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Abstract. When three beads slide on a ring and collide with elastic collisions, if their velocities are chosen carefully, they undergo periodic motion. We compare this problem in mechanics to a geometric problem in billiard dynamics in a right-triangle. The billiard problem uses specular reflection at the boundaries (angle in equals angle out), whereas the boundary conditions for bead collisions follow from the conservation of energy and momentum which depend on the relative velocities of the beads. For the billiard problem, using a sequence of reflections of the right triangle, we highlight techniques to find special families of periodic orbits that are parallel to one of the boundaries, or oblique to the hypotenuse at certain discrete angles. For the bead collision problem, we show how this approach can be generalized to one in which the triangles are not only reflected across boundaries, but also rotated around the point of collision to adjust for the fact that the boundary condition is non-specular. All of the results described in the paper are accessible to high school or college level pre-calculus students since the techniques rely only on geometry, trigonometry, symmetries and reflections, algebra, simple number theory, and basic laws of mechanics.

1. Introduction

When N-beads slide along a frictionless ring, their dynamics give rise to a sequence of collisions. If the initial positions and velocities of the beads are chosen appropriately, the dynamics are periodic and the sequence of collisions and bead positions repeat indefinitely due to conservation of energy. In this note, we study the problem of three beads on a ring, with equal masses, whose (relative) initial positions are denoted by their angular separations $(\theta_1, \theta_2, \theta_3)$, and whose initial velocities are given by $\vec{v} = (v_1, v_2, v_3)$. The problem is a simple one-dimensional example of the more general system called a gas of hard spheres developed in the 1930's [2, 3, 4, 6, 8, 9, 11, 12, 13] to understand atomic interactions in an idealized setting.

The initial set-up is depicted in Figure 1 with the convention that positive velocities are clockwise, negative are counterclockwise. We note also that the constraint $\theta_1 + \theta_2 + \theta_3 = 2\pi$ allows us to view a trajectory as a sequence of line segments in an isosceles right triangle, with sides representing the primary angles $0 \leq \theta_1 \leq 2\pi$; $0 \leq \theta_2 \leq 2\pi$ as shown in

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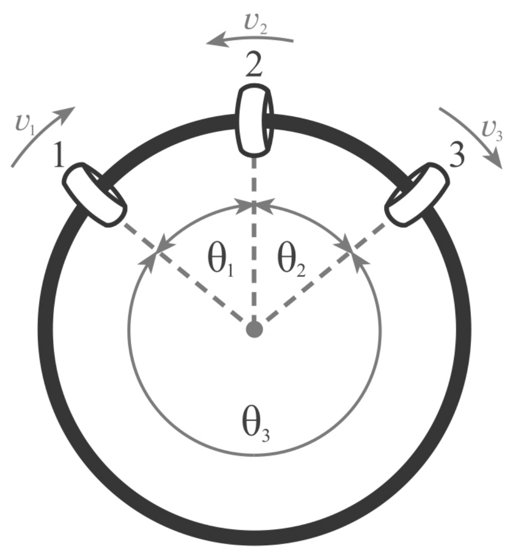


Figure 1. Schematic diagram of three beads on a ring. We assume beads have equal mass and collisions are perfectly elastic with no friction as the beads slide on the ring. Gravity points perpendicular to the ring so plays no role. The initial conditions are specified by the angles $(\theta_1, \theta_2, \theta_3)$ and the initial velocities $\vec{v} = (v_1, v_2, v_3)$. We use the convention that positive velocities are clockwise, negative velocities are counterclockwise. There is the constraint that $\theta_1 + \theta_2 + \theta_3 = 2\pi$.

Figure 2. We note that Glashow and Mittag [3] formulate the problem in an acute triangle (equilateral when all the masses are equal) as opposed to an isosceles right triangle. As the beads slide around the ring, they move along a straight line segment in the (θ_1, θ_2) triangle with fixed slope, denoted by $S_{(i-1)}$ in Figure 2 for the first segment. A collision of two beads represents the line segment hitting one of the triangle boundaries. As shown in the figure, a collision with the hypotenuse corresponds to $\theta_1 + \theta_2 = 2\pi$, which implies a collision between beads 1 and 3, since $\theta_3 = 0$. When the beads collide, subject to the laws of conservation of energy and momentum, their relative velocities change, and a new segment, with slope S_i, describes the relative motion. The next collision with the boundary is on the θ_1 leg of the triangle, which implies a collision between beads 2 and 3 since θ_2 is zero. This gives rise to the third line segment depicted in Figure 2, with slope $S_{(i+1)}$. What are the equations for these slopes and how do they relate to the reflection laws upon collision?

