cup \{1 + \gcd(x_1, y_1), 2 + \gcd(x_1, y_1), \ldots, z\}$. First, notice that every c_{i_λ}, $\lambda \in \{2, \ldots, k-1\}$ contains 2 elements from $\{1, 2, \ldots, z-\gcd(x_1, y_1)\} \cup \{1+\gcd(x_1, y_1), 2+\gcd(x_1, y_1), \ldots, z\}$, namely $i_{\lambda-1} + \gcd(x_1, y_1)$ and i_λ. Similarly, c_{i_1} contains i_1 and c_{i_k} contains $i_{k-1} + \gcd(x_1, y_1)$.

Suppose that c_{i_k} contains another element besides $i_{k-1} + \gcd(x_1, y_1)$, call it x. We claim that x must be i_k. First, note that $i_{k-1} + \gcd(x_1, y_1) \in \{1+\gcd(x_1, y_1), 2+\gcd(x_1, y_1), \ldots, z\}$ and each $z-\gcd(x_1, y_1)$ consecutive integers is distinct modulo $\gcd(x_2, y_2)$, so x is in $\{1, 2, \ldots, z-\gcd(x_1, y_1)\}$.

Again, using that any consecutive $z - \gcd(x_1, y_1)$ integers are distinct modulo $\gcd(x_2, y_2)$ and noting that $i_k \in \{1, \ldots, \gcd(x_2, y_2)\}$ is the smallest representative of its modulo $\gcd(x_2, y_2)$ equivalence class, then $i_k \in \{1, 2, \ldots, z - \gcd(x_1, y_1)\}$ and $x = i_k$. Now, we consider the element $i_k + \gcd(x_1, y_1)$. It cannot be in some c_λ that is not in the chain, or else $c_{i_k} \sim c_\lambda$, contradicting that $\{c_{i_1}, c_{i_2}, \ldots, c_{i_k}\}$ forms an equivalence class. However, c_{i_1} is the only cycle on the chain with fewer than 2 elements from $\{1, 2, \ldots, z-\gcd(x_1, y_1)\} \cup \{1+\gcd(x_1, y_1), 2+\gcd(x_1, y_1), \ldots, z\}$. Thus, $i_k + \gcd(x_1, y_1) \in c_{i_1}$.

Thus, for every $j = 1,\ldots,k$, $i_{j+1} - i_j \equiv \gcd(x_1,y_1) \mod \gcd(x_2,y_2)$, where we define $i_{k+1} = i_1$. Taking the sum of this equality for $j = 1,\ldots,k$, we see that $0 = i_1 - i_1 \equiv k\gcd(x_1,y_1) \mod \gcd(x_2,y_2)$. Because there is a correspondence between cycles in the chain $c_{i_1}, c_{i_2},\ldots,c_{i_k}$ and transpositions in the cycle decomposition of the top restricted permutation, and there are $z - \gcd(x_1,y_1)$ transpositions in the top restricted permutation, then $k \le z - \gcd(x_1,y_1)$. By hypothesis, $z - \gcd(x_1,y_1) < \gcd(x_2,y_2)$. Thus, $k < \gcd(x_2,y_2)$. By assumption, $\gcd(\gcd(x_1,y_1),\gcd(x_2,y_2)) = \gcd(x_1,y_1,x_2,y_2) = 1$. Hence, the equation $0 \equiv k\gcd(x_1,y_1) \mod \gcd(x_2,y_2)$ cannot hold. Thus, our original assumption that c_{i_k} contains another element of $\{1,2,\ldots,z-\gcd(x_1,y_1)\} \cup \{1+\gcd(x_1,y_1), 2+\gcd(x_1,y_1),\ldots,z\}$ besides i_{k-1} was incorrect.

Suppose that c_{i_1} contains another element besides i_1, call it x. We claim x must be $i_k + \gcd(x_1,y_1)$. First, note that $i_1 \in \{1,2,\ldots,z-\gcd(x_1,y_1)\}$ and each $z-\gcd(x_1,y_1)$ consecutive integers are distinct modulo $\gcd(x_2,y_2)$, so x is in $\{1+\gcd(x_1,y_1), 2+\gcd(x_1,y_1),\ldots,z\}$. If $x = i_\lambda + \gcd(x1,y1)$, where λ is an element of $\{1,...,k-1\}$, then the cycles in the chain are not disjoint, which is a contradiction. Likewise, if $x = \mu + \gcd(x1,y1)$ for some $\mu \in c_\mu$ which is not in the chain, then c_μ is in the same equivalence class as c_{i_1}, a contradiction. Hence, c_{i_1} must also contain only i_1 from $\{1,2,\ldots,z-\gcd(x_1,y_1)\} \cup \{1+\gcd(x_1,y_1), 2+\gcd(x_1,y_1),\ldots,z\}$.

