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Abstract. For a fixed $b \in \mathbb{Z}^+$, a point $(r,s) \in \mathbb{Z}\times\mathbb{Z}$ is b-visible from the origin if there exists a power function $f(x) = ax^b$ with $a \in \mathbb{Q}$ such that $f(0) = 0$ and $f(r) = s$, and no other point in the integer lattice belongs to the graph of f. In this article, we extend the definition of b-visibility given by Goins, Harris, Kubik, and Mbirika [1] to the study of weak visibility. For a fixed $b \in \mathbb{Z}^+$, we say that a point $Q = (h,k)$ in the array $\Delta_{m,n} = \{1,2,\ldots,m\}\times\{1,2,\ldots,n\}$ is *weakly b-visible* from a point $P = (r,s) \in \mathbb{Z}^+\times\mathbb{Z}^+$ such that $P \notin \Delta_{m,n}$ if no other point in $\Delta_{m,n}$ lies on the curve $f(x) = \frac{(s-k)}{(r-h)^b}(x-h)^b + k$ between Q and P. In this paper we give necessary and sufficient conditions for determining if a point in $\Delta_{m,n}$ is weakly b-visible by an external point. We also show that for any point $P = (r,s)$ with $r > m$ and $s > n$, there exists a $b \geq 1$ such that every point in $\Delta_{m,n}$ is weakly b-visible from P. Our last result considers a fixed $b > 1$ and specifies the coordinates of a point P that weakly b-views every point in $\Delta_{m,n}$, and as a corollary we provide a way to determine the coordinates of the closest point to the array satisfying such a condition. We conclude by providing a few directions for future research.

1. Introduction

We can contextualize results in the study of weak lattice point visibility by imagining a photographer commissioned to take pictures of a marching band. The photographer must be able to take a picture where every band member is visible, and where all members are in an evenly spaced, rectangular formation. To formally describe this, we imagine the band members standing on lattice points in the rectangular array $\Delta_{m,n} = \{1,2,3,\ldots,m\}\times\{1,2,3,\ldots,n\}$ with $m \geq n$, while the photographer stands on a positive integer lattice point somewhere outside of the region encompassed by the band.

Assuming that the photographer's camera only takes pictures linearly (i.e in a straight line), we say that a point, $Q \in \Delta_{m,n}$, is *weakly visible* by $P \notin \Delta_{m,n}$ if there are no other points in $\Delta_{m,n}$ which lie on the line segment connecting Q and P. If all $Q \in \Delta_{m,n}$ are weakly visible from P, then we say that the array $\Delta_{m,n}$ is *weakly visible* from P. Laison and Schick [3] showed that in order to guarantee that $\Delta_{m,n}$ is weakly visible from P, the

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lattice point $P = (r,s)$ must satisfy $r > m$ and $s > n$, but this point might be rather far away from the formation. Nicholson and Sharp [5], further determined that such a lattice point P must lie at least $\sqrt{m^2+1}$ units away from $\Delta_{m,n}$. More recently, Nicholson and Rachan [4] gave necessary and sufficient conditions, depending only on the parameters m and n, such that a particular lattice point $Q \in \Delta_{m,n}$, is weakly visible from the point P . However, giving coordinates to the closest point P from which every point is weakly visible remains unknown.

Of course, not all physical behavior acts along a linear trajectory. In a more general situation, consider our scenario from before, but instead of taking pictures, we wish to throw a Frisbee to one of the band members. In order to avoid tossing the disk to a band member directly obstructing our target we decide that the Frisbee must be thrown with a curved trajectory, which we assume is a polynomial function of a fixed degree.

As before, we think of each member of the band as an integer lattice point in $\Delta_{m,n}$ and we suppose that we are located outside of this rectangular array. Thus, the question of interest is:

> *If the trajectory of the disk follows the trajectory of a polynomial function (with a fixed degree), will we be able to throw the disk to every member of the band without having another band member obstruct our throw?*

This question has been fully analyzed when the person throwing the disk is standing at the origin and the team comprises all integer lattice points in a $\mathbb{Z}^+ \times \mathbb{Z}^+$ array. In particular, Goins, Harris, Kubik and Mbirika [1], and later Harris and Omar [2], determined whether the band member standing at the location (r,s) can receive the disk thrown via a trajectory of the form $f(x) = ax^b$, for a fixed $b \in \mathbb{Z}^+$ and $a \in \mathbb{Q}$. Furthermore, they determined necessary and sufficient conditions for a band member standing at $(r,s) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ to be able to receive the disk (when thrown from the origin) to be solely dependent on the existence or non-existence of an integer $k > 1$ satisfying $k|r$ and $k^b|s$. Lattice points satisfying such a divisibility condition were said to be b-invisible from the origin.