2. Specular and Non-specular reflections

Consider the motion of beads 1 and 2 with masses m_1, m_2 as they head towards a collision. Before they collide, we denote their velocities v_1,v_2. After collision, we denote their velocities v_1' and v_2'. The beads must obey conservation of kinetic energy:

$$m_1 v_1^2 + m_2 v_2^2 = m_1 {v'}_1^2 + m_2 {v'}_2^2, \tag{1}$$

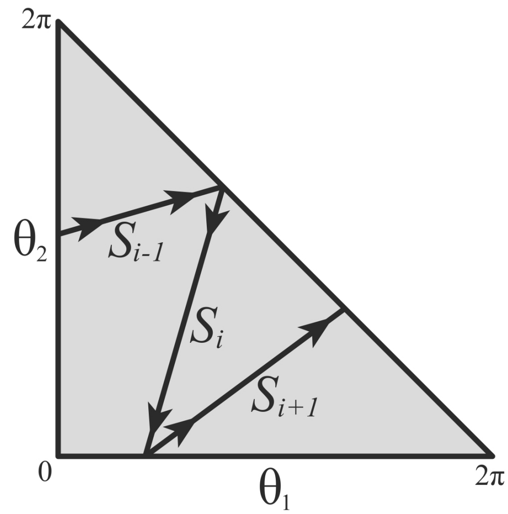


Figure 2. The three beads dynamics can be viewed as trajectory generated by a sequence of straight line segments in a right triangle. The ith segment has slope S_i. The change in slope in the ith and $(i+1)$st segment, which we denote $\Delta S_i = S_{i+1} - S_i$, is determined by the conservation of energy and momentum of the system. A collision on the θ_1 side represents a collision between beads 2 and 3. A collision on the θ_2 side represents a collision between beads 1 and 2, while a collision on the hypotenuse represents a collision between beads 1 and 3.

and conservation of momentum:

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'. \tag{2}$$

Equation (1) can be re-arranged as:

$$m_1(v_1^2 - v'^2_1) = -m_2(v_2^2 - v'^2_2),$$

$$m_1(v_1 + v_1')(v_1 - v_1') = -m_2(v_2 + v_2')(v_2 - v_2'). \tag{3}$$

Equation (2) can be re-arranged as:

$$m_1(v_1 - v_1') = -m_2(v_2 - v_2'). \tag{4}$$

Then we can divide equation (3) by equation (4):

$$(v_1 + v_1') = (v_2 + v_2'). \tag{5}$$

But equation (4) implies:

$$(v_1 - v_1') = -\frac{m_2}{m_1}(v_2 - v_2'),$$

and if we assume all masses are equal (which we do in the remainder of the paper), we have:

$$(v_1 - v_1') = -(v_2 - v_2'). \tag{6}$$

Adding equations (6) and (5) and subtracting equation (6) from (5) gives the important relations:

$$\begin{aligned} v_1 &= v_2', \\ v_2 &= v_1'. \end{aligned}$$

These equations state that, with equal masses, the velocity of beads 1 and 2 simply exchange their values upon colliding. The same holds true when beads 2 and 3 collide, and when beads 3 and 1 collide.

We next consider the slopes of each of the line segments in Figure 2. The axes are the relative angles θ_1 and θ_2, and we denote the change in each by $\Delta\theta_1$ and $\Delta\theta_2$, and the corresponding slopes as $\Delta\theta_2/\Delta\theta_1$. We know that:

$$\begin{aligned} \Delta\theta_1 &= v_2 - v_1, \\ \Delta\theta_2 &= v_3 - v_2. \end{aligned}$$

This gives rise to a formula for the slope of any line segment approaching a boundary (before collision):

$$\text{Slope}_{in} = \frac{\Delta\theta_2}{\Delta\theta_1} = \frac{v_3 - v_2}{v_2 - v_1}. \tag{7}$$

To obtain the slope of a line segment heading away from a boundary (after collision) we use the fact that the velocities of the two beads simply exchange their values upon colliding. So, for a collision between beads 1 and 2, we exchange v_1 and v_2 in equation (7) to obtain the formula for the slope after collision with the θ_2 wall:

$$\text{Slope}_{out}^{(1,2)} = (v_3 - v_1)/(v_1 - v_2). \tag{8}$$

Likewise, for a collision between beads 2 and 3, we exchange v_2 and v_3 in equation (7) to obtain the formula for the slope after collision with the θ_1 wall:

$$\text{Slope}_{out}^{(2,3)} = (v_2 - v_3)/(v_3 - v_1). \tag{9}$$

For a collision between beads 1 and 3, we obtain the formula for the slope after collision with the hypotenuse:

$$\text{Slope}_{out}^{(1,3)} = (v_1 - v_2)/(v_2 - v_3). \tag{10}$$

Notice from equation (10) and equation (7) that $\text{Slope}_{out}^{(1,3)} = 1/\text{Slope}_{in}$ for collisions with the hypotenuse.