We have shown that if the chain $c_{i_1}, c_{i_2},\ldots,c_{i_k}$, $k > 1$, is an equivalence class, then c_{i_1} and c_{i_k} each contain only one element from the set $\{1,2,\ldots,z-\gcd(x_1,y_1)\} \cup \{1+\gcd(x_1,y_1), 2+\gcd(x_1,y_1),\ldots,z\}$.

We now show that

$$\left(c_{i_1} \circ c_{i_2} \circ \cdots \circ c_{i_k}\right) \circ ((i_1, i_1 + \gcd(x_1,y_1)) \circ \cdots \circ (i_{k-1}, i_{k-1} + \gcd(x_1,y_1)))$$

is a single cycle by induction on k. In the case $k = 2$, we have the permutation $(c_{i_1} \circ c_{i_2}) \circ (i_1, i_1 + \gcd(x_1,y_1))$, where $i_1 \in c_{i_1}$ and $i_1 + \gcd(x_1,y_1) \in c_{i_2}$. If $c_{i_1} = (j_1, i_1)$ and $c_{i_2} = (j_2, i_1 + \gcd(x_1,y_1))$, where j_1 and j_2 are cycles that are disjoint and each j_i is disjoint from all other cycles in the chain, then $(c_{i_1} \circ c_{i_2}) \circ (i_1, i_1 + \gcd(x_1,y_1)) = (i_1, j_2, i_1 + \gcd(x_1,y_1), j_1)$ and has one cycle. For the inductive step, consider

$$\left(c_{i_1} \circ c_{i_2} \circ \cdots \circ c_{i_k}\right) \circ ((i_1, i_1 + \gcd(x_1,y_1)) \circ \cdots \circ (i_{k-1}, i_{k-1} + \gcd(x_1,y_1))).$$

Again, letting $c_{i_1} = (j_1, i_1)$ and $c_{i_2} = (j_2, i_1 + \gcd(x_1,y_1))$, where j_1 and j_2 are strings of consecutive integers, then we have the following equalities, where, for the sake of easing notational issues, $\sigma = (i_2, i_2 + \gcd(x_1,y_1)) \circ \cdots \circ (i_{k-1}, i_{k-1} + \gcd(x_1,y_1))$:

$$\begin{aligned}
&\left(c_{i_1} \circ c_{i_2} \circ \cdots \circ c_{i_k}\right) \circ ((i_1, i_1 + \gcd(x_1,y_1)) \circ \sigma) \\
&= \left(c_{i_3} \circ \cdots \circ c_{i_k}\right) \circ c_{i_1} \circ c_{i_2} \circ (i_1, i_1 + \gcd(x_1,y_1)) \circ \sigma \\
&= \left(c_{i_3} \circ \cdots \circ c_{i_k}\right) \circ (i_1, j_2, i_1 + \gcd(x_1,y_1), j_1) \circ \sigma \\
&= \left((i_1, j_2, i_1 + \gcd(x_1,y_1), j_1) \circ c_{i_3} \circ \cdots \circ c_{i_k}\right) \circ \sigma
\end{aligned}$$

As the cycle $(i_1, j_2, i_1 + \gcd(x_1, y_1), j_1)$ contains only i_2 among the set

$$\{i_2, i_3, \ldots, i_k\} \cup \{i_2 + \gcd(x_1, y_1), i_3 + \gcd(x_1, y_1), \ldots, i_k + \gcd(x_1, y_1)\}$$

the inductive hypothesis tells us that, the final permutation in the sequence of equalities is a cycle.

