

Finding Euler Characteristics of Hilbert Schemes using Colored Young Diagrams

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The Minnesota Journal of Undergraduate Mathematics

Volume 5 (2019-2020 Academic Year)

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ABSTRACT. The Hilbert schemes of the singular space $(\mathbb{C}^2/\mathbb{Z}_n)$ and of the orbifold $[\mathbb{C}^2/\mathbb{Z}_n]$ are two structures that contain geometric data about group actions on a polynomial ring. Our goal is to understand this geometry by finding the Euler characteristics of these spaces. The problem is equivalent to counting Young diagrams that are based on the group action. For the singular case, we count all zero generated Young diagrams that contain a certain number of 0 colored squares, and we prove a theorem greatly reducing the problem, sometimes into already solved cases. For the Hilbert scheme of the orbifold, we count all Young diagrams with a given coloring, and we develop a procedure to obtain the desired generating function, as well as closed form generating functions for special cases. We also explore the method of vertex operator algebras.

1. INTRODUCTION

In this paper, we seek to further understand the geometry of certain Hilbert schemes by calculating the Euler characteristic, an important topological invariant. We begin with some background on Hilbert schemes, the rich interplay between Hilbert schemes and Young diagrams, and the Euler characteristic. Expanding on previous research, we find a generating function for the Euler characteristic of certain Hilbert schemes via Young diagrams. A good reference for background material is *Ideals, Varieties and Algorithms* by Cox, Little and O'Shea [2].

1.1. Hilbert Schemes. The Hilbert scheme of m points on the plane, $\text{Hilb}^m(\mathbb{C}^2)$, is an algebro-geometric object parametrizing the possible arrangements of m points in a plane. That is, each point of the Hilbert scheme is an ideal corresponding to a configuration of points, and the internal scheme structure reflects the geometric information of the possible arrangements and the ways the points can collide.

More recently, mathematicians have applied the notion of Hilbert schemes to more complicated surfaces arising from a finite group of symmetries acting on a plane. Given a finite group G action on $\mathbb{C}[x, y]$, the quotient space \mathbb{C}^2/G is defined by identifying elements

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that lie in the same orbit of the group action. Note that the resulting surface may contain singularities.

Another surface arising from such a group action is the orbifold $\left[\frac{\mathbb{C}^2}{G}\right]$. It also arises by identifying points in the same orbit, but carries more geometric structure than the quotient space as it also encodes the isotropic subgroups — that is, at each point the subgroup of elements that fix that point. For a more thorough introduction to quotient spaces and orbifolds, see Chapter 13 of [14].

We can define the Hilbert scheme of points on both the quotient space and the orbifold, which parametrize configurations of m points on each surface respectively. We will only consider the cases where the surfaces arise from the torus actions of cyclic groups (actions that affect x and y separately), which are represented by multiplication by $G_{k,n} = \begin{bmatrix} \omega & 0 \\ 0 & \omega^k \end{bmatrix} \in \mathbb{Z}_n$, where ω is the n^{th} root of unity.

The Hilbert scheme of m points on the quotient space is defined as:

$$\text{Hilb}^m\left(\frac{\mathbb{C}^2}{G}\right) = \left\{ I \subset \mathbb{C}[x, y]^G \mid \dim \frac{\mathbb{C}[x, y]^G}{I} = m \right\}.$$

Here, G is a cyclic group, and I is an ideal of $\mathbb{C}[x, y]$. Since I is fixed by the torus action of G , it is a monomial ideal. For each m , we are interested in those ideals I such that when the fixed points of the group action are quotiented by I , the result forms an m dimensional vector space \mathbb{C} .

Due to the greater structure of the orbifold, the Hilbert scheme of points on the orbifold will depend not just on the number of points, but on a representation of the group action. It is defined as:

$$\text{Hilb}^{v=m_0\rho_0+m_1\rho_1+\dots+m_{k-1}\rho_{k-1}}\left[\frac{\mathbb{C}^2}{G}\right] = \left\{ I \mid \frac{\mathbb{C}[x, y]}{I} \simeq v \right\}.$$

Here, v is a representation of the group action, decomposed into irreducible representations ρ_i . For each representation, we are interested in those ideals which when quotiented into the polynomial ring are isomorphic to that representation.

Further elaboration and explanation for these definitions and the related concepts can be found in [7].

1.2. Young Diagrams. Young diagrams are important combinatorial objects with wide use in representation theory and other subfields of mathematics. Young diagrams are *stable* configurations of boxes in the plane. Informally, one may think of physically stacking boxes in the plane, with “gravity” acting in the direction $(-1, -1)$. A configuration is then *stable* if it is supported by (i.e. touching) either a box or one of the lines $y = 0$, $x = 0$ to the left and below. See Figure 1 for examples and non-examples of Young diagrams:

More general background on Young diagrams can be found in [4].