Throughout this manuscript we assume that every integer lattice point lies on the positive quadrant of $\mathbb{Z} \times \mathbb{Z}$. Furthermore, when we talk about the integer lattice, saying that a point (x,y) is between two points, (p,q) and (r,s), we mean that $\min\{p,r\} < x < \max\{p,r\}$ and $\min\{q,s\} < y < \max\{q,s\}$. We recall here the definition of b-visibility of a lattice point, as given in [1].

Definition 1.1. Fix $b \in \mathbb{Z}^+$. A point $(r,s) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ is said to be *b-invisible* with respect to the unique function $f(x) = ax^b$, such that the point (r,s) lies on the graph of $f(x)$, if the following condition holds: $f(x)$ intersects at least one point on the integer lattice between $(0,0)$ and (r,s). The point is said to be *b-visible* if it lies on the graph of $f(x)$, and is not b-invisible.

Proposition 1.2 (Proposition 3 in [1])**.** *A point $(r,s) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ is b-visible iff $\widetilde{\gcd}_b(r,s) = 1$, where $\widetilde{\gcd}_b(r,s) := \max\{k \in \mathbb{Z}^+ \mid k|r$ and $k^b|s\}$.*

Proposition 1.2 says that a lattice point (r,s) is b-visible if and only if there is no number k such that $k > 1$, $k|r$, and $k^b|s$. This is because such a k would give us the point $(\frac{r}{k}, \frac{s}{k^b})$ which lies in the integer lattice between $(0,0)$ and (r,s), and lies on the graph of $f(x)$. Furthermore, suppose there is at least one point that lies on the graph of $f(x)$ between $(0,0)$ and (r,s). Take the smallest x value of any such point, let's say μ. Then we can conclude that $k = \frac{r}{\mu}$ is an integer greater than 1 that divides r and such that $k^b|s$. We know that k is an integer as the prime factorization of μ is equal to that of the denominator of a, and so must also appear in the factorization of r. We also know that k is greater than one because $\mu < r$, and we know that $k|r$ as $k \cdot \mu = r$. To see that $k^b|s$, note that μ is the smallest value capable of canceling the lowest-terms denominator of a, the rational coefficient of our function, $f(x)$. Thus, $a \cdot \mu^b$ is an integer, and $s = a \cdot \mu^b \cdot k^b$, thus $k^b|s$.

In this article, we extend the definition of b-visibility given by Goins et al. to the study of *weak b-visibility.*

Definition 1.3. Fix $b \in \mathbb{Z}^+$. A point $Q = (h,k) \in \Delta_{m,n}$ is said to be *weakly b-visible* from a point $P = (r,s) \notin \Delta_{m,n}$ if the curve $f(x) = \frac{(s-k)}{(r-h)^b}(x-h)^b + k$ exists, and there does not exist $Q' \in \Delta_{m,n}$ $(Q' \neq Q)$ such that Q' lies on the curve $f(x)$ between Q and P. If every $Q \in \Delta_{m,n}$ is weakly b-visible from $P \notin \Delta_{m,n}$, then we say that $\Delta_{m,n}$ is *weakly b-visible* from P or that *P weakly b-views* $\Delta_{m,n}$.

The curve $f(x)$ in Definition 1.3 will exist in all cases except when $r = h$, which is to say, when the point P lies directly above Q. Note as well that by setting $b = 1$ we recover the standard definition of weak visibility.

We continue by describing our results on weak b-visibility. For the following lemma, we define $P - Q := (r-h, s-k)$ to be the point found by subtracting the coordinates of Q from the coordinates of P.

Lemma 1.4. *Fix $b \in \mathbb{Z}^+$, and let $Q = (h,k) \in \Delta_{m,n}$ and $P = (r,s) \notin \Delta_{m,n}$, such that $r > m$ and $s > n$. Then Q is weakly b-visible from P if and only if one of the following holds*

(1) $P - Q$ is b-visible,

(2) $P - Q$ is b-invisible and $\left(\dfrac{r-h}{\ell}, \dfrac{s-k}{\ell^b}\right) \notin \Delta_{m-h,n-k}$ for any $\ell > 1$ satisfying $\ell|(r-h)$ and $\ell^b|(s-k)$.

