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Abstract. We study a variant of Chomp which we call a Diet Chomp, where the total number of squares allowed to be removed is limited. We also discuss other games, which can be considered as stepping stones from Nim to Chomp.

1. Introduction and Background

We study impartial games with two players where the same moves are available to both players, see [1, 2]. Players alternate moves. In a *normal* play, the person who does not have a move loses. In a *misère* play, the person who makes the last move loses.

Example 1.1. Consider a game played on a heap of tokens, where a player can take any number of tokens from the heap. The first player wins under normal play by taking all the tokens. Moreover, the first player wins under misère play if there is more than one token in the heap: the player takes all but one tokens.

A $\mathcal{P}$-*position* is a position from which the *previous* player wins, assuming perfect play. An $\mathcal{N}$-*position* is a position from which the *next* player wins given perfect play. When we play we want to end our move with a $\mathcal{P}$-position and want to see an $\mathcal{N}$-position before our move. A *terminal position* is a position where neither player can move. We can deduce

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that terminal positions are $\mathcal{P}$-positions under normal play and $\mathcal{N}$-positions under misère play.

In our Example 1.1, the only $\mathcal{P}$-position under normal play is the terminal position. The only $\mathcal{P}$-position under misère play is a heap with one token.

1.1. **Introduction to Nim.** Many introductions to combinatorial game theory start with the game of Nim, which is played on several heaps of tokens. A move consists of taking some tokens from one of the heaps. In the terminal position, all the tokens are taken. Our Example 1.1 was the game of Nim on one heap. There is a natural definition of a *sum* of games, where Nim on several heaps is a sum of games of Nim on one heap [2].

Nim is important because every impartial game is equivalent to playing a game of Nim with one heap of a certain size. Thus, every game can be assigned a non-negative integer—the size of the heap—called *a nimber*, *nim-value*, or *a Grundy number*. For example, the game of Nim itself, even when played with many heaps, is equivalent to playing the game of Nim with one heap.

In many cases researchers find the $\mathcal{P}$-positions first, then solve the game rigorously. The proof is often easy when the positions are known. It is enough to show that any move from a non-terminal $\mathcal{P}$-position leads to an $\mathcal{N}$-position and that from any $\mathcal{N}$-position there exists a move to a $\mathcal{P}$-position.

1.2. **Introduction to Chomp.** The game of Chomp is played on a rectangular m by n chocolate bar with grid lines dividing the bar into mn squares. A move consists of chomping a square out of the chocolate bar along with all the squares to the right and above the chosen piece. The player eats the chomped squares. Players alternate moves. The lower left square is poisoned. The player who is forced to chomp it has to eat it. Such a player loses while dying a slowly and painful death.

The game of Chomp is a misère game. One could make it a normal play, if the lower left square is not poisoned. Such a game is not particularly interesting, as the first player can just eat the whole bar and win.

The game of Chomp is not completely solved [5], but the first player wins (in a non-trivial game when $mn > 1$). This can be proven by a *strategy-stealing argument*. Suppose that in the first move the first player chomps only the top right square. If the second player has a winning response to this, then the same move is also a legal first move for the first player. The first player can 'steal' this move and win the game.

We study a new variant of Chomp which we call Diet Chomp.

Definition 1.2 (Diet Chomp)**.** In this Chomp variant, players are not allowed to eat too much chocolate in one move. That is, the number of squares that can be removed in one move is restricted by a parameter k. The players are allowed to make a move the same way as in the game of Chomp with a condition that they can only Chomp away not more than k small chocolate squares at a time. When k is given, we call this variant k-Diet Chomp. Unlike regular Chomp, the normal play becomes interesting here.

To start, we review known facts about subtraction games. We also discuss several known games that serve as a bridge between Nim and Chomp.

2. Nim, Subtraction($[k]$) and Poker-Nim

In the game of Nim there are several heaps of tokens. The players are allowed to take any number of tokens from a single heap. The solution to Nim is well known, and we describe it next [3, 2, 1].