Every transposition in the top permutation is associated to some equivalence class of chains with length greater than 1, because the the transposition $(i, i + \gcd(x_1, y_1))$ is associated with the chain containing $\{c_i, c_{i'}\}$, where $i' \equiv i + \gcd(x_1, y_1) \mod \gcd(x_2, y_2)$. In this equivalence class, we have shown that taking the composition with such a transposition decreases the number of cycles by 1. Hence, composing all $z - \gcd(x_1, y_1)$ transpositions in the top restricted permutation with the bottom restricted permutation, we obtain a permutation with $\gcd(x_2, y_2) - (z - \gcd(x_1, y_1)) = \gcd(x_2, y_2) + \gcd(x_1, y_1) - z$ cycles. □

Using these results, we can determine the component number of all twisted torus links which correspond to general braids of the form $GB(\pm x_1, y_1, \mp x_2, y_2, z)$ with $z \in \{0, 1, 2\}$. In the next section, we will show that if $z = 0$, then $T((r_1, \pm s_1), (r_2, \mp s_2))$ is the split link of two torus links, which have well known component number. When $z = 1$ or $z = 2$, we automatically have $\gcd(x_1, y_1) + \gcd(x_2, y_2) \geq z$, a situation which was fully described in Theorem 3.3. Unfortunately, once we let $z = 3$, Theorem 3.3 no longer describes all the possibilities. It is possible that $\gcd(x_1, y_1) = \gcd(x_2, y_2) = 1$. Here, the upper and lower restricted permutations can be either $(1, 2, 3)$ or $(1, 3, 2)$. If both are the same, their composition is a 3-cycle, and thus the twisted torus link is a knot, but if they are different, their composition is the identity permutation, meaning the twisted torus link has three components. We have yet to find a specific formula which tells us, in general, what the upper and lower restricted permutations will be in this case.

Thus far, we have shown that the formulas given for the component number of the closure of a general braid only held if that general braid has an inverse image. In the future, we hope to relax this condition to any general braid. We conjecture that Theorem 3.3 holds for any general braid. Unfortunately, if $GB(x_1, y_1, x_2, y_2, z)$ does not have an inverse image, Lemmas 3.4, 3.5, and 3.7 no longer hold. In some cases (for example when $\gcd(x_1, y_1) \geq z$), this does not affect the proof of Theorem 3.3, but other portions of the proof become more difficult when it is no longer true that $\gcd(x_2, y_2) | z$. In the future, we would like to rework the proof of Theorem 3.3 so that it no longer relies on the three lemmas.

4. Twisted Torus Links

Component number is a useful tool for classifying links; however, it contains no information about the interaction between the components of a given link. In this section, we give a more thorough classification of several infinite families of twisted torus links with help from their general braid representations. The results presented here are presented only for twisted torus links of the form $T((r_1, s_1), (r_2, -s_2))$, although Corollary 2.4 would provide results for twisted torus links of the form $T((r_1, -s_1), (r_2, s_2))$. We begin by examining when two twisted torus links are represented by the same general braid:

Corollary 4.1. *Suppose $r_1, s_1, r_2, s_2 \in \mathbb{N}$ such that*

(1) $r_2 \geq r_1 \geq s_1 + s_2$,

(2) $r_1 \bmod (s_1+s_2) \le \min(s_1,s_2)$,

(3) $r_2+2(r_1 \bmod (s_1+s_2)) \ge r_1+\frac{2s_2}{s_1}(r_1 \bmod (s_1+s_2)) \in \mathbb{N}$, *and*

(4) $(r_1(\frac{2s_2}{s_1}+1)) \bmod (s_1+s_2) \ge \max(s_1,s_2)$.

Then $T((r_1,s_1),(r_2,-s_2))$ *and* $T((r_1+\frac{2s_2}{s_1}(r_1 \bmod (s_1+s_2)),s_1),(r_2+2(r_1 \bmod (s_1+s_2)),-s_2))$ *are represented by the same general braid.*

Note that, for two twisted torus links, being represented by the same general braid is equivalent to being isotopic as links in S^3.