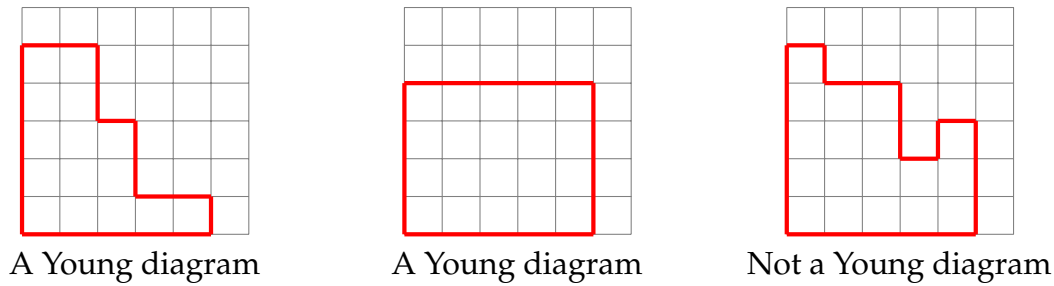


FIGURE 1. Young diagrams and non-Young diagrams

1.2.1. *Interpretations of Young Diagrams.* One classical problem is to count the number of Young diagrams with a certain number of boxes. It is easy to see that Young diagrams of size n are in bijection with partitions of size n : given a Young diagram, let λ_i be the height of the i th column, then $\lambda = \{\lambda_i\}$ is a partition of n . Conversely, given any partition $\lambda = \{\lambda_i\}$ of n , we can order λ_i from greatest to least, then take the Young diagram with i th column of height λ_i . See Figure 2 for an example.

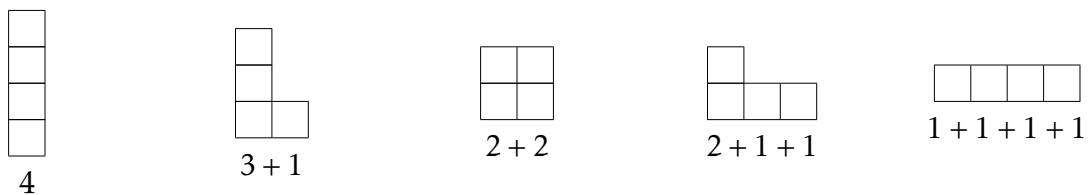


FIGURE 2. Young diagrams corresponding to partitions of 4

Although there is no known closed formula for the number of Young diagrams of size n , the coefficient of q^n in the series expansion of the following gives the number of Young diagrams of size n :

$$\prod_{i=1}^{\infty} \frac{1}{1 - q^i} = (1 + q + q^2 + q^3 \dots)(1 + q^2 + q^4 + q^6 \dots)(1 + q^3 + q^6 + q^9 \dots) \dots$$

(for a proof of this expansion, see, for example, [13]). One can then calculate the number of partitions of size n by expanding the series to the n th term. For example, for $n = 5$:

$$1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 \dots$$

The coefficient of q^5 is 7, so there are 7 Young diagrams with five blocks. A function like the one above which stores combinatorial data in its coefficients is called a *generating function*. They will be key for presenting solutions to our problem.

Another important interpretation of Young diagrams is their correspondence with monomial ideals of $\mathbb{C}[x, y]$. An ideal of $\mathbb{C}[x, y]$ is *monomial* if it has monomial generators. First, we assign to each square (m, n) the monomial $x^m y^n$. Then we identify the squares representing the generators of the ideal. All squares above and to the right of these generating squares will be members of the ideal. The remaining squares form a Young diagram. Thus, there is a bijection between monomial ideals and Young diagrams. For example, the Young diagram in Figure 1 corresponds to the ideal generated by y^3, xy^2, x^3 .

y^3	xy^3	x^2y^3	x^3y^3
y^2	xy^2	x^2y^2	x^3y^2
y	xy	x^2y	x^3y
1	x	x^2	x^3

FIGURE 3. Monomials in a grid

Note that this notion can naturally be extended to monomial ideals in n variables using Young diagrams of higher dimension. For more on these correspondences and their applications, see [12].

1.2.2. *Coloring of Young Diagrams via Torus Actions.* In addition to regular Young diagrams, one may also want to count (for reasons that will soon become apparent) the number of Young diagrams with the boxes filled with “colors” in different patterns. Given three natural numbers a, b, n , the coloring associated to (a, b, n) assigns to the box (i, j) the value $(ai + bj) \bmod n$. Note that we use zero indexing, meaning that the bottom left most square to be box $(0, 0)$ For some examples, see Figure 4:

1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1

(1,1,2) coloring

1	2	0	1	2	0
2	0	1	2	0	1
0	1	2	0	1	2
1	2	0	1	2	0
2	0	1	2	0	1
0	1	2	0	1	2

(1,2,3) coloring

2	3	0	1	2	3
0	1	2	3	0	1
2	3	0	1	2	3
0	1	2	3	0	1
2	3	0	1	2	3
0	1	2	3	0	1

(1,2,4) coloring

FIGURE 4. Some grid colorings

More background for the colorings described in this paper can be found in the introduction of [1] and in [6], [5]. Note that there are other ways to fill in Young diagrams with numbers, like with Young tableaux in [4], which are completely different from the colorings we describe here.