Suppose $A = (a_1, a_2, \dots, a_n)$ is a position in this game: a_i is the number of tokens in the i-th heap. Let us denote the XOR operation as $\oplus$. XORing a_i is done by representing a_i in binary, then summing them without carry, then representing the result as a decimal. The following theorem is true.

Theorem 2.1. *For normal play* Nim, *the Grundy value of a position* $(a_1, a_2, \dots, a_n)$ *is*

$$a_1 \oplus a_2 \oplus \dots \oplus a_n.$$

The $\mathcal{P}$-positions correspond to Grundy value zero.

Corollary 2.2. *A* $\mathcal{P}$*-position in normal play* Nim *satisfies:*

$$a_1 \oplus a_2 \oplus \dots \oplus a_n = 0.$$

Similarly, the $\mathcal{P}$-positions for misère play are known [2]:

Theorem 2.3. *For the misère play if* $\max a_i > 1$, *a* $\mathcal{P}$*-position satisfies:*

$$a_1 \oplus a_2 \oplus \dots \oplus a_n = 0,$$

otherwise:

$$a_1 \oplus a_2 \oplus \dots \oplus a_n = 1.$$

A *subtraction game*, denoted Subtraction(S), is played with heaps of tokens and depends on a set S of positive integers. A move is defined by choosing a heap and removing any number of tokens, such that this number is in set S. If S equals a set of all natural numbers, the resulting subtraction game is Nim. That is, subtraction games are a natural generalization of Nim.

The subtraction games are well-studied [2, 1], and we restrict ourselves to the case when S is equal to $[k]$, where the latter denotes the range of integers from 1 to k inclusive: $[k] = \{1, 2, \dots, k\}$. We might call Subtraction($[k]$) game the k-Diet Nim game in an analogy to k-Diet Chomp. We will not use this name in this paper as our tokens, sadly, are not made of chocolate.

Example 2.4. Consider Subtraction($[k]$) under normal play on one heap. The $\mathcal{P}$-positions are the heaps of sizes that are multiples of $k+1$. Indeed, when a losing player is given a position which is a multiple of $k+1$, whatever they do the result is not a multiple of $k+1$. When a winning player is given a position that is not a multiple of $k+1$, they can always move to a multiple of $k+1$. As the terminal position is also a multiple of $k+1$, this strategy works. Under misère play the one-heap game has $\mathcal{P}$-positions equal to 1 plus a multiple of $k+1$.

The Grundy values and $\mathcal{P}$-positions for Subtraction($[k]$) game are known [2, 1].

Theorem 2.5. *For* Subtraction*($[k]$) normal play, the Grundy value for a position $(a_1, a_2, \dots, a_n)$ is*

$$(a_1 \pmod{k+1}) \oplus (a_2 \pmod{k+1}) \oplus \cdots \oplus (a_n \pmod{k+1}).$$

Therefore the $\mathcal{P}$-positions are such that

$$(a_1 \pmod{k+1}) \oplus (a_2 \pmod{k+1}) \oplus \cdots \oplus (a_n \pmod{k+1}) = 0.$$

We are mostly interested in Subtraction($[2]$).

Example 2.6. Consider a position $A = (a_1, \dots, a_n)$ in Subtraction($[2]$) under normal play. It is a $\mathcal{P}$-position if and only if $A = (a_1, \dots, a_n) \pmod 3$ has an even number of ones as well as an even number of twos.

Theorem 2.7. *The $\mathcal{P}$-positions in misère* Subtraction*($[k]$) considered modulo $k+1$ are:*

- *If there is a heap that is more than 1, then XOR is zero.*
- *If every heap is zero or one, then there is an odd number of ones, that is, XOR is 1.*

Example 2.8. Consider a position $A = (a_1, \dots, a_n)$ in Subtraction($[2]$) under misère play. It is a $\mathcal{P}$-position if and only if $A = (a_1, \dots, a_n) \pmod 3$ either has no twos and an odd number of ones, or it has an even number of ones as well as an even number of twos.