Proof. Let $r_1,s_1,r_2,s_2 \in \mathbb{N}$ satisfying the hypotheses above. Theorem 2.3 then tells us that $T((r_1,s_1),(r_2,-s_2))$ is isotopic to the closure of $GB(x_1,y_1,x_2,y_2,z)$ with parameters

$$\begin{aligned}
x_1 &= r_1-r_2-s_2\left\lfloor\frac{r_1}{s_1+s_2}\right\rfloor-(r_1 \bmod (s_1+s_2))\\
y_1 &= s_2\\
x_2 &= s_1\left\lfloor\frac{r_1}{s_1+s_2}\right\rfloor+(r_1 \bmod (s_1+s_2))\\
y_2 &= s_1\\
z &= r_1 \bmod (s_1+s_2)
\end{aligned}$$

By Corollary 2.6, we note that $GB(x_1,y_1,x_2,y_2,z)$ satisfies the second set of criteria for having an inverse image because we have that $r_2+2(r_1 \bmod (s_1+s_2)) \ge r_1+\frac{2s_2}{s_1}(r_1 \bmod (s_1+s_2)) \in \mathbb{N}$ and we have that $(r_1(\frac{2s_2}{s_1}+1)) \bmod (s_1+s_2) \ge \max(s_1,s_2)$. Thus, $GB(x_1,y_1,x_2,y_2,z)$ has closure isotopic to the twisted torus link $T((r_1+\frac{2s_2}{s_1}(r_1 \bmod (s_1+s_2)),s_1),(r_2+2(r_1 \bmod (s_1+s_2)),-s_2))$. This means

$$T((r_1,s_1),(r_2,-s_2)) \approx T((r_1+\frac{2s_2}{s_1}(r_1 \bmod (s_1+s_2)),s_1),(r_2+2(r_1 \bmod (s_1+s_2)),-s_2)).$$

□

We now present two cases in which $T((r_1,s_1),(r_2,-s_2))$ is a *composite knot* of two torus knots. A composite knot is one that can be created from two knots, K and J, by removing a small arc of both K and J and then attaching the lose ends of K and J together, respecting the orientation of each knot. This process is called taking the *connected sum* of the knots K and J, and the resulting knot is denoted $K\#J$. A knot that cannot be expressed as the connected sum of two nontrivial knots is called *prime*.

Corollary 4.2. *Let* $r_1,s_1,r_2,s_2 \in \mathbb{N}$ *with*

(1) $r_2 \ge r_1 \ge s_1+s_2$,

(2) $r_1 \bmod (s_1+s_2) = 1$, *and*

(3) $\gcd(r_2-r_1+1,s_2)=1$.

Then $T((r_1,s_1),(r_2,-s_2)) \approx T(s_2,r_1-r_2-s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor-1)\#T(s_1,s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor+1)$.

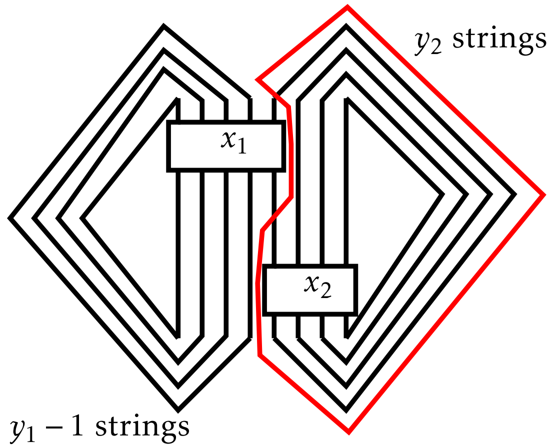


FIGURE 17. $GB(x_1, y_1, x_2, y_2, 1) \approx T(y_1, x_1) \# T(y_2, x_2)$ if $T(y_1, x_1)$ and $T(y_2, x_2)$ are knots.

Proof. Assume the hypotheses as stated. Theorem 2.3 tells us that $T((r_1, s_1), (r_2, -s_2))$ is isotopic to the closure of the braid $GB(r_1 - r_2 - s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor - 1, s_2, s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1, s_1, 1)$.

We note that $\gcd(r_1 - r_2 - s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor - 1, s_2)$ is the same as $\gcd(r_2 - r_1 + 1, s_2) = 1$ and $\gcd(s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1, s_1)$ is the same as $\gcd(1, s_1) = 1$. Theorem 3.3 then tells us that $T((r_1, s_1), (r_2, -s_2))$ is a knot. The closure of $GB(r_1 - r_2 - s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor - 1, s_2, s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1, s_1, 1)$ is of the form shown in Figure 17 and is isotopic to $T(s_2, r_1 - r_2 - s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor - 1) \# T(s_1, s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1)$. □

Corollary 4.3. *Let* $r_1, s_1, r_2, s_2 \in \mathbb{N}$ *with*

(1) $r_2 \geq r_1 \geq s_1 + s_2$,

(2) $r_1 \bmod (s_1 + s_2) = s_1 + s_2 - 1$, *and*

(3) $\gcd(r_2 - r_1 - 1, s_2) = 1$.