In our interpretation of Young diagrams as monomial ideals of $C[x, y]$, the coloring corresponds to an action of (the algebraic torus) $(C^*)^2$ on $C[x, y]$ via

$$(s, t) \cdot f(x, y) = f(sx, ty)$$

More specifically, we consider a cyclic subgroup of $(C^*)^2$, generated by (ω^a, ω^b) , where ω is an n th root of unity.

Note that only monomial ideals are fixed under torus actions (since multiplying by generators by units does not change the ideal). Therefore, monomial ideals are key to understanding the geometry of these Hilbert schemes.

We can use a “coloring” of the grid to represent the group action. The “color” of each square is the power of ω by which the corresponding monomial is multiplied when x and y are acted on by the the group action. For example, Figure 5 shows a Young diagram corresponding to the action of $G_{1,2} = \begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} \in \mathbb{Z}_3$. The bottom left square is colored 0 because the monomial 1 is unaffected by the group action, the square above and to the right is colored 2 because xy maps to $(\omega x)(\omega y) = \omega^2 xy$, and so on.

0	1	2	0
2	0	1	2
1	2	0	1
0	1	2	0

FIGURE 5. A Young diagram of an action in \mathbb{Z}_3

1.3. The Euler Characteristic. The Euler characteristic is an old and fundamental topological invariant related to the curvature of a space. Finding the Euler characteristic of Hilbert schemes gives us greater understanding of the geometric information they represent.

In some cases, computing the Euler characteristic reduces to a purely combinatorial problem. For example, it has been known for a long time that the Euler characteristic of the Hilbert scheme of m points can be computed by counting the number of partitions of m , which correspond to Young diagrams. It is well-known also that the Euler characteristic of the Hilbert scheme of quotient spaces and orbifolds can also be computed by counting certain types of colored Young diagrams.

Broadly, this follows from the fact that a point of Hilbert scheme is an ideal, and the points that are fixed by the group action are monomial ideals, which we have seen correspond to Young diagrams. For more details on this derivation, see [8] and [9]. For explanations specific to the quotient space and orbifold cases, see [3], [6], and [7]. And for more background on this topic, see [11].

For our purposes, the Euler characteristic of the Hilbert scheme of m points on the quotient space corresponds to the number of zero generated Young diagrams containing m 0-colored squares. A Young diagram is zero generated if each square in a corner position on the outside boundary (i.e. a square outside of the Young diagram whose left and bottom edges lie on the boundary of the Young diagram or the axes) is occupied by a 0-colored square — for instance, the highlighted squares in Figure 6 are in outside corner positions, and note that they all are 0-colored. Our goal was to find a generating function to count such diagrams.

Again, since the orbifold has more structure, the Euler characteristic of the Hilbert scheme in this case will depend not just on a number m , but on a list of m colors. In particular, if

2	3	0	1	2	3	0	1	2	3
0	1	2	3	0	1	2	3	0	1
2	3	0	1	2	3	0	1	2	3
0	1	2	3	0	1	2	3	0	1
2	3	0	1	2	3	0	1	2	3
0	1	2	3	0	1	2	3	0	1
2	3	0	1	2	3	0	1	2	3
0	1	2	3	0	1	2	3	0	1

FIGURE 6. A zero-generated Young diagram

2	3	0	1	2	3
0	1	2	3	0	1
2	3	0	1	2	3
0	1	2	3	0	1
2	3	0	1	2	3
0	1	2	3	0	1

FIGURE 7. A Young diagram with coloring $3\rho_0 + 2\rho_1 + 4\rho_2 + 3\rho_3$

the color i occurs c_i times in the list (for $i = 0, 1, \dots$) the corresponding Euler characteristic is equal to the number of Young diagrams that contain c_0 0-colored squares, c_1 1-colored squares, and so on. We will refer to such Young diagrams as having the coloring $c_0\rho_0 + c_1\rho_1 + \dots + c_{n-1}\rho_{n-1}$. Our goal was to find a generating function which counts such colored Young diagrams. The coefficient of the monomial $q_0^{c_0} q_1^{c_1} \dots q_{n-1}^{c_{n-1}}$ in our generating function will equal the number of Young diagrams with coloring $c_0\rho_0 + c_1\rho_1 + \dots + c_{n-1}\rho_{n-1}$. For example, the Young diagram in Figure 3 for the action of $\begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix} \in \mathbb{Z}_4$ has the coloring $3\rho_0 + 2\rho_1 + 4\rho_2 + 3\rho_3$. In the generating function, it would contribute to the $q_0^3 q_1^2 q_2^4 q_3^3$ term.

1.4. **Previous Work.** Both of these problems have been solved for the actions of $\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$ and $\begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix} \in \mathbb{Z}_n$ on $\mathbb{C}[x, y]$ in [6] and [5] by Ádám Gyenge, András Némethi, and Balázs Szendrői. They used a bijection with motions on a combinatorial abacus (sometimes called a Dirac sea representation in the simplest cases) to count each coloring for the orbifold problem, and then strategically substituted roots of unity to count zero generated Young diagrams.