Example 2.9. Consider a two-heap Subtraction($[2]$) under misère play. In this case, the $\mathcal{P}$-positions modulo 3 are $(0,1)$, $(1,0)$, and $(2,2)$. Notice that these are exactly the positions that have the total number of tokens equal to 1 plus a multiple of 3. Hence we can describe the $\mathcal{P}$-positions the same way for both one heap and two heaps Subtraction($[2]$): $\mathcal{P}$-positions have 1 plus a multiple of 3 tokens.

In the next variant of Nim we want to allow the players to put tokens back into a heap. We start with several heaps of tokens. We allow two types of moves. A player can take any number of tokens from a heap or add any number of tokens to a heap. To prevent the players from being tired of an infinite number of moves, one should add constraints on putting tokens back. For example, Poker Nim has an additional bag of tokens; and only the tokens from the bag can be used to increase the size of a heap [1]. This way an infinite loop in the game is prevented by limiting the total number of tokens that can be put back during the game.

As with other games we want to put Poker Nim on a diet. We consider the game k-Poker Nim which is like Subtraction($[k]$) where, in addition, the players are allowed to put up to k tokens back into any one of the heaps in one move, given that the total number of tokens that are put back during the game is limited by some number.

Most books leave it as an exercise to the reader to show that Poker Nim has the same Grundy values as Nim. Similarly, k-Poker Nim has the same Grundy values as Subtraction($[k]$). We decided to follow the tradition and leave the next theorem without a proof.

Theorem 2.10. *Poker games have the same $\mathcal{P}$-positions as the non-poker equivalents and the same Grundy values.*

3. Monotonic Games

We can define a position A in Chomp as $(a_1, a_2, \ldots, a_n)$, where a_i is the number of chocolate squares in the i-th row from the top. The rules of Chomp force the sequence a_i to be non-decreasing.

To build a bridge from Nim to Chomp, we consider Nim variants where a position $A = (a_1, a_2, \ldots, a_n)$ is allowed only if the sequence is non-decreasing, that is $a_i \le a_{i+1}$, for $1 \le i < n$. We call such games *monotonic*. For example, Monotonic Nim is a monotonic game where we can take any number of tokens from one heap, given that the resulting sequence is still non-decreasing.

If, in addition, we put a limit of k on the total number of tokens that can be taken, we get a game that we call Monotonic Subtraction($[k]$). A player can subtract any number of tokens between 1 and k inclusive from one heap, given that the resulting sequence is non-decreasing.

Monotonic Nim is similar to the Silver Dollar without the Dollar game [2], also called Sliding [1]. The Silver Dollar without the Dollar game is played on a strip of squares numbered 1, 2, ..., n. Coins are put on some squares, at most one coin per square. On a move, a player can slide a coin to a lower-numbered square. The coins cannot jump over or collide with other coins. As usual, the last player to be able to make a legal move wins under normal play, and loses under misère play. Positions $A = (a_1, a_2, \ldots, a_n)$ in Silver Dollar without the Dollar game are strictly increasing: $a_i < a_{i+1}$, for $1 \le i < n$. The name of this game, Silver Dollar without the Dollar, sounds weird because there exists a game called the Silver Dollar game [2]. The Silver Dollar game is related to the Silver Dollar without the Dollar game, but is out of the scope of our discussion.

We are interested in monotonic games because their legal positions match legal positions in Chomp. We describe the Grundy values for Monotonic Nim and Monotonic Subtraction($[k]$) by using the same method that is used in solving the Silver Dollar game [1, 2].

Suppose we have a position $A = (a_1, a_2, \ldots, a_{2n})$ with an even number of heaps. We map it to a vector $B = (b_1, b_2, \ldots, b_n)$, where $b_i = a_{2i} - a_{2i-1}$. For a position with an odd number of heaps we first extend it to a position with an even number of heaps, by adding a zero heap in front. We call the vector B the *difference vector*.