Moreover, we prove that the behavior of weak b-visibility is rather different from classical weak visibility (the $b = 1$ case). For example, Nicholson and Rachan established that for a point P to be able to weakly view all points in $\Delta_{m,n}$, P must lie outside of a specified region of the plane adjacent to $\Delta_{m,n}$. We restate their result below for ease of reference.

Theorem 1.5 (Theorem 3 in [4]). *The point $Q = (x_0, y_0) \in \Delta_{m,n}$ is not weakly visible by the point $P = (x_1, y_1)$ if and only if all the following hold*

(1) $\gcd(x_1 - x_0, y_1 - y_0) > 1$

(2) $m - x_0 \geq (x_1 - x_0)/\gcd(x_1 - x_0, y_1 - y_0)$

(3) $n - y_0 \geq (y_1 - y_0)/\gcd(x_1 - x_0, y_1 - y_0)$.

In the weakly b-visible setting, the result is rather different.

Theorem 1.6. *Fix $b \in \mathbb{Z}^+$. Then the point $Q = (h,k) \in \Delta_{m,n}$ is not weakly b-visible by the point $P = (r,s)$, with $r > m, s > n$ if and only if all the following hold*

(1) $\widetilde{\gcd}_b(r-h, s-k) > 1$

(2) $m - h \geq (r-h)/\widetilde{\gcd}_b(r-h, s-k)$

(3) $n - k \geq (s-k)/(\widetilde{\gcd}_b(r-h, s-k))^b$

Since $\widetilde{\gcd}_b(r-h, s-k) \leq \gcd(r-h, s-k)$ for $b > 1$, Theorem 1.6 shows that the point P can lie within the adjacency region for any $b > 1$.

Next, we show that for any $P = (r,s)$ with $r > m$ and $s > n$, we can always find a $b > 1$ such that $\Delta_{m,n}$ is weakly b-visible from P.

Theorem 1.7. *Let $P = (r,s)$, where $r > m$ and $s > n$. Then there exists a $b_0 \in \mathbb{Z}^+$ such that $\Delta_{m,n}$ is weakly b-visible by P for all $b \geq b_0$.*

Our last result considers a fixed $b > 1$ and specifies a point P such that $\Delta_{m,n}$ is weakly b-visible by P, and as a corollary we provide a way to determine the coordinates of the closest point to the array satisfying such a condition.

Theorem 1.8. *Fix $b \in \mathbb{Z}^+$ and let $m,n \in \mathbb{Z}^+$ satisfy $m-1 > n+1$. Then the set $\Delta_{m,n}$ is weakly b-visible from the point $P = \left(\left\lceil \sqrt[b]{m^{b+1} - 2m^b - m + 3}\right\rceil, n+1\right)$.*

Our last result describes the location of the point P closest on average to $\Delta_{m,n}$ that weakly b-views all of $\Delta_{m,n}$.

Corollary 1.9. *Fix $m-1 > n+1$ and $b \in \mathbb{Z}^+$ and let $f(x) = \frac{1}{(m^b-1)}(x^b - 1) + 1$. If $m^* > m$ is the smallest positive integer such that $n+1 < f(m^*)$, then $\Delta_{m,n}$ is weakly b-visible from $P = (m^*, n+1)$.*

In Section 2, we establish all of the above results related to weak b-visibility, presenting illustrative examples throughout. We conclude with Section 3, where we provide a few possible directions for future research.

2. Results

We begin our analysis with the proof of Lemma 1.4, which establishes that weakly b-visible and b-visible are non-equivalent notions. In general, there are more weakly b-visible points.

Proof of Lemma 1.4. ($\Rightarrow$) Suppose $Q \in \Delta_{m,n}$ is weakly b-visible from P, and suppose that $P - Q = (r-h, s-k)$ is not b-visible, and hence is b-invisible. Now we assume that there exists an $\ell > 1$ such that $(\frac{r-h}{\ell}, \frac{s-k}{\ell^b}) \in \Delta_{m-h,n-k}$, because if no such ℓ existed, we'd be done with our proof. From this assumption, it follows that $(\frac{r-h}{\ell} + h, \frac{s-k}{\ell^b} + k) \in \Delta_{m,n}$. However, this contradicts that Q is weakly b-visible from P, as this point lies on the function $f(x) = \frac{(s-k)}{(r-h)^b}(x-h)^b + k$ and is a point of $\Delta_{m,n}$ strictly between Q and P.