We are now in a position to derive formulas for the angles of a trajectory heading into and out from a collision with a wall. For this, we refer to Figure 3, and we call the angle into a wall the incident angle, α_i, while the angle out from the wall is called the reflected angle α_r. The simplest reflection law at a wall is called specular reflection, when $\alpha_i = \alpha_r$, as shown in Figure 3(a). This reflection law is used when a ray of light hits a mirror, when a standard rubber ball bounces off a wall, or when a billiard ball bounces off one of the sides of a billiard table (assuming no spin is imparted on the ball).

More complicated reflection laws sometimes occur in nature, and we call these non-specular reflection, as shown in Figure 3(b). Here, the reflected angle α_r is a more general

Figure 3. Relationship between the incident angle α_i and reflected angle α_r for specular and non-specular reflection.(a) Specular reflection: $\alpha_r = \alpha_i$; (b) Non-specular reflection: $\alpha_r = f(\alpha_i, \vec{v})$.

function of the incident angle, or even the incoming velocity:

$$\alpha_r = f(\alpha_i, \vec{v}). \tag{11}$$

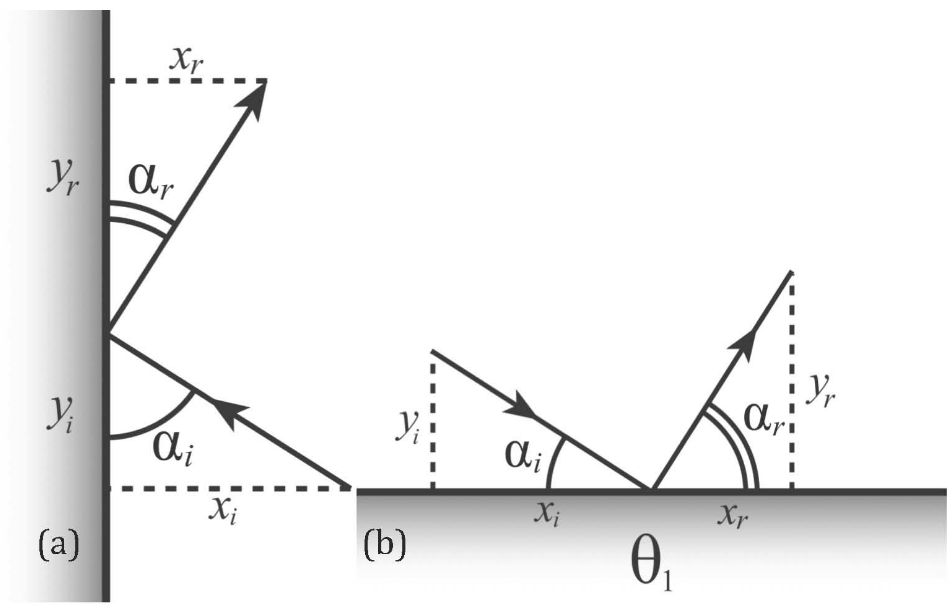


Figure 4. Non-specular reflection off sides for three-beads on a ring. (a) Collision on θ_2 side representing a collision between beads 1 and 2; (b) Collision on θ_1 side representing a collision between beads 2 and 3.

What are the appropriate reflection laws corresponding to bead collisions? For this, we refer to Figure 4 and consider reflections off each of the three legs of the right triangle. In Figure 4(a) we consider the collision between beads 1 and 2 on the θ_2 wall. Using trigonometry, we have the following:

$$\text{Slope}_{in} = \frac{y_i}{x_i} = 1/\tan(\alpha_i^{(1,2)}) = (v_3 - v_2)/(v_2 - v_1),$$

which yields:

$$\alpha_i^{(1,2)} = \arctan[(v_2 - v_1)/(v_3 - v_2)]. \tag{12}$$

Similarly,

$$\text{Slope}_{out}^{(1,2)} = \frac{y_r}{x_r} = 1/\tan(\alpha_r^{(1,2)}) = (v_3 - v_1)/(v_1 - v_2),$$

which yields:

$$\alpha_r^{(1,2)} = \arctan[(v_1 - v_2)/(v_3 - v_1)]. \tag{13}$$

Using Figure 4(b), by similar reasoning we can derive the angle of incidence and reflection for collisions between beads 2 and 3 on the θ_1 wall:

$$\alpha_i^{(2,3)} = \arctan[(v_3 - v_2)/(v_2 - v_1)], \tag{14}$$

$$\alpha_r^{(2,3)} = \arctan[(v_2 - v_3)/(v_3 - v_1)]. \tag{15}$$

By setting equation (12) equal to equation (13), one can easily prove that only if $v_3 = \frac{1}{2}(v_1 + v_2)$ is the reflection law specular. For general bead velocities, the angle in and out are not the same off the θ_2 wall. Likewise, by setting equation (14) equal to equation (15), one can prove that only if $v_1 = \frac{1}{2}(v_2 + v_3)$ is there specular reflection off the θ_1 wall.