For the zero generated problem, the generating function for the case $\begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix} \in \mathbb{Z}_n$ was

$$\left(\prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{n+1} \cdot \sum_{\vec{m}=(m_1, \dots, m_n) \in \mathbb{Z}_n} \zeta^{m_1+m_2+\dots+m_n} (q^{1/2})^{\vec{m}^T \cdot C_{\Delta} \cdot \vec{m}} \tag{1}$$

where $\zeta = \exp \frac{2\pi i}{1+h^V}$, h^V is the (dual) Coxeter number of the corresponding finite root system (one less than the dimension of the corresponding simple Lie algebra divided by n), and C_{Δ} is the finite type Cartan matrix corresponding to Δ .

For the orbifold problem, the generating function for that same case is:

$$\left(\prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{n+1} \cdot \sum_{\vec{m}=(m_1, \dots, m_n) \in \mathbb{Z}_n} q_1^{m_1} \dots q_n^{m_n} (q^{1/2})^{\vec{m}^T \cdot C_{\Delta} \cdot \vec{m}} \tag{2}$$

where $q = \prod_{i=0}^n q_i^{d_i}$ with $d_i = \dim \rho_i$, and C_{Δ} is the finite type Cartan matrix corresponding to Δ .

Additionally, Benjamin Young [1] found results about colored three dimensional Young diagrams using techniques of vertex operator algebras. He sliced three dimensional Young diagrams into two dimensional Young diagrams, and then used the operators to describe how they interlace together. This kind of interlacing is useful, because it allows us to break Young diagrams into simpler Young diagrams and reconstruct them.

1.5. **Results.** Our goal was to find the Euler characteristic of two types of Hilbert Schemes arising from the torus action of a cyclic group on $\mathbb{C}[x, y]$: the Hilbert Scheme of the quotient surface, and the Hilbert scheme of points on an orbifold. This group action can be represented as multiplication by $\begin{bmatrix} \omega & 0 \\ 0 & \omega^k \end{bmatrix} \in \mathbb{Z}_n$, where ω is the n^{th} root of unity, and we call this group action $G_{n,k}$.

The Euler characteristic (χ) of both types of Hilbert schemes has a combinatorial interpretation in terms of colored Young diagrams. We worked on finding generating functions to count the relevant Young diagrams.

Our first result reduces the problem for the quotient surface to the same problem for a simpler group action $G_{n,k}$ where n and k are relatively prime:

$$\chi \left(\text{Hilb}^m \frac{\mathbb{C}^2}{G_{k,n}} \right) = \chi \left(\text{Hilb}^m \frac{\mathbb{C}^2}{G_{\frac{k}{d}, \frac{n}{d}}} \right)$$

where $d = \text{GCD}(n, k)$

In the orbifold case we found that the same simplification applies for representations where in the corresponding Young diagrams, the distribution of colors is the same in each column (these are discussed in [5] where they are called "balanced" Young diagrams).

Furthermore, we found a method of obtaining the generating function for the orbifold problem: The following procedure results in the generating function in q_0, q_1, \dots, q_{n-1} for the Euler characteristic of the orbifold $G_{k,n}$ acting on $\mathbb{C}[x, y]$. Let $d := \text{GCD}(n, k)$ and let $u \bmod v$ mean $u - v \lfloor \frac{u}{v} \rfloor$. Take the series expansion of $\prod_{a=1}^{\infty} \frac{1}{1 - \prod_{b=1}^a t_b}$ and replace each term $c \prod_{w=0}^m (t_w)^{e_w}$ with

$$c \prod_{i=0}^{n-1} q_i \sum_{j=0}^{j_{\max}} \sum_{l=0}^{\frac{n}{d}-1} \left[1 + \frac{-1 - \left(\lfloor \frac{i}{d} \rfloor - l \right) \left(\frac{k}{d} \right)^{-1} \bmod \frac{n}{d} + e_{jn+ld+(i \bmod d)}}{\binom{\frac{n}{d}}{}} \right]$$

where j_{\max} is the greatest natural number such that $jn + ld + (i \bmod d) \leq m$.

Although this is not a closed form, as it requires modification of a series in an infinite number of variables, this formula applies to all group actions $G_{k,n}$.

We also used vertex operator algebras to try and develop a closed form of the orbifold generating function for certain group actions. For example, using the notation of those operators, the generating function for $G_{2,4}$ would be:

$$\sum_{\lambda} Q_{1,3} \Gamma_+ Q_{0,2} \lambda.$$

Here, λ ranges over all Young diagram, the operators $Q_{0,2}$ and $Q_{1,3}$ describe the coloring of a Young diagram, and the operator Γ_+ finds all Young diagrams whose columns can be alternately interlaced with the columns of a given diagram to yield a valid Young diagram.