Then $T((r_1, s_1), (r_2, -s_2)) \approx T(s_2, r_1 - r_2 - s_2(\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1) - 1) \# T(s_1, s_1(\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1) - 1)$.

Proof. Assume the hypotheses as stated. Theorem 2.3 tells us that $T((r_1, s_1), (r_2, -s_2))$ is isotopic to the closure of $GB(r_1 - r_2 - s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor - s_2 - 1, s_2, s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor + s_1 - 1, s_1, 1)$.

Here we point out that $\gcd(r_1 - r_2 - s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor - s_2 - 1, s_2)$ is equal to $\gcd(r_2 - r_1 - 1, s_2) = 1$ and that $\gcd(s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor + s_1 - 1, s_1) = \gcd(1, s_1) = 1$. By Theorem 3.1, $T((r_1, s_1), (r_2, -s_2))$ is a knot. The closure of the braid $GB(r_1 - r_2 - s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor - s_2 - 1, s_2, s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor + s_1 - 1, s_1, 1)$ is of the form in Figure 17 and is isotopic to $T(s_2, r_1 - r_2 - s_2(\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1) - 1) \# T(s_1, s_1(\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1) - 1)$. □

We now observe that if $r_1 \bmod (s_1 + s_2) = 0$, then $T((r_1, s_1), (r_2, -s_2))$ is a split link:

Corollary 4.4. *Let* $r_1, s_1, r_2, s_2 \in \mathbb{N}$ *with*

(1) $r_2 \geq r_1 \geq s_1 + s_2$ *and*

(2) $r_1 \bmod (s_1 + s_2) = 0$

Then $T((r_1, s_1), (r_2, -s_2)) \approx T(s_2, r_1 - r_2 - s_2\frac{r_1}{s_1+s_2}) \sqcup T(s_1, s_1\frac{r_1}{s_1+s_2})$.

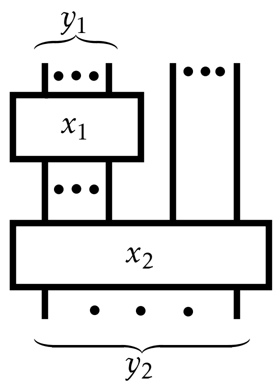


FIGURE 18. $GB(x_1, y_1, x_2, y_2, y_1)$ which has closure $T((y_1, x_1), (y_2, x_2))$

Proof. Let $r_1, s_1, r_2, s_2 \in \mathbb{N}$ with $r_2 \geq r_1 \geq s_1 + s_2$ and $r_1 \bmod (s_1 + s_2) = 0$. By Theorem 2.3, the link $T((r_1, s_1), (r_2, -s_2))$ is isotopic to the closure of $GB(r_1 - r_2 - s_2\frac{r_1}{s_1+s_2}, s_2, s_1\frac{r_1}{s_1+s_2}, s_1, 0)$. Since no strings are shared between the two sets of twists, the closure of this braid is $T(s_2, r_1 - r_2 - s_2\frac{r_1}{s_1+s_2}) \sqcup T(s_1, s_1\frac{r_1}{s_1+s_2})$. □

Remark. It follows from Corollary 4.4 that if $r_2 \geq r_1 \geq s_1 + s_2$ with $r_1 \bmod (s_1 + s_2) = 0$, then

$$T((r_1, s_1), (r_2, -s_2)) \approx T((s_2, r_1 - r_2 - s_2\frac{r_1}{s_1 + s_2}) \sqcup T(s_1, s_1\frac{r_1}{s_1 + s_2})$$

has $\gcd(s_2, r_2 - r_1 + s_2\frac{r_1}{s_1+s_2}) + \gcd(s_1, s_1\frac{r_1}{s_1+s_2}) = \gcd(s_2, r_2 - r_1) + s_1$ components.