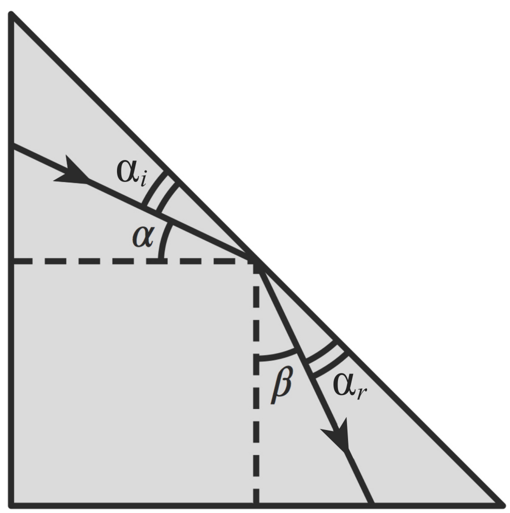


Figure 5. Specular reflection off the hypotenuse for three-beads on a ring. Angle in, α_i equals angle out, α_r.

To see what happens to reflections off the hypotenuse, when beads 1 and 3 collide, consider Figure 5. We know that:

$$\alpha + \alpha_i = \pi/4 = \beta + \alpha_r.$$

Also:

$$\alpha = \arctan(\text{Slope}_{in}) = \arctan((v_3 - v_2)/(v_2 - v_1)),$$

$$\beta = \arctan(1/\text{Slope}_{out}) = \arctan((v_3 - v_2)/(v_2 - v_1)),$$

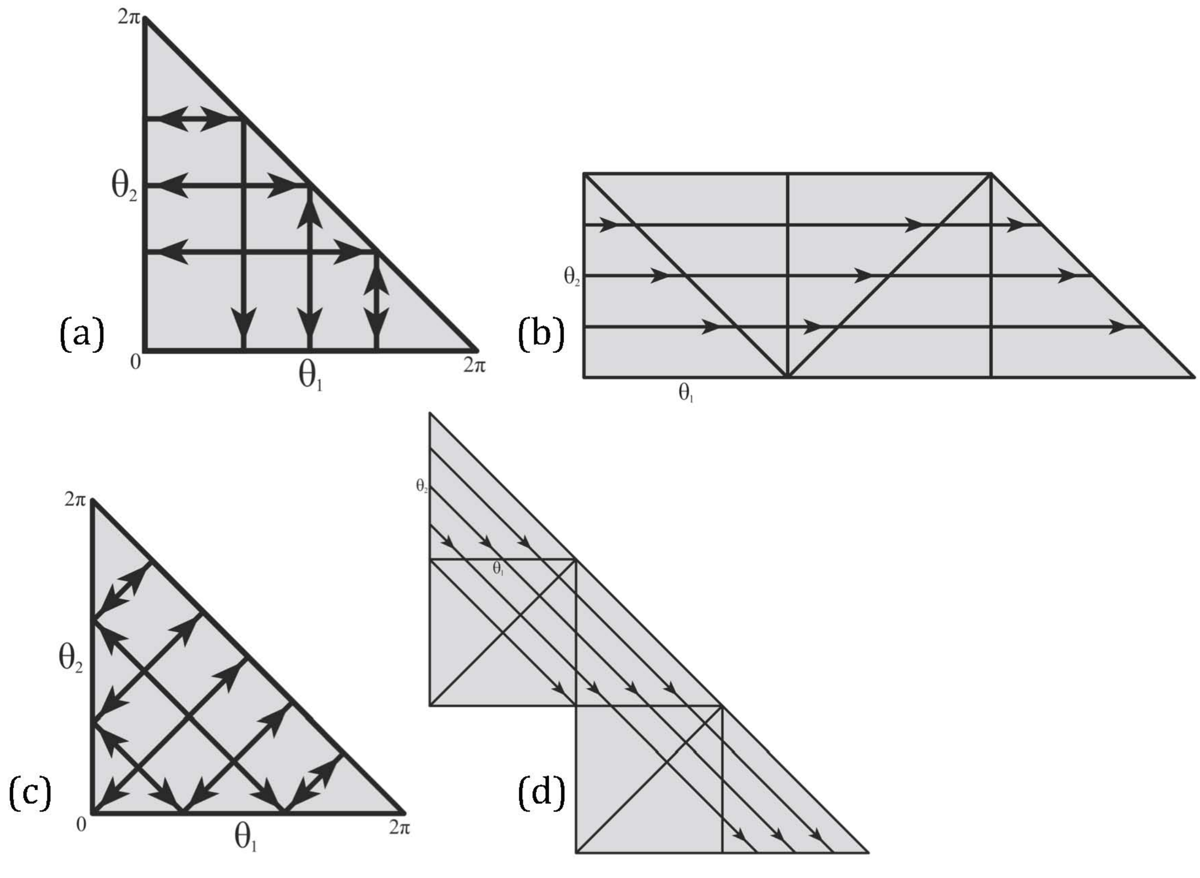
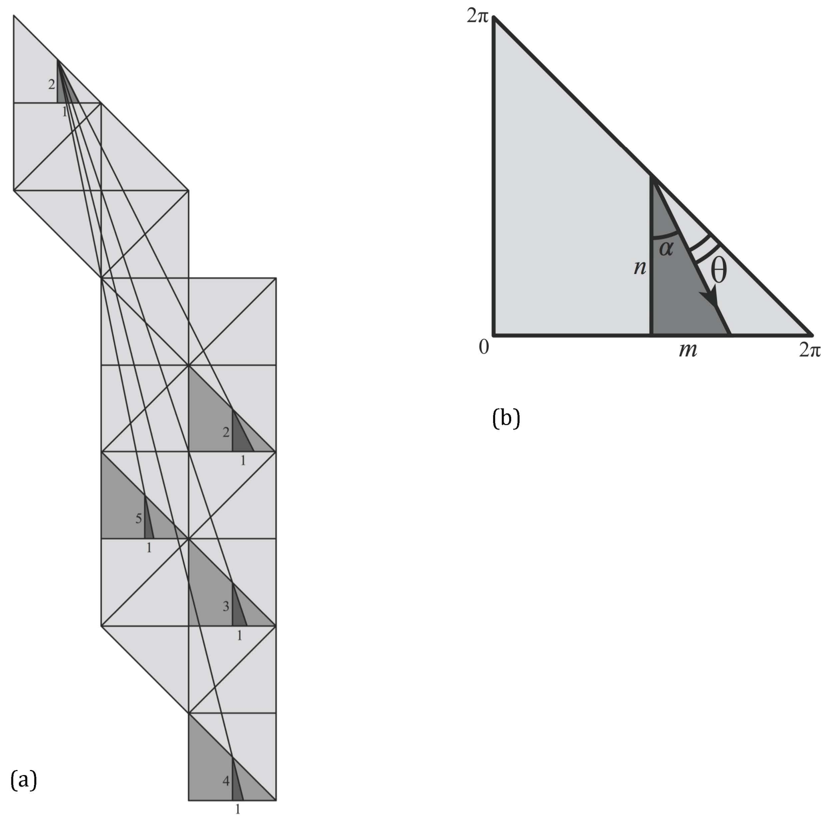


Figure 6. Periodic billiard orbits parallel to a side. (a) Dense family of 4-bounce orbits parallel to a leg; (b) Family of orbits in (a) in the sequence of reflected triangles; (c) Dense family of 6-bounce orbits parallel to the hypotenuse; (d) Family of orbits in (c) in the sequence of reflected triangles.

from which we get that $\alpha = \beta$, hence $\alpha_i = \alpha_r$, proving that reflections off the hypotenuse are always specular, regardless of the velocities.

3. Periodic billiard orbits by the method of reflection

A classical problem with specular reflections off all boundaries is the mathematical problem of billiard dynamics, which has been studied for billiard tables with different shapes, including right triangles [1, 5, 7, 10]. In Cipra et al. [1], a method was described to find periodic trajectories for the billiard problem in a right triangle using a simple method of reflection which we describe here. Consider a trajectory shown in Figure 6(a) that starts parallel to one of the two legs of the right triangle. Upon collision with any of the three sides, in order to enforce the angle in equals angle out specular reflection law, simply reflect the triangle across the boundary, as shown in Figure 6(b). Each collision with a subsequent boundary requires the same reflection, with the overall collection of line segments forming a straight line. When the sequence of reflections are such that the straight line trajectory reaches the same point on the boundary on which it started, with the same slope (guaranteed because the global trajectory is a straight line), and with the same orientation, the trajectory is periodic. Figures 6(a), 6(b) depict a family of periodic trajectories that are parallel to the θ_1 side, undergo a sequence of 4 bounces before returning to their original location. We call these trajectories 4-bounce orbits. By construction, it is also easy to see that one of these periodic orbits emanates from any point on the θ_2 side, as long as it is parallel to the θ_1 side. This implies that there is a dense family of initial conditions associated with these 4-bounce trajectories since the initial point could take on any real value between 0 and 2π. Figure 6(c), 6(d) depict a dense family of 6-bounce orbits in the triangle that are parallel to the hypotenuse. The one special (non-dense) orbit is the bottom one shown in Figure 6(d) which begins at the bottom left corner of the triangle. This special 2-bounce orbit is associated with an isolated initial condition.