These operators allow us to build the desired colored Young diagrams out of simpler diagrams that are better understood. Using identities associated with these operators, it may be possible to express such formulas explicitly as a generating function. For more on these methods, see [1] and our Section 4.

2. REDUCING THE ZERO GENERATED PROBLEM

To find the Euler characteristic of the Hilbert scheme of the quotient space, we must find a generating function where the coefficient of x^n is the number of zero generated Young diagrams containing n 0-colored squares.

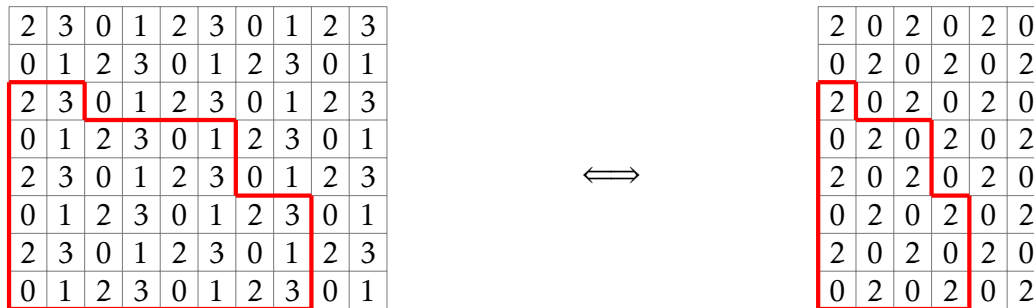


FIGURE 8. Reducing a zero-generated Young diagram

2.1. First Reduction Theorem. Our contribution was to find a method of reducing group actions to simpler group actions with the same number of zero generated Young diagrams of each size. An example of such a reduction from $\begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix} \in \mathbb{Z}_4$ to $\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} \in \mathbb{Z}_2$ is shown in Figure 8.

Theorem 2.1. *There is a bijection between the zero-generated Young diagrams of $\mathbb{C}[x, y]$ under $\begin{bmatrix} \omega & 0 \\ 0 & \omega^k \end{bmatrix} \in \mathbb{Z}_n$, and the zero-generated Young diagrams of $\mathbb{C}[x, y]$ under $\begin{bmatrix} \omega & 0 \\ 0 & \omega^{\frac{k}{d}} \end{bmatrix} \in \mathbb{Z}_{\frac{n}{d}}$, where $d = \text{GCD}(k, n)$.*

Proof. Note that in the grid determined by $\mathbb{C}[x, y]$ under $\begin{bmatrix} \omega & 0 \\ 0 & \omega^k \end{bmatrix} \in \mathbb{Z}_n$ the bottom row consists of: $0, 1, 2, \dots, n - 1, 0, 1, 2, \dots$. This is the case because the bottom left corner square is 0 (since constants are invariant under Torus actions) and moving right by a square corresponds to increasing the power of x by 1, and thereby multiplying by ω .

Now we will show that a column contains a 0 only if its bottom entry is divisible by d . Suppose there is a column containing a 0, and the bottom entry of that column is a . Moving up a column corresponds to increasing the power of y by 1 and thereby multiplying by ω^k . Therefore, we have $a + tk \equiv (0 \pmod n)$, where t is the height of the square marked 0. We can write this as $a + tk = un$ for some integer u , or $a = un - tk$. Since both d divides both un and tk , it divides the right hand side, and so must also divide a .

Next, consider moving along the boundary of a 0 generated Young diagram (the staircase-like line separating the boxes in the diagram from the rest of the grid, excluding the axes), starting from the top left corner and alternately moving down and to the right until the bottom right corner is reached. We see that every "down" move must end in the bottom-left corner of a square marked with 0, because this ending square will be a generator of the ideal. Thus, a "down" move can only occur to the left of a column which contains 0's. We have shown that only every d^{th} column contains a 0, so the boundary will always move "right" on columns which do not contain a 0. This means that in 0 generated Young diagrams, columns whose bottom-most square is not divisible by d will be the same height as the first column to the left whose bottom-most square is divisible by d .

The implication of this fact is that 0 generated Young diagrams are composed of rectangular blocks of width d . Given this specific rule, we can delete the columns whose bottom squares are not divisible by d without losing any information. That is, this deletion is reversible, because we know that the deleted columns must be the same height as the remaining column to the left. Note that since all 0's remain, this deletion will map 0 generated Young diagrams to 0 generated Young diagrams.

Once we delete those columns, moving one square right on the grid means adding d , while moving one square up means adding k . Thus every square is determined by $cd + wk \pmod n$ for some integers c and w . By dividing everything by d , we find that this board is isomorphic to one determined by $c + w\frac{k}{d} \pmod{\frac{n}{d}}$ which is the board resulting from the group action of $\begin{bmatrix} \omega & 0 \\ 0 & \omega^{\frac{k}{d}} \end{bmatrix} \in \mathbb{Z}_{\frac{n}{d}}$ on $\mathbb{C}[x, y]$.