($\Leftarrow$) If $P - Q = (r-h, s-k)$ is b-visible, then there are no lattice points on the curve $g(x) = \frac{(s-k)}{(r-h)^b}x^b$ lying strictly between $(0,0)$ and $P-Q$. Thus, there are no such points in $\Delta_{r-h,s-k}$. It follows that there are no points in $\Delta_{r,s}$ that lie on $f(x) = \frac{(s-k)}{(r-h)^b}(x-h)^b + k$ and are strictly between P and Q. As $r > m$ and $s > n$, the same holds for $\Delta_{m,n}$ and Q is weakly b-visible from P. See Figure 2 for an illustration.

Likewise, if $P-Q$ is b-invisible and $(\frac{r-h}{\ell}, \frac{s-k}{\ell^b}) \notin \Delta_{m-h,n-k}$ for any $\ell > 1$ satisfying $\ell|(r-h)$ and $\ell^b|(s-k)$, then suppose for contradiction that Q is not weakly b-visible from P. Then there exists a $Q' \in \Delta_{m,n}$ strictly between Q and P which lies on the curve $f(x) = \frac{(s-k)}{(r-h)^b}(x-h)^b + k$. Suppose $Q' = (\alpha, \beta)$. Then α must be equal to $\frac{r-h}{\ell} + h$ for some integer $\ell > 1$. This is because $(\alpha - h) < (r-h)$ and $\frac{(s-k)}{(r-h)^b}(x-h)^b$ must be an integer. Then, It follows that $\beta = \frac{s-k}{\ell^b} + k$, and as α and β are integers, $\ell|(r-h)$ and $\ell^b|(s-k)$. Finally, as $(\alpha, \beta) \in \Delta_{m,n}$, we have that $(\frac{r-h}{\ell}, \frac{s-k}{\ell^b}) \in \Delta_{m-h,n-k}$, which is a contradiction as desired. $\square$

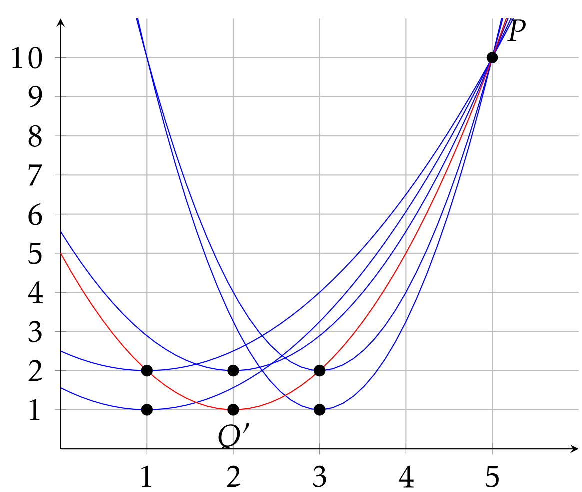


Figure 1. Weakly 2-visibility of points in $\Delta_{3,2}$ from $P = (5,10)$.

We illustrate Lemma 1.4 in the following example.

Example 2.1. Consider the case of 2-visibility ($b = 2$ in the definition of b-visibility), $\Delta_{3,2}$, and $P = (5,10)$. We compute the 2-visibility of each of the points $P - Q$ with $Q \in \Delta_{3,2}$ and use this to determine if $Q \in \Delta_{m,n}$ is weakly 2-visible from P. Note

$$\begin{aligned}
&P-(1,1)=(4,9) \text{ is 2-visible,}\\
&P-(2,1)=(3,9) \text{ is 2-invisible and } (3/3, 9/3^2)=(1,1)\in\Delta_{3-2,2-1}=\Delta_{1,1},\\
&P-(3,1)=(2,9) \text{ is 2-visible,}\\
&P-(1,2)=(4,8) \text{ is 2-invisible and } (4/2, 8/2^2)=(2,2)\notin\Delta_{3-1,2-2}=\Delta_{2,0}=\emptyset,\\
&P-(2,2)=(3,8) \text{ is 2-visible, and}\\
&P-(3,2)=(2,8) \text{ is 2-invisible and } (2/2, 8/2^2)=(1,2)\notin\Delta_{3-3,2-2}=\Delta_{0,0}=\emptyset.
\end{aligned}$$

From the above computations and Lemma 1.4 we know that all of the points in $\Delta_{3,2}$, with the exception of $(2,1)$, are weakly 2-visible from $P=(5,10)$. Figure 1 provides a visualization of this example. Note that if $Q \in \Delta_{3,2}$ and $Q \neq (2,1)$, then no point in $\Delta_{3,2}$ blocks Q from being weakly 2-visible from P, but the point $(3,2)\in\Delta_{3,2}$ blocks $Q'=(2,1)$ from being weakly 2-visible from P.