(a)

(b)

Figure 7. Periodic billiard orbits emanating from the center of the hypotenuse ($\theta_1 = \theta_2 = \pi$) from the hypotenuse at angle θ. (a) The orbits are characterized by the small right triangle whose ratio of sides determine the angle θ; (b) Small right triangle insert inside initial triangle, defining θ, α and sides with integer ratios n and m.

More complex periodic billiard orbits that are not parallel to a side can be constructed as well. Figures 7(a), 7(b) show how to construct orbits that emanate from the center of the hypotenuse at angle θ. Figure 7(b) depicts the beginning leg of the trajectory at angle θ, forming a right triangle with sides $n : m$, where n and m are integers. We show in this diagram the 2:1 orbit (10 bounces), 3:1 orbit (12 bounces), 4:1 orbit (18 bounces), and 5:1

orbit (16 bounces). We can see from this construction that a periodic orbit exists for any choice of integers n and m. Table 1 summarizes properties of these orbits, along with the 6:1 orbit, 4:3 orbit, and the associated discrete angles θ obtained using the fact that:

$$\theta = \pi/4 - \alpha, \quad \alpha = \arctan(n/m).$$

n	m	Bounces	θ (deg)
1	2	10	18.4349
1	3	12	26.5651
1	4	18	30.9638
1	5	16	33.6901
1	6	26	35.5377
3	4	22	8.1301

Table 1. Periodic billiard trajectories associated with Figure 7. $n : m$ are the integers shown in Figure 7(b), while $\theta = \pi/4 - \arctan(n/m)$.

4. Periodic orbits of three beads on a ring

We now construct periodic orbits of the three bead problem using the boundary reflection laws derived in section II. The simplest family of solutions are shown in Figure 8, all bouncing twice before repeating. Figure 8(a) shows the three separate isolated families in the right triangle, oscillating back and forth from one of the corners (all three beads colliding) to the opposite side. Figure 8(b) depicts the orbit that hits the hypotenuse, and since collisions with the hypotenuse use the specular reflection law, we can use the method of reflection, as shown in the figure. A schematic diagram of the three separate solutions on the ring are shown in Figure 8(c). Figure 9 depicts a (dense) family of 6-collision orbits. As shown in Figure 9(b), after 3 collisions, the beads are in the same relative positions, with the same relative velocities as their initial state, but the configuration is rotated by π from its initial state. This represents one cycle around the loop in Figure 9(a). It requires traversing the cycle one more time around to return to the initial configuration without the π phase shift. Figure 10 depicts a (dense) family of 6-collision orbits with a dense family of line segments in the right triangle parallel to one of the sides.

5. Three runners on a track

As long as the masses are equal, the three beads are indistinguishable. Since they exchange velocities upon collision, we can view the beads as *passing through each other* upon collision. This allows us to view the problem as three runners on a track running with constant velocities, where each bead collision represents two of the runners passing each other (never mind that the runners are all in the same lane, can run in opposite directions, manage to avoid each other, and never get tired!). Consider a track of length L. Runner 1 completes one lap in time T_1 (the fundamental period), given by $T_1 = L/|v_1|$. She completes n laps in time nT_1, where n is any integer. Now imagine we take a snapshot of the runner's positions at the end of each period $t = T_1, 2T_1, 3T_1, \ldots$. For the system