It also follows from Theorem 2.3 that if $r_1 \bmod (s_1 + s_2) = \min(s_1, s_2)$ or $r_1 \bmod (s_1 + s_2) = \max(s_1, s_2)$, then the closure of the corresponding general braid is another twisted torus link:

Corollary 4.5. *Let $r_1, s_1, r_2, s_2 \in \mathbb{N}$ with $r_2 \geq r_1 \geq s_1 + s_2$ and $r_1 \bmod (s_1 + s_2) = \min(s_1, s_2)$, then*

$$T((r_1, s_1), (r_2, -s_2)) \approx \begin{cases} T((s_1, s_1(\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1)), (s_2, r_1 - r_2 - s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor - s_1)) & \text{if } s_1 < s_2 \\ T((s_2, r_1 - r_2 - s_2(\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1)), (s_1, s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor + s_2)) & \text{if } s_1 > s_2 \\ T(s_1, r_1 - r_2) & \text{if } s_1 = s_2 \end{cases}$$

Proof. Assume the stated hypotheses.

First assume $s_1 < s_2$. Theorem 2.3 then tells us that $T((r_1, s_1), (r_2, -s_2))$ is isotopic to the closure of $GB(r_1 - r_2 - s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor - s_1, s_2, s_1(\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1), s_1, s_2)$, which has the same closure as the braid $GB(s_1(\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1), s_1, r_1 - r_2 - s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor - s_1, s_2, s_2)$, which is of the form shown in Figure 18. It is clear that the closure of this braid is $T((s_1, s_1(\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1)), (s_2, r_1 - r_2 - s_2\lfloor\frac{r_1}{s_1+s_2}\rfloor - s_1))$.

Now assume $s_1 > s_2$. Theorem 2.3 then tells us that $T((r_1, s_1), (r_2, -s_2))$ is isotopic to the closure of $GB(r_1 - r_2 - s_2(\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1), s_2, s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor + s_2, s_1, s_1)$, which is of the form shown in Figure 18. It is clear that the closure of this braid is $T((s_2, r_1 - r_2 - s_2(\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1)), (s_1, s_1\lfloor\frac{r_1}{s_1+s_2}\rfloor + s_2))$.

Finally assume $s_1 = s_2$. Theorem 2.3 then tells us that $T((r_1, s_1), (r_2, -s_2))$ is isotopic to the closure of $GB(r_1 - r_2 - s_1(\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1), s_1, s_1(\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1), s_1, s_1)$. This braid is made up of $r_2 - r_1 + s_1(\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1)$ negative twists followed by $s_1(\lfloor\frac{r_1}{s_1+s_2}\rfloor + 1)$ positive twists on s_1 strings.

Since the latter is a set of full twists, Lemma 2.1 allows to cancel the positive twists with $s_1(\lfloor \frac{r_1}{s_1+s_2} \rfloor + 1)$ negative twists. The resulting braid consists of $r_2 - r_1$ negative twists on s_1 strings, which has closure $T(s_1, r_1 - r_2)$. □

Remark. Corollary 4.5 tells us if $r_1, r_2, s \in \mathbb{N}$ with $r_2 \geq r_1 \geq 2s, r_1 \bmod 2s = s$, and $r_2 = r_1 + 1$, then $T((r_1, s), (r_2, -s)) \approx T(s, -1)$, which is the unknot.

Remark. Corollary 4.5 tells us if $r_1, r_2, s \in \mathbb{N}$ with $r_2 \geq r_1 \geq 2s, r_1 \bmod 2s = s$, and $r_2 = r_1$, then $T((r_1, s), (r_2, -s) \approx T(s, 0)$, which is the unlink of s components.

Corollary 4.6. *Let $r_1, s_1, r_2, s_2 \in \mathbb{N}$ with $r_2 \geq r_1 \geq s_1 + s_2$ and $r_1 \bmod (s_1 + s_2) = \max(s_1, s_2)$, then*

$$T((r_1, s_1), (r_2, s_2)) \approx \begin{cases} T((s_1, s_1\lfloor \frac{r_1}{s_1+s_2} \rfloor), (s_2, r_1 - r_2 - s_2(\lfloor \frac{r_1}{s_1+s_2} \rfloor + 1) - s_1)) & \text{if } s_1 < s_2 \\ T((s_2, r_1 - r_2 - s_2(\lfloor \frac{r_1}{s_1+s_2} \rfloor + 2)), (s_1, s_1(\lfloor \frac{r_1}{s_1+s_2} \rfloor + 1) - s_2)) & \text{if } s_1 > s_2 \\ T(s_1, r_1 - r_2) & \text{if } s_1 = s_2 \end{cases}$$