Theorem 3.1. *Playing a monotonic game with a starting position A is equivalent to playing the matching poker game starting on the A's difference vector.*

Proof. We start with a non-terminal position $A = (a_1, a_2, \ldots, a_{2n})$ that is mapped to a vector of n non-negative integers $B = (b_1, b_2, \ldots, b_n)$. Suppose in our move we take x tokens from an even-numbered heap $2i$, where $1 \le x \le a_{2i} - a_{2i-1}$. The resulting position is mapped to B with x subtracted from $b_i = a_{2i} - a_{2i-1}$. Suppose in our move we take x tokens from an odd-numbered heap $2i-1$, where $1 \le x \le a_{2i-1} - a_{2i-2}$. The resulting position is mapped to B with x added to $b_i = a_{2i} - a_{2i-1}$.

If we look at changes in the difference vector we see that we are playing Poker Nim on it. The tokens that we add in this Poker Nim match the tokens that are removed from an odd-numbered heap in the monotonic game. That means the total is bounded by $\sum_{i=1}^{n} a_{2i-1}$.

Now we consider a Poker Nim move in the difference position B and see if we can find a matching move in the monotonic game in A. Suppose in our move we take x tokens from heap i in B. This is equivalent to taking x tokens from heap $2i$ in A. As $1 \leq x \leq b_i$, it follows that the corresponding move in A is legal, that is, the resulting position is monotonic. Suppose in our move we add x tokens to heap i in B, where $x \leq \sum_{i=1}^{n} a_{2i-1}$. If $x \leq a_{2i-1} - a_{2i-2}$, then there is a matching legal move from a position A that adds x tokens to heap $2i-1$. If $x > a_{2i-1} - a_{2i-2}$, then there is no such move. That means the monotonic game is not exactly equivalent to the corresponding Poker Nim. It is almost equivalent: we have to add additional constraints on how many tokens we can add to a particular heap at a particular time in the game. The good news is that these extra constraints do not change the analysis of these variants of Poker Nim. In particular, the Grundy values of a position in such variants of Poker Nim are the same as in Nim.

Therefore, the Grundy value of a position A in a monotonic game is the same as the Grundy value of the difference vector, considered as a position in Nim. □

Notice that the theorem establishes a correspondence between the moves in a monotonic game and the moves in a restricted version of a corresponding Poker Nim. Therefore, the theorem holds for both normal and misère plays.

Example 3.2. In a two-heap Monotonic Subtraction([2]) under misère play, a position (a_1, a_2) is a $\mathcal{P}$-position, if and only if $a_2 - a_1 = 3k+1$, for $k \geq 0$. In a three-heap Monotonic Subtraction([2]) under misère play, a position (a_1, a_2, a_3) is a $\mathcal{P}$-position, if and only if $a_3 - a_2 + a_1 = 3k+1$, for $k \geq 0$.

2-Diet Chomp under Normal Play

Now we move to Chomp for health-conscious players. Namely, we study a variant of Chomp where a player makes a Chomp move that is limited to one or two chocolate squares. The positions in our game are $A = (a_1, a_2, \ldots, a_n)$, so that the sequence is non-decreasing: $a_i \leq a_{i+1}$, where $1 \leq i < n$. For convenience, we assume that $a_0 = 0$. As before, a_i corresponds to the number of chocolate squares in the i-th row from the top.

In one move we are allowed to:

- subtract 1 from a_i if $a_i > a_{i-1}$.
- subtract 2 from a_i if $a_i > a_{i-1} + 1$.
- subtract 1 from a_i and a_{i+1} if $a_{i+1} = a_i > a_{i-1}$.

For 2-Diet Chomp, the $\mathcal{P}$-positions depend on the total number of tokens.

Lemma 3.3. *For 2-Diet Chomp, the $\mathcal{P}$-positions are such that the total number of tokens $\left(\sum_{i=1}^{n} a_i\right)$ is divisible by 3.*

Proof. The terminal position, $(0,0,\ldots,0)$, is a $\mathcal{P}$-position. $\mathcal{P}$-positions differ by multiples of 3, therefore there is no move from a $\mathcal{P}$-position to a $\mathcal{P}$-position. What is left to show is that every $\mathcal{N}$-position has a move to a $\mathcal{P}$-position.