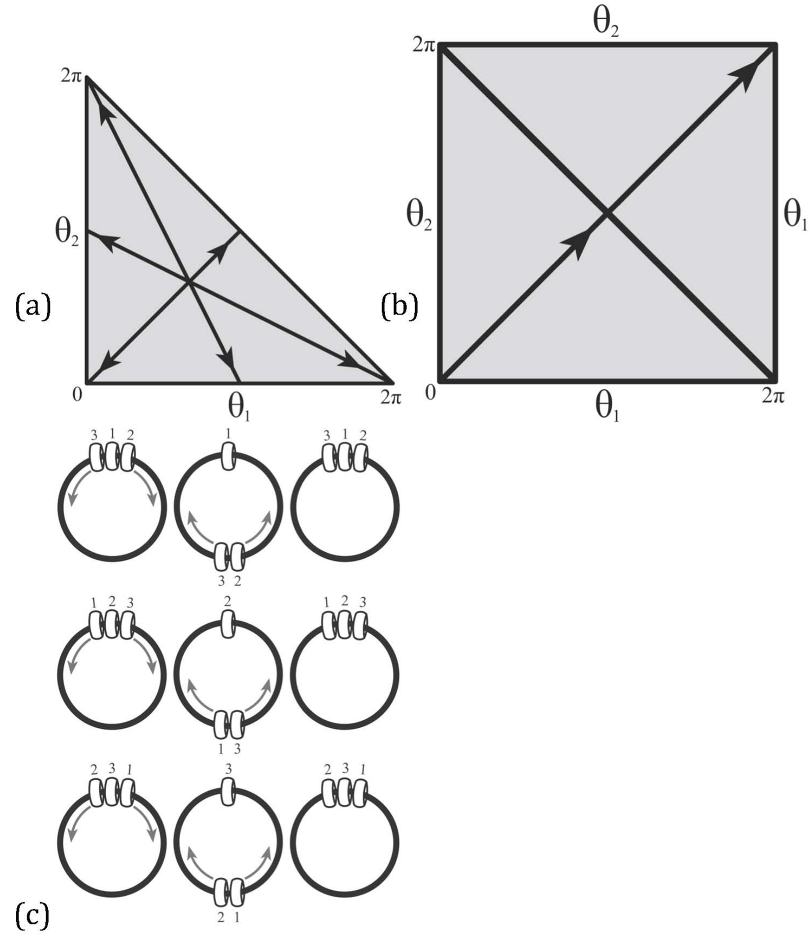


Figure 8. Periodic 2-bounce bead orbits and schematic collision diagrams. (a) The three different 2-bounce orbits in the triangle; (b) The 2-bounce orbit in the reflected triangle; (c) Schematic diagrams of the three bead 2-bounce orbits on the ring.

to be periodic, each of the other two runners must be at their original starting points as well. So, let T_2 be the fundamental period of runner 2, and T_3 the fundamental period of runner 3. In order for the system to be periodic, nT_1 must equal mT_2, and pT_3 for three integers (n, m, p). If this condition holds, the system is periodic. This condition can just as easily be written as a relation among the three velocity ratios of the runners:

$$\frac{v_1}{v_2} = \frac{n}{m}; \quad \frac{v_1}{v_3} = \frac{n}{p}. \tag{16}$$

It is also straightforward to see that if condition (16) is *not* met for any integer triplets, then the three runners will never be in their starting positions at the same exact time, so the system cannot be periodic. Thus, condition (16) is a necessary and sufficient condition for periodicity. Another way to state this condition is that the three velocities must be *rational* multiples of each other, since every rational number can be written as a ratio of integers. Since the slope of the trajectory, as shown in (7), is a ratio of the velocity

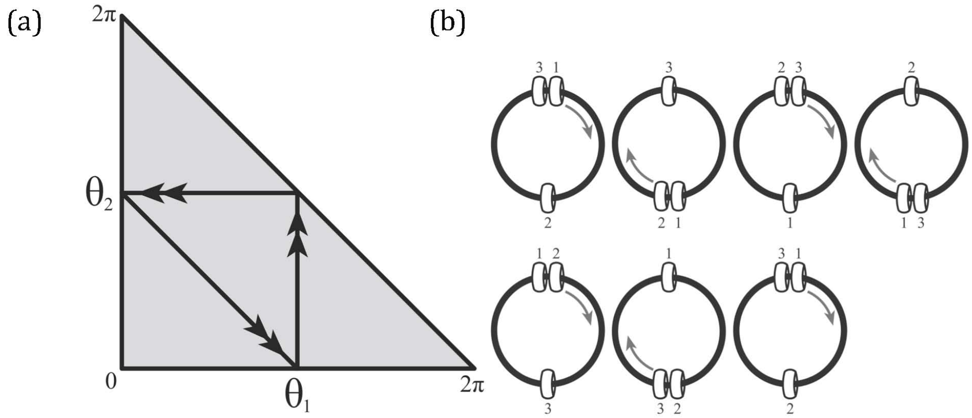


Figure 9. Periodic 6-bounce orbit sequence. (a) The orbit repeats after 3-bounces, but the beads are rotated by angle π from their original configuration; (b) After three more bounces (two loops around the inner triangle), the original configuration is back to where it started, with no change in orientation.