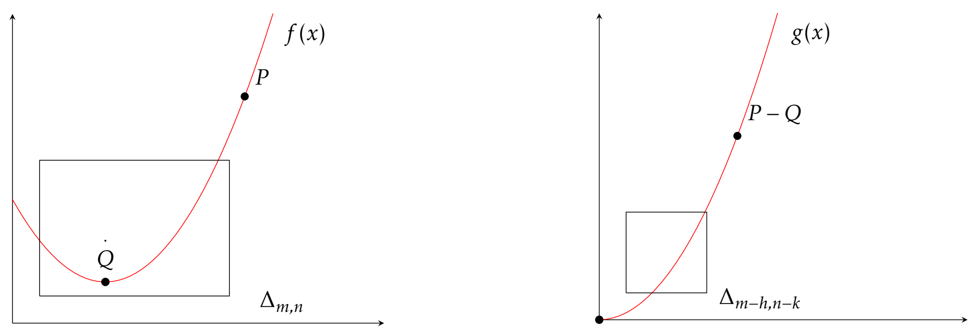


FIGURE 2. The left shows P attempting to weakly b-view Q. The right shows the origin attempting to b-view $P-Q$. Note $f(x)=\frac{s-k}{(r-h)^b}(x-h)^b+k$ and $g(x)=\frac{s-k}{(r-h)^b}x^b$.

Nicholson and Rachan [4] provided necessary and sufficient conditions for a point $Q \in \Delta_{m,n}$ to not be weakly visible from a point $P \notin \Delta_{m,n}$ as stated in Theorem 1.5.

We now present a proof to Theorem 1.6, the analogous result in the weak b-visibility setting.

Proof of Theorem 1.6. Fix b, $Q=(h,k)\in\Delta_{m,n}$, and $P=(r,s)$ such that $r>m, s>n$. $(\Rightarrow)$ Suppose that Q is not weakly b-visible by P. Therefore, by Lemma 1.4, $P-Q$ cannot be b–visible. By Proposition 1.2, this gives us that $\widetilde{\gcd}_b(r-h,s-k)>1$. Now, by some careful study of Lemma 1.4, we see by contrapositive of the lemma that $\left(\frac{r-h}{\widetilde{\gcd}_b(r-h,s-k)}, \frac{s-k}{\widetilde{\gcd}_b(r-h,s-k)}\right)$ is in $\Delta_{m-h,n-k}$ as desired.

$(\Leftarrow)$ Now suppose that $\widetilde{\gcd}_b(r-h,s-k)>1$. By Proposition 1.2, we know that $P-Q$ is not b-visible. Suppose further that $\left(\frac{r-h}{\widetilde{\gcd}_b(r-h,s-k)}, \frac{s-k}{\widetilde{\gcd}_b(r-h,s-k)}\right)\in\Delta_{m-h,n-k}$. Then, by Lemma 1.4, Q is not weakly b-visible from P. $\square$

Our next result establishes restrictions on the coordinates of P so that $\Delta_{m,n}$ can be weakly b-visible from P. In particular, we can conclude that if $P=(r,s)$ weakly b-views $\Delta_{m,n}$, then $r>m$ and $s>n$.

Lemma 2.2. *Fix $b\in\mathbb{Z}^+$, and let $P\notin\Delta_{m,n}$, with $m\geq 2$. If $P=(r,s)$ weakly b-views $\Delta_{m,n}$, then $r>m$ and $s>n$.*

Proof. We show that if $r\leq m$, or $s\leq n$, then P does not weakly b-view $\Delta_{mn,}$, starting with the case $s\leq n$. Let $P=(m+i,j)$ with $i\in\mathbb{Z}^+$ and $1\leq j\leq n$. We claim that $Q=(m-1,j)\in\Delta_{m,n}$ is not weakly b-visible from $P=(m+i,j)$. Note that P and Q lie on the curve $f(x)=\frac{(j-j)}{(m+i-(m-1))^b}(x-(m-1))^b+j=j$ and $Q'=(m,j)\in\Delta_{m,n}$ lies between Q and P. Hence Q is not weakly b-visible from P.