Thus, we have a one to one correspondence between 0 generated Young diagrams of these two group actions, which gives us a way to simplify the task of counting them. \square

2.2. Generating Functions. Since the columns deleted never contain 0's, the number of 0's is the same in the old and new Young diagram. Therefore, given the above bijection, the generating functions for zero generated Young diagrams are exactly the same for the two group actions. In cases where the grid reduces to $\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$ or $\begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix}$, we can use previous results to obtain closed form generating functions. In general, this occurs when $k|n$ or $(n - k)|n$. Therefore, we can apply Equation 1 with a simple change of variables to find the number of 0-generated Young diagrams of a given size in many more cases. Recall that Equation 1 is:

$$\left(\prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{n+1} \cdot \sum_{\vec{m}=(m_1, \dots, m_n) \in \mathbb{Z}_n} \zeta^{m_1+m_2+\dots+m_n} (q^{1/2})^{\vec{m}^T \cdot C_{\Delta} \cdot \vec{m}}$$

where $\zeta = \exp \frac{2\pi i}{1+h^V}$, h^V is the (dual) Coxeter number of the corresponding finite root system (one less than the dimension of the corresponding simple Lie algebra divided by n), and C_{Δ} is the finite type Cartan matrix corresponding to Δ .

3. TOWARDS AN ORBIFOLD GENERATING FUNCTION

To find the Euler characteristic of the Hilbert scheme of an orbifold, we must find a generating function such that the coefficient of $q_0^{e_0} q_1^{e_1} q_2^{e_2} \dots$ is the number of Young diagrams with e_0 squares of color 0, e_1 squares of color 1, and so on.

3.1. Procedure for Orbifold Generating Function. We first give a theorem showing how we can arrive at such a generating function. Although it is not a closed form (as it requires manipulation of a generating function with infinitely many variables) it applies for all torus actions of cyclic groups, $G_{k,n}$.

Theorem 3.1. *The following procedure results in the generating function in q_0, q_1, \dots, q_{n-1} for the Euler characteristic of the orbifold $\begin{bmatrix} \omega & 0 \\ 0 & \omega^k \end{bmatrix} \in \mathbb{Z}_n$ acting on $\mathbb{C}[x, y]$. Let $d := \text{GCD}(n, k)$ and let $u \bmod v$ mean $u - v \lfloor \frac{u}{v} \rfloor$. Take the series expansion of $\prod_{a=1}^{\infty} \frac{1}{1 - \prod_{b=1}^a t_b}$ and replace each term $c \prod_{w=0}^m (t_w)^{e_w}$ with*

$$c \prod_{i=0}^{n-1} q_i \left[1 + \frac{-1 - \left(\lfloor \frac{i}{d} \rfloor - l \right) \binom{k}{d}^{-1} \bmod \frac{n}{d} + e_{jn+ld+(i \bmod d)}}{\binom{n}{d}} \right]$$

where j_{\max} is the greatest natural number such that $jn + ld + (i \bmod d) \leq m$.

Proof. We apply the identity $\frac{1}{1-\zeta} = 1 + \zeta + \zeta^2 + \dots$ to each factor $\frac{1}{1 - \prod_{b=1}^a t_b}$ in the original function $\prod_{a=1}^{\infty} \frac{1}{1 - \prod_{b=1}^a t_b}$. This identity is the formula for the sum of an infinite geometric series, and we can apply it without worrying about convergence since we are treating our function simply as a formal sum. Using this, we can write the original function as

$$\prod_{f=0}^{\infty} \sum_{g=0}^{\infty} \prod_{h=1}^f (t_h)^g = (1 + t_0 + t_0^2 + \dots)(1 + (t_0 t_1) + (t_0 t_1)^2 + \dots)(1 + (t_0 t_1 t_2) + (t_0 t_1 t_2)^2 + \dots) \dots$$

This can be interpreted as a generating function for Young diagrams, where for each term in the full expansion (using the same notation as before), c counts the number of Young diagrams whose w^{th} column has e_w boxes. In fact, there is a unique such Young diagram, and so $c = 1$ for each term.

Since the coloring of the grid by the group action is fixed, the number of boxes in each column of a Young diagram uniquely determines the number of boxes of each color in that diagram. We will show that the replacement procedure takes each term corresponding to a Young diagram with columns of particular sizes, and returns a term which corresponds to the coloring of that Young diagram.

Any given color will not necessarily appear in every column of the grid. In fact, the color numbered α will only appear in a column numbered γ (starting from 0) if and only if $\alpha \equiv \gamma \bmod d$. For each color i , our procedure affects all of the columns with this property.

In each column, the colors repeat every $\frac{n}{d}$ boxes. If a color α first appears in a column in row numbered β (again starting from 0), then the number of boxes marked α in the first s boxes of that column will be $\lfloor 1 + \frac{s-\beta}{(\frac{n}{d})} \rfloor$.