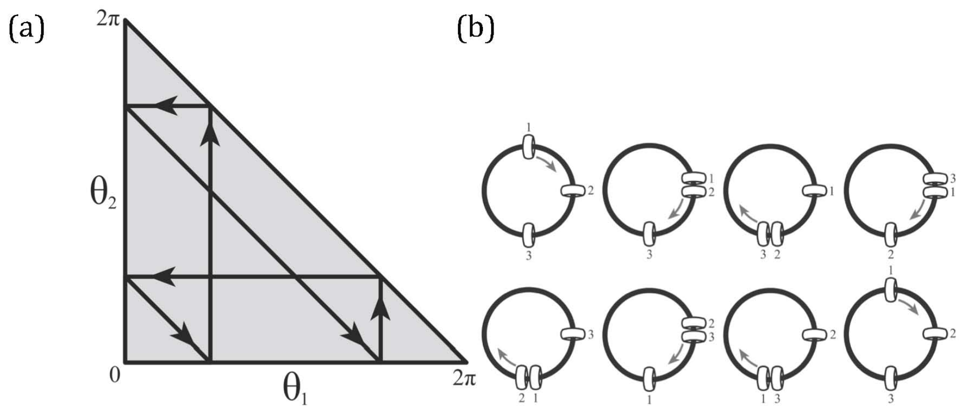


Figure 10. Periodic 6-bounce bead orbits and schematic collision diagrams. (a) Dense family of orbits parallel to the triangle sides; (b) Schematic sequence of collisions on the ring.

differences, this condition implies that the slope must also be rational:

$$\text{Slope}_{in} = \frac{p-m}{m-n}.$$

The corresponding rational slopes after collision can also be obtained using (16) along with the formulas (8), (9), (10). A basic fact about rational and irrational numbers (any number that cannot be written as a ratio of integers) is that both are *dense* on the real line. Between any two rational numbers is another rational number, and between any two irrational numbers is another irrational number. One important difference between rational numbers and irrational numbers is that the rationals are *countable* and dense, while the irrationals are *uncountable* and dense. All of these statements then directly apply to periodic (rational velocity multiples) and non-periodic (irrational velocity multiples) solutions to three (or any number) equal mass beads colliding elastically on a ring. Since there are many more irrational numbers (uncountably infinite) than rational (countably infinite), it follows that there are many more non-periodic orbits than periodic ones.

6. Method of reflections and rotation

Generalizing the method of reflection for the billiards problem, one can develop a similar method to construct periodic orbits for the three-bead collision problem. Consider the diagram in Figure 11 for an orbit starting at point A. When a trajectory hits one of the two legs of triangle, it is necessary both to reflect and rotate the new triangle in order to continue the trajectory as a straight line. By reflecting the triangle across the side, one implements the specular reflection law appropriate for the billiard problem. To adjust so that the non-specular law is enforced, it is necessary to also rotate the triangle around the collision point by angle $\alpha_r - \alpha_i$ where α_i is the incident angle, and α_r is the reflected angle given by formula (13) or (15) depending on which of the two legs the collision occurs. With the reflected and rotated triangle, the trajectory ends at point B at the same location on the hypotenuse as in the original triangle. The trajectory can then be continued indefinitely as a straight line, reflecting and rotating upon each boundary collision.

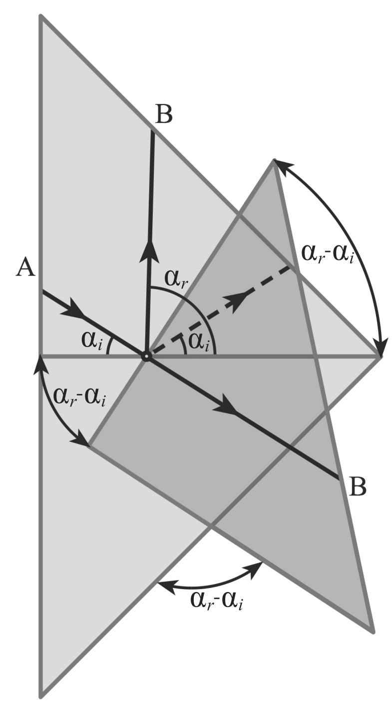


Figure 11. Method of reflection and rotation for the bead collisions. The orbit is drawn as a straight line with two segments. The second segment is drawn in a reflected and rotated triangle, where the reflected triangle is about the collision side, just like the billiard problem. However since the reflection is non-specular, the triangle must be rotated by angle $\alpha_r - \alpha_i$ to compensate for the difference between the specular reflection and the non-specular reflection.

7. Discussion

The method of reflection and rotation described in this note could be implemented for any non-specular reflection law, as long as the law is specified as a functional relationship as in equation (11). The rotation angle would simply be the appropriate angle α_r associated with that functional relation, minus the incident angle α_i since that was accounted for using the reflection step. One can imagine generalizing the billiard problem described in this paper to finding periodic orbits in arbitrary N-gon domains, using the method of reflection about a side. Modeling more complicated effects at collisions, such as imparting spin on the billiard ball, could be handled by introducing an appropriate rotation of the N-gon around the collision point after the reflection step. Inelastic collisions have also been studied in Cooley et al. [9] and Grossman et al. [4] and methods could be generalized to handle those as well, but the equations become more complicated and are not as simple to describe with the high school level mathematics used in this note.

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