For the other case, let $P=(\ell,j)$ with $n<j$ and $1\leq\ell\leq m$. P doesn't weakly b-view any of the points in the lattice with the same x-coordinate, because, as was mentioned after Definition 1.3, we do not have a curve in this case. $\square$

We are now ready to show how to find a family of exponents, such that $Q\in\Delta_{m,n}$ is weakly b-visible by a fixed point $P=(r,s)$ with $r>m$ and $s>n$. This was the statement of Theorem 1.7.

Proof of Theorem 1.7. By Lemma 1.4 it suffices to find $b_0\in\mathbb{Z}^+$ such that $P-Q$ is b_0-visible for any $Q\in\Delta_{m,n}$. Let $Q=(i,j)\in\Delta_{m,n}$ with $1\leq i\leq m$ and $1\leq j\leq n$. Note $P-Q=(r-i,s-j)$ and let $F(s-j)$ denote the maximum exponent appearing in the prime factorization of $s-j$, or if $s-j=1$, then $F(s-j)=0$. Then setting $b_0=\left[\max\limits_{1\leq j\leq n}F(s-j)\right]+1$ ensures that $k^b\nmid(s-j)$ for any $k>1$, $1\leq j\leq n$, and any $b\geq b_0$. Hence $\Delta_{m,n}$ is weakly b-visible by P. Note that we do not need to consider the prime factorization of $r-i$ since showing that $k^b\nmid(s-j)$ for any $k>1$ and any Q is enough to guarantee that Q is already weakly b-visible from P. This holds for any $b\geq b_0$, which gives us a family of exponents. $\square$

To illustrate Theorem 1.7, we present the following example.

Example 2.3. Consider $\Delta_{3,2}$ and $P=(5,10)\notin\Delta_{3,2}$. The prime factorization of $P-Q$, for each $Q\in\Delta_{3,2}$, is given by:

$$\begin{aligned}
(5-1,10-1)&=(4,9)=(4,3^2)\\
(5-1,10-2)&=(4,8)=(4,2^3)\\
(5-2,10-1)&=(3,9)=(3,3^2)\\
(5-2,10-2)&=(3,8)=(3,2^3)\\
(5-3,10-1)&=(2,9)=(2,3^2)\\
(5-3,10-2)&=(2,8)=(2,2^3).
\end{aligned}$$

Notice that the highest power appearing in the factorization of the y-coordinates in the set of points $P-Q$ for $Q\in\Delta_{3,2}$ is 3. So selecting $b=4$ would make $\Delta_{3,2}$ weakly 4-visible by P, as there would not exist a $k > 1$ such that the y-coordinate of $P - Q$ is divisible by k^4 for any $k > 1$ and any $Q \in \Delta_{3,2}$. Thus by Lemma 1.4, $\Delta_{3,2}$ is weakly b-visible from P for any $b \geq 4$.

For any $b \geq 2$, Theorem 1.8 provides a $P = (r, s)$ such that $\Delta_{m,n}$ is weakly visible from P, provided that $m > n + 2$. We present the proof of this statement below.

Proof of Theorem 1.8. We proceed using a similar proof technique as in [3, Theorem 1] and Figure 3 illustrates the point of interest $P = \left(\left\lceil \sqrt[b]{m^{b+1} - 2m^b - m + 3}\right\rceil, n+1\right)$.

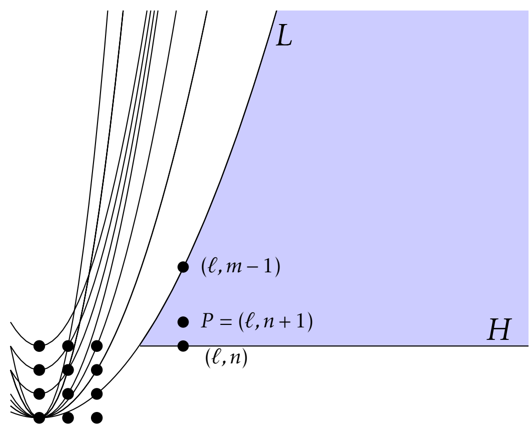


FIGURE 3. The point $P = (\ell, n+1)$ (where $\ell = \left\lceil \sqrt[b]{m^{b+1} - 2m^b - m + 3}\right\rceil$) weakly b-views all of $\Delta_{m,n}$.