Suppose $\left(\sum_{i=1}^{n} a_i\right) \equiv 1 \mod 3$. We can always remove one square, so it moves to a $\mathcal{P}$-position. If $\left(\sum_{i=1}^{n} a_i\right) \equiv 2 \mod 3$, removing two squares moves it to a $\mathcal{P}$-position, except there could be a position such that there is no valid move that removes two squares.

The only positions for which it is not allowed to remove two squares are "perfect stairs" positions: $(1,2,\ldots,n)$. However, the total number of tokens in such a position is a triangular number $T_n = \sum_{i=1}^{n} i = \binom{n}{2}$. It is widely known, or one might check it as an exercise, that triangular numbers do not have remainder 2 modulo 3. That means we can always move from an $\mathcal{N}$-position to a $\mathcal{P}$-position. □

Interestingly, in this case the game is equivalent to playing Subtraction([2]) under normal play on one heap with the same total of tokens.

2-Diet Chomp under Misère play

This game becomes more difficult under misère play than under normal play.

We can explicitly describe $\mathcal{P}$-positions under misère play for narrow rectangles.

Lemma 3.4. *For $1\times n$ rectangles, the $\mathcal{P}$-positions are $3k+1$. For $2\times n$ rectangles, the $\mathcal{P}$-positions are $(a, a+3k+1)$, where $k \geq 0$.*

Proof. For $1 \times n$ rectangles, the game is equivalent to Subtraction([2]) on one heap, also under misère play.

For $2 \times n$ rectangles, we prove the lemma by showing that all moves from a $\mathcal{P}$-position go to an $\mathcal{N}$-position, and from any $\mathcal{N}$-position there exists a move to a $\mathcal{P}$-position.

Notice that we cannot have a move that changes both values from a $\mathcal{P}$-position. By subtracting 1 or 2 from each coordinate we change the difference modulo 3. That means every move from a $\mathcal{P}$-position goes to an $\mathcal{N}$-position.

On the other hand, from an $\mathcal{N}$-position $(a, a+3k+2)$, we can move to $(a, a+3k+1)$, which is a $\mathcal{P}$-position. From an $\mathcal{N}$-position $(a, a+3k)$, where $a > 0$, we can move to $(a-1, a+3k) = (a-1, (a-1)+3k+1)$, which is a $\mathcal{P}$-position. From an $\mathcal{N}$-position $(0, 3k)$, where $k > 0$, we can move to $(0, 3k-2) = (0, 3(k-1)+1)$, which is a $\mathcal{P}$-position.

Additionally, $(0,1)$ is a $\mathcal{P}$-position, which completes the proof. □

We are fascinated by the fact that for $2 \times n$ rectangles the game is equivalent to playing Monotonic Subtraction([2]) on two heaps with the same total of tokens.

For $3\times n$ rectangles, the situation is more complicated. We wrote a program and observed that $\mathcal{P}$-positions are periodic with period 12. That is, position (a_1, a_2, a_3) is the same type as (a_1+12, a_2+12, a_3+12). We know that positions in layer $a_1 = k$ depends only on positions in layers $a_1 = k-1$ and $a_1 = k-2$, because a move removes two chocolate squares at the most. Therefore, if the two layers $a_1 = 12$ and 13 are the same as layers $a_1 = 0$ and $a_1 = 1$ respectively, then layer $a_1 = k$ is the same as layer $a_1 = k-12$ for any $k \geq 12$.

Figure 1 shows $\mathcal{P}$-positions for values a_1 ranging from 0 to 11 inclusive. The i-th diagram, going in order left to right top to bottom, describes positions $(i-1, a_2, a_3)$. The bottom left corner of the i-th diagram corresponds to $(i-1, i-1, i-1)$. Given that $a_3 \geq a_2$, the diagrams have triangular shapes. In addition, the value of $a_3 - a_1$ is determined by the column, and the value of $a_2 - a_1$ is determined by the row. Also note that $a_3 - a_2$ is constant along NE diagonals. The $\mathcal{P}$-positions are black, while the $\mathcal{N}$-positions are gray.