Proof. The proof is analogous to that of Corollary 4.5. We note that when $s_1 = s_2, \min(s_1, s_2) = \max(s_1, s_2)$, so the third case in Corollary 4.5 and in Corollary 4.6 are the same. □

We have shown that all those twisted torus knots which correspond to general braids of the form $GB(\pm x_1, y_1, \mp x_2, y_2, 1)$ are composite; however, we have not proven that all other twisted torus knots are prime. To attempt to answer this question, we calculated the hyperbolic volume of several twisted torus links using SnapPy. Our experiments suggest that twisted torus knots $T((r_1, s), (r_2, -s))$ for $1 < 2s \leq r_1 \leq r_2 \leq 30$ are hyperbolic. It follows that these knots are also prime. In lieu of this discovery, we propose the following conjecture:

Conjecture 4.7. *Let $r_1, s_1, r_2, s_2 \in \mathbb{N}$ with $r_2 \geq r_1 \geq s_1 + s_2$. If $T((r_1, s_1), (r_2, -s_2))$ is a knot and $r_1 \bmod (s_1 + s_2) \in \{2, ..., \min(s_1, s_2), \max(s_1, s_2), ..., s_1 + s_2 - 2\}$, then $T((r_1, s_1), (r_2, -s_2))$ is a hyperbolic knot or torus knot. In particular, $T((r_1, s_1), (r_2, -s_2))$ is prime.*

5. Acknowledgements

This research was completed during the Sam Houston State University Knots and Links REU program and supported by NSF grant DMS-1262897. We would like to thank Sriyan Wickramasuriya for working alongside us throughout this research project. We would also like to thank Michelle Berry, Victoria Diaz, Gage Martin, and Emily Winn, the other members of the Sam Houston State University Knots and Links REU program, for engaging us in helpful discussions. We would also like to extend our heartfelt thanks to the referee for thoughtful and thorough comments.

References

[1] J. Birman, T. Brendle: Braids: A Survey, in *Handbook of Knot Theory*, ed. W. Menasco and M. Thistlethwaite (2005).

[2] J. Birman, I. Kofman, A new twist on Lorenz links. *J. Topology* 2 (2009), pp. 227-248. DOI: 10.1112/jtopol/jtp007

[3] R. S. Bowman, S. Taylor, A. Zupan, Bridge spectra of twisted torus knots. *Int. Math. Res. Notices* (2014). DOI: 10.1093/imrn/rnu162.

[4] B. Guntel Doleshal, Fibered and primitive/Seifert twisted torus knots. *J. Knot Theory Ramifications* 22 Issue 1 (2013). DOI: 10.1142/S0218216512501416

[5] S. Lee, Twisted torus knots that are unknotted. *Int. Math. Res. Notices* (2013). DOI: 10.1093/imrn/rnt087.

[6] S. Lee, Twisted torus knots $T(p,q;kq,s)$ are cable knots. *J. Knot Theory Ramifications* 21, 1250005 (2012). DOI: 10.1142/S0218216511009753

[7] Y. Moriah, E. Sedgwick, Heegaard splittings of twisted torus knots. *Topology Appl.*, 01/2009. DOI: 10.1016

[8] K. Morimoto, On composite twisted torus knots. *Tokyo J. Math.* 35, No. 2 (2012), 499-503.

[9] K. Morimoto, On tangle decompositions of twisted torus knots. *J. Knot Theory Ramifications*, 22, 1350049 (2013). DOI: 10.1142/S0218216513500491

[10] H. Morton, The Alexander polynomial of a torus knot with twists. *J. Knot Theory Ramifications*, 15, 1037 (2006). DOI: 10.1142/S0218216506004920

Student biographies

Eric J. Hanson: (*Corresponding author:* ehanson4@brandeis.edu) Eric graduated from Gustavus Adolphus College in 2016 with a BA in mathematics. He is currently pursuing his PhD in mathematics at Brandeis University. Eric's mathematical interests focus on applications of topology to representation theory.

Max Zhou: Max graduated from Indiana University Bloomington in 2016 with a BS in Mathematics. He is currently pursuing his PhD in mathematics at the University of California Los Angeles, and is interested in analysis.