Let $\mathcal{L}$ denote the set of all curves of the form

$$f(x) = \left(\frac{y_1 - y_0}{x_1^b - x_0^b}\right)x^b + \frac{y_0 x_1^b - y_1 x_0^b}{x_1^b - x_0^b}$$

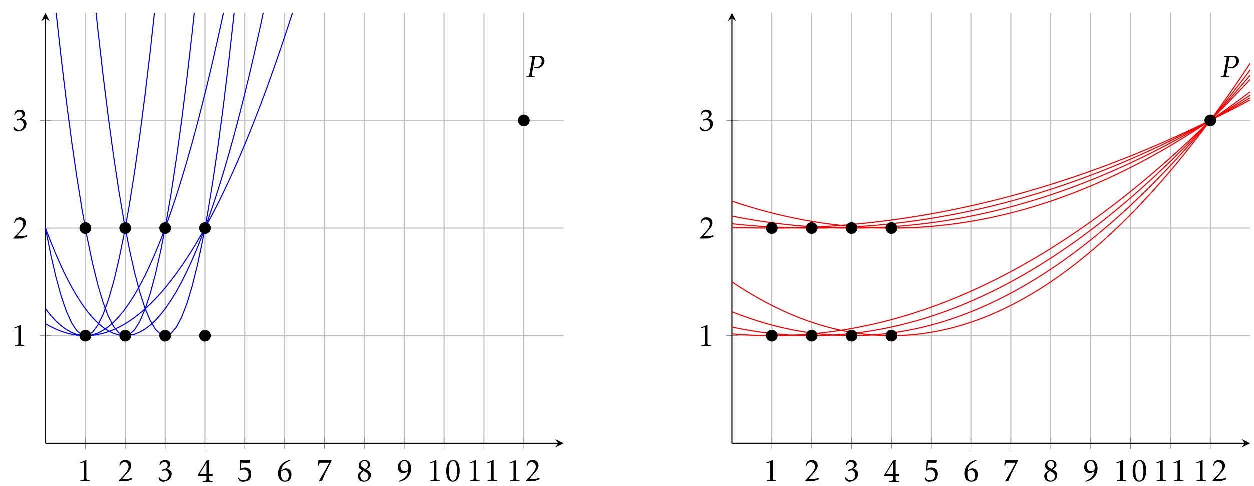
uniquely connecting $Q = (x_0, y_0)$ to $Q' = (x_1, y_1)$, where Q' and Q are points in $\Delta_{m,n}$ such that the coordinates of $Q' - Q$ are positive integers (Figure 4a depicts some of the curves in $\mathcal{L}$ when $b = 2$) . Showing that the point $P = \left(\left\lceil \sqrt[b]{m^{b+1} - 2m^b - m + 3}\right\rceil, n+1\right)$ does not lie on any of the curves in $\mathcal{L}$, will establish the result. This is because we could then find curves between P and any point in $\Delta_{m,n}$, and those curves won't have any other points in $\Delta_{m,n}$ on them.

Note that the curve $L \in \mathcal{L}$ connecting the points $(1,1)$ and $(m,2)$ in $\Delta_{m,n}$ is the graph of the function $f(x) = \frac{1}{(m^b-1)}((x)^b - 1) + 1$, and $f(x)$ has the smallest possible positive rate of change on the interval $[1, \infty)$ from all of the curves in $\mathcal{L}$. That is, the power function $f(x)$ through the points $(1,1)$ and $(m,2)$ results in the smallest positive leading coefficient for the power functions in $\mathcal{L}$.

Now note that no curves in $\mathcal{L}$ cross through the region below L and above the line H with equation $H(x) = n$. But by evaluating $f(x)$ at a chosen x value, we see that L contains

the point $\left(\left\lceil \sqrt[b]{m^{b+1}-2m^b-m+3}\right\rceil, \mathcal{M}\right)$ (where $\lceil x \rceil$ indicates the least integer greater than or equal to x) and $\mathcal{M} \geq m-1$. Furthermore, H contains the point $\left(\left\lceil \sqrt[b]{m^{b+1}-2m^b-m+3}\right\rceil, n\right)$, and $m-1 > n+1 > n$.
Therefore, the point $P = \left(\left\lceil \sqrt[b]{m^{b+1}-2m^b-m+3}\right\rceil, n+1\right)$ is below L and above H, as illustrated in Figure 3. Thus $\Delta_{m,n}$ is weakly b-visible by P. □

Although Theorem 1.8 provides a point that weakly b-views all of $\Delta_{m,n}$, this point might be quite far from $\Delta_{m,n}$. The following example illustrates this issue.