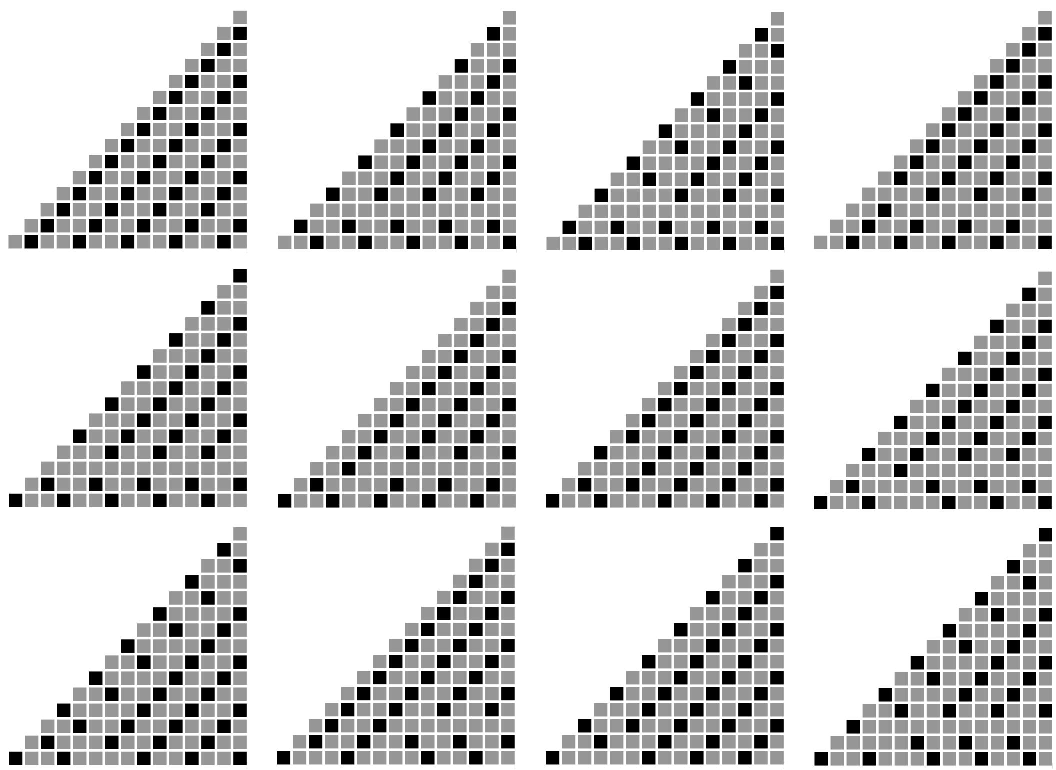

Figure 1. $\mathcal{P}$-positions for 2-Diet Chomp with 3 rows and a_1 ranging from 0 to 11

We can make the following observation from these pictures:

- If we remove three bottom rows and three top NE diagonals, all the pictures have the same pattern. More precisely, for $a_2 - a_1 \geq 3$ and $a_3 - a_2 \geq 3$ the $\mathcal{P}$-positions correspond to values $a_1 + a_3 - a_2 \equiv 1 \pmod 3$.
- Each of the three bottom rows eventually becomes periodic too with period either 3 or 1. In particular, for $a_2 - a_1 < 3$, for positions with $a_2 + a_3 - a_1 > 7$, the position (a_1, a_2, a_3) is the same type as $(a_1, a_2, a_3 + 3)$.
- Each of the three top diagonals going NE becomes periodic with period either 2 or 1. In particular, for $a_3 - a_2 < 3$, for positions with $a_3 - a_1 > 7$, the position (a_1, a_2, a_3) is the same type as $(a_1, a_2 + 2, a_3 + 2)$.

We also use the function of a position: $d(A) = a_2 + a_3 - a_1$ in our future discussion. Notice that if $d(A) > 7$, the position A belongs to one of the three regions above that follow a periodic pattern. We call the region, where $d(A) > 7$, the *periodic region*.