This row number β for column γ is the smallest nonnegative number such that $k\beta + \gamma \equiv \alpha \bmod n$. In our function, $\alpha = i$ and $\gamma = jn + ld + (i \bmod d)$. Thus we solve for β in the equation

$$k\beta + jn + ld + (i \bmod d) \equiv i \bmod n$$

$$k\beta + ld \equiv i - (i \bmod d) \equiv d \left\lfloor \frac{i}{d} \right\rfloor \bmod n$$

$$\frac{k}{d}\beta + l \equiv \left\lfloor \frac{i}{d} \right\rfloor \pmod{\frac{n}{d}}$$

$$\beta \equiv \left(\left\lfloor \frac{i}{d} \right\rfloor - l \right) \left(\frac{k}{d} \right)^{-1} \pmod{\frac{n}{d}}$$

and take β to be the smallest nonnegative integer satisfying the congruence. Plugging in this value for β , and plugging the row index of the top square in column γ of the diagram $s = e_{jn+ld+(i \bmod d)} - 1$ into $\lfloor 1 + \frac{s-\beta}{(\frac{n}{d})} \rfloor$ yields the desired formula. \square

3.2. Orbifold Special Case Reduction Theorem. For some special cases, we were able to find a closed form of the generating function for the orbifold. We use a reduction method much like that used with the zero generated Young diagrams.

Theorem 3.2. Consider all Young diagrams for the action of $\begin{bmatrix} \omega & 0 \\ 0 & \omega^k \end{bmatrix} \in \mathbb{Z}_n$ on $\mathbb{C}[x, y]$ whose coloring $\sum_0^{n-1} m_0 \rho_0$ obeys the property: The sum of all m_i for $0 \leq i \leq n - 1$ in a particular residue class mod $\text{GCD}[n, k]$ is the same for all mod $\text{GCD}[n, k]$ residue classes. These Young diagrams are in bijection with Young diagrams of the action of $\begin{bmatrix} \omega & 0 \\ 0 & \omega^{\frac{k}{\text{GCD}[n, k]}} \end{bmatrix} \in \mathbb{Z}_{\frac{n}{\text{GCD}[n, k]}}$ on $\mathbb{C}[x, y]$.

Proof. Each column of the grid of a torus group action contains numbers of a particular residue class mod $\text{GCD}[n, k]$. Therefore, the Young diagrams characterized in the statement have the same number of boxes in each type of column. Since no column of a Young diagram can be taller than the column to its left, such Young diagrams are composed of blocks which are $\text{GCD}[n, k]$ wide. Deleting all but one of the columns in each block does not lose information, and so is reversible. Performing this deletion results in a Young diagram in a grid corresponding to the group action of $\begin{bmatrix} \omega & 0 \\ 0 & \omega^{\frac{k}{\text{GCD}[n, k]}} \end{bmatrix} \in \mathbb{Z}_{\frac{n}{\text{GCD}[n, k]}}$ on $\mathbb{C}[x, y]$. \square

3.3. Generating Functions. If the generating function for the reduced action is the function $f(q_0, q_1, \dots, q_{\frac{n}{\text{GCD}[n, k]}-1})$, we can use the bijection to obtain the generating function for the original action by substituting for q_i the product of all q_j with $0 \leq j \leq n - 1$ and $j \equiv i \pmod{\text{GCD}[n, k]}$. In cases where the grid reduces to that of $\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix}$ or $\begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix}$, we can use previous results to obtain closed form generating functions. In general, this occurs when $k|n$ or $(n - k)|n$. Again, we find that with a simple change of variables, we can apply Equation 2 to many more cases. Recall that Equation 2 is:

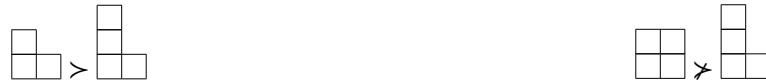
$$\left(\prod_{m=1}^{\infty} (1 - q^m)^{-1} \right)^{n+1} \cdot \sum_{\vec{m}=(m_1, \dots, m_n) \in \mathbb{Z}_n} q_1^{m_1} \cdots q_n^{m_n} (q^{1/2})^{\vec{m}^T \cdot C_{\Delta} \cdot \vec{m}}$$

where $q = \prod_{i=0}^n q_i^{d_i}$ with $d_i = \dim \rho_i$, and C_{Δ} is the finite type Cartan matrix corresponding to Δ .

4. VERTEX OPERATOR ALGEBRAS

Another possibly useful tool is that of operators, which were used to solve many problems involving 3D Young diagrams. For more background information on vertex operator algebras, see [1].