(A) Curves in $\mathcal{L}$. (B) Quadratics through P and $Q \in \Delta_{4,2}$.

FIGURE 4. Figure for Example 2.4.

Example 2.4. Fix $b = 2$. Consider $\Delta_{4,2}$ and define $P = (12,3)$. Let $\mathcal{L}$ denote the set of all curves of the form $f(x) = \frac{(y_1-y_0)}{(x_1-x_0)^2}(x-x_0)^2 + y_0$ connecting $Q = (x_0, y_0)$ to $Q' = (x_1, y_1)$, where Q' and Q are points in $\Delta_{4,2}$ such that the coordinates of $Q'-Q$ are positive integers. Figure 4a depicts the curves in $\mathcal{L}$ in blue.

Then the curve $L \in \mathcal{L}$ connecting the points $(1,1)$ and $(4,2)$ is the graph of the function $f(x) = \frac{1}{3^2}(x-1)^2 + 1$. Moreover, this function f has the smallest possible positive rate of change from the lines in $\mathcal{L}$. That is, the function $f(x)$ through the points $(1,1)$ and $(4,2)$ results in the smallest value of a possible for the power functions in $\mathcal{L}$.

We now claim that $\Delta_{4,2}$ is weakly 2-visible by $P = (12,3)$. To show this we proceed as in Example 2.3 by computing the prime factorization of $P - Q$ for each $Q \in \Delta_{4,2}$, which are given by:

$$\begin{array}{ll}
(12-1, 3-1) = (11, 2), & (12-1, 3-2) = (11, 1), \\
(12-2, 3-1) = (2\cdot 5, 2), & (12-2, 3-2) = (2\cdot 5, 1), \\
(12-3, 3-1) = (3^2, 2), & (12-3, 3-2) = (3^2, 1), \\
(12-4, 3-1) = (2^3, 2), & (12-4, 3-2) = (2^3, 1).
\end{array}$$

Let $P-Q=(r,s)$ and notice that there is no $k>1$ such that $k|r$ and $k^2|s$ for any $Q\in\Delta_{4,2}$. Thus $\Delta_{4,2}$ is weakly 2-visible by $P=(12,3)$, by Lemma 1.4. Figure 4b shows the quadratics through Q and P for each $Q\in\Delta_{4,2}$, so that Q is weakly 2-visible from P. Note that there is a point closer to $\Delta_{4,2}$ that weakly b-views all of $\Delta_{4,2}$. That point is $P'=(6,3)$.

We now provide a proof of Corollary 1.9, which describes the location of the point P closest on average to $\Delta_{m,n}$ that weakly b-views all of $\Delta_{m,n}$.

Proof of Corollary 1.9*.* Let L and H be as in the proof of Theorem 1.8. Note that $f(x)=L$. If $P=(m^*,n+1)$, where $m^*>m$ is the smallest positive integer such that $n+1<f(m^*)$, then P satisfies the conditions of Theorem 1.8 - namely, $n<n+1<f(m^*)$ and $m<m^*$. Thus P weakly b-views $\Delta_{m,n}$.

The coordinates of P will be minimal among any points lying below L and above H. This means that it is the point closest to (m,n) that is able to weakly b-view $\Delta_{m,n}$. Thus the average distance from $P=(m^*,n+1)$ to $\Delta_{m,n}$ will be minimal, as any P' able to weakly b-view the lattice must be at least as far horizontally and vertically from any point in the lattice as P is. □

3. Future work

In this paper, we fixed a nonnegative integer b and we considered polynomial functions of degree d going through the points $Q=(h,k)\in\Delta_{m,n}$ and $P=(r,s)$ with $r>m$ and $s>n$. In particular, we considered the functions $f(x)=\frac{s-k}{(r-h)^b}(x-h)^b+k$, as this allowed us to extend the definition of b-visibility to weak lattice point visibility. However, one could consider other types of functions, such as rational, exponential, logarithmic, or even sinusoidal functions, albeit some of the computations would be much more difficult to analyze.

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