Given a list of $\mathcal{P}$-positions, to prove that the list is correct, one needs to check that any move from a $\mathcal{P}$-position goes to an $\mathcal{N}$-position, and from any $\mathcal{N}$-position, there exists a move to a $\mathcal{P}$-position. The next lemma shows that we need to only make a finite number of checks.

Lemma 3.5. *To prove the pattern we need to check the moves from positions in layers 0 through 13, and for positions A such that* $d(A) = a_2 + a_3 - a_1 < 14$.

Proof. Consider a position $A = (a_1, a_2, a_3)$ so that $a_2 + a_3 - a_1 \geq 14$, that is $d(A) \geq 14$. Let us consider positions $A' = (a_1, a_2, a_3 - 3)$ or $A'' = (a_1, a_2 - 2, a_3 - 2)$. Note that if each of these positions exists, it belongs to the periodic region. Namely, $d(A') \geq 11$, and $d(A'') \geq 10$. In any case, at least one of these two positions exists, and is the same type as A.

Let us assume that it is A''. For any move b, the position $A'' - b$ is in the periodic region, because $d(A'' - b) \geq 8$. Therefore, for any move b, positions $A'' - b$ and $A - b$ are the same type. Therefore, the positions A and A'' are the same type. The case of position A' is similar. □

According to the lemma, to prove that the pattern we discovered continues, we need to carry out a finite number of calculations. We performed all these calculations manually and painstakingly, thus proving the following theorem.

Theorem 3.6. *Position* (a_1, a_2, a_3) *is the same type as* $(a_1 + 12, a_2 + 12, a_3 + 12)$. *The positions in the first 12 layers are described by the diagrams in Figure 1.*

We also made a program that calculates the $\mathcal{P}$-positions for up to 5 rows. They show a similar behavior. The diagrams are available online at [4].

To sum up, this game played on $1 \times n$ and $2 \times n$ rectangles is equivalent to Subtraction([2]) with 1 and 2 heaps correspondingly. This game on $3 \times n$ rectangles is more complicated, but still, for positions that correspond to middle areas of the diagram, the game is equivalent to playing Subtraction([2]) under normal play on three heaps with the same total of tokens.

4. Conclusion and Future Research

This research was done as part of MIT PRIMES STEP program. The program allows students in grades seven through nine to do research in mathematics. Tanya Khovanova is the mentor in this program. The program functions like a math club with some portion of the time devoted to research.

For our research topic, the students studied combinatorial game theory. The students reinvented and solved Poker Nim and Silver Dollar Game without the Dollar. The Silver Dollar Game without the Dollar is a natural intermediate game between Nim and Chomp. Poker Nim is needed to solve the Silver Dollar Game without the Dollar.

The main focus of this paper is Diet Chomp. To the best of our knowledge this game was invented by us and never studied before. The game of Chomp is still unsolved and Diet Chomp seems to be more tractable.

The results show that Diet Chomp's $\mathcal{P}$-positions might be eventually periodic. We can argue that Diet Chomp on $k \times n$ rectangles have a finite number of moves. That means the Grundy values are bounded. It is natural to assume that some periodic patterns might emerge.

Our results show that for positions (a_1, a_2, a_3), where the two consecutive numbers differ at least by 3, the results become periodic. One explanation is that such positions are several moves away from positions with the same number of squares in two neighboring rows, that is, from positions that allow moves that are different from subtraction moves. As a result the game becomes similar to the Monotonic Subtraction([2]) game.

We hope that our initial analysis of Diet Chomp will encourage other researchers to explore further.

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Student biographies

June Ahn, Benjamin Chen, Richard Chen, Ezra Erives, Jeremy Fleming, Michael Gerovitch, Tejas Gopalakrishna (*corresponding author:* tejgop@gmail.com), Neil Malur, Nastia Polina, and Poonam Sahoo: The paper was collectively authored by ten participants of the PRIMES STEP program for middle and high school students at the Massachusetts Institute of Technology, and their mentor, Dr. Tanya Khovanova. The students were in the 8th and 9th grades at public schools in Acton, Belmont, Brookline, Lexington, Natick, and Weston, MA, or homeschooled.