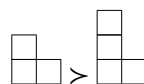
Two Young diagrams λ, μ interlace if $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ for every i ; note that this relation is not necessarily symmetric. More colloquially, λ, μ interlace if we can build a new Young diagram by alternately taking columns from λ and μ , from left to right. If λ, μ interlace, then we write $\lambda > \mu$. For example:



Γ_- finds all the partitions that interlace over a given partition. For example:

$$\Gamma_- \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \dots + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} + \dots$$

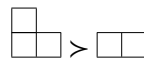
where



Γ_+ finds all the partitions that interlace into a given partition. For example:

$$\Gamma_+ \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} + \begin{array}{|c|} \hline \square \\ \hline \end{array}$$

where



Q identifies the coloring of a given partition. For example:

$$Q_{0,2} \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = q_0 q_2^2 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$$

Known commutator identities allow us to obtain closed formulas in many circumstances, but we have not yet been able to do so for our cases. For instance, the following is operator notation of the generating function for the orbifold problem of the action $\begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix} \in \mathbb{Z}_4$:

$$\sum_{\lambda} Q_{1,3} \Gamma_+ Q_{0,2} \lambda.$$

5. CONCLUSION

We have found a way to reduce the quotient space problem to that of simpler group actions, many of which have been solved. We have also obtained a procedure to obtain the generating function for the orbifold problem, and have a closed form for some special cases. We also have investigated additional methods, which may be useful, either to our problems, or other related questions. Looking ahead, we hope to solve more cases of the quotient space problem, and to find closed forms for more cases of the orbifold problem.

6. APPENDIX A: ABACUS METHOD

Though the abacus method has not yet brought us close to a desired generating function, it was used to solve the orbifold problem for the action of $\begin{bmatrix} \omega & 0 \\ 0 & \omega \end{bmatrix} \in \mathbb{Z}_n$ in [7]. It involves using an abacus – that is, a list of columns of numbers with circles marking some of the numbers. The columns are called “runners” and the circles are called “beads”, and the beads can be moved to change positions within each runner. We construct such an abacus configuration in a way that records the boundary of a Young diagram, and then move the beads to record modification of the diagram. The number of runners corresponds to the number of colors. For example, if we let marked places represent “down” moves and unmarked places represent “right” moves, the following Young diagram is represented by the following abacus:

0						⑤	④
1	0	1				③	2
0	1	0				①	0
						-1	②
						③	-4
						-5	-6

Using the abacus, we found a way to generate all Young diagrams which interlace with a given Young diagram, that is, which can be spliced with the original, by alternating columns, to form a valid Young diagram. Interlacing diagrams correspond to moving each bead (starting from the top) at most once, either to the left or diagonally up and right. This fact may be useful because Young diagrams for many group actions can be broken up into interlacing Young diagrams for simpler group actions.

A more thorough introduction to the combinatorial abacus can be found in Chapter 11 of [10].

7. APPENDIX B: CHECKING FORMULAS COMPUTATIONALLY

We also used a computer program to test the procedure for obtaining the orbifold generating function. Given values for n and k and a maximum size, the following Mathematica code will output a list of n -tuples which represent all of the colorings of Young diagrams up to the maximum size. In the code below, we have set $n = 3$ and $k = 2$, and the maximum size is 8.

```

partitions = {};
For[l = 1, l <= 8, l++,
  For[m = 1, m <= PartitionsP[l], m++,
    AppendTo[partitions, Part[IntegerPartitions[l], m]];
  ]
]
n = 3;
k = 2;
d = GCD[n, k];

```

```

h = PowerMod[k/d, -1, n/d];
colorings = {};
For[s = 1, s <= Length[partitions], s++,
  young = Part[partitions, s];
  term = {};
  For[i = 0, i < n, i++,
    j = 0;
    c = 0;
    While[j + Mod[i, d] < Length[young],
      c = c +
        Floor[1 + ((Part[young, j + Mod[i, d] + 1] - 1 -
          Mod[h*(Floor[i/d] - (j/d)), n/d])/(n/d))];
      j = j + d;
    ]
    AppendTo[term, c];
  ]
  AppendTo[colorings, term];
]
colorings

```

To obtain a generating series from this, we have only to convert each n -tuple into a monomial in n variables, and sum the results into a polynomial. After applying the preceding program, we use the following:

```

series = 0;
For[r = 1, r <= Length[colorings], r++,
  series =
  series +
  x^Part[Part[colorings, r], 1]*
  y^Part[Part[colorings, r], 2]*
  z^Part[Part[colorings, r], 3];
]
series

```

And we get the first few terms of the generating series:

$$\begin{aligned}
& x + xy + xz + 3xyz + 3x^2yz + xy^2z + 3x^2y^2z + x^3y^2z + \\
& xyz^2 + 3x^2yz^2 + x^3yz^2 + xy^2z^2 + 9x^2y^2z^2 + 9x^3y^2z^2 + x^4y^2z^2 + \\
& 3x^2y^3z^2 + 9x^3y^3z^2 + 3x^2y^2z^3 + 9x^3y^2z^3 + 3x^2y^3z^3
\end{aligned}$$

The term $9x^3y^2z^3$, for example, means that for the action of $\begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix} \in \mathbb{Z}_3$ on $\mathbb{C}[x, y]$, there are 9 Young diagrams with the coloring $3\rho_0 + 2\rho_1 + 3\rho + 2$. Indeed, the first several terms of our generating series for this action match those found in [6].

STUDENT BIOGRAPHIES

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