The Minnesota Journal of Undergraduate Mathematics

Chord Inverses of Real-Valued Parametric Functions

Lawrence Dongilli and Dr. Michael McClendon

University of Central Oklahoma

The Minnesota Journal of Undergraduate Mathematics

Volume 5 (2019-2020 Academic Year)

Sponsored by
School of Mathematics
University of Minnesota
Minneapolis, MN 55455

The Minnesota Journal of Undergraduate Mathematics

Volume 5 (2019-2020 Academic Year)

Chord Inverses of Real-Valued Parametric Functions

Lawrence Dongilli* and Dr. Michael McClendon

University of Central Oklahoma

Abstract. It is commonly known from elementary geometry that if a point P is exterior to a circle, there are exactly two lines tangent to the circle which intersect at P. Consequently, the points of tangency between the circle and these lines identify a specific chord of the circle, which in turn identifies a distinct line in the plane. If we let P traverse some parametric function $f(t)$, we obtain a series of chord-containing lines which correspond to a series of points belonging to $f(t)$. In this paper, we treat these lines as the tangent lines of a parametric function $g(t)$ which we call the *chord inverse* of $f(t)$. We also derive chord inverses of various functions and discuss both their general and specific properties.

1. Defining the Chord Inverse

Let us suppose that the twice differentiable parametric function $f(t) = (x(t), y(t))$ does not pass through the interior of the unit circle U and that the point $P = (a, b)$ is on $f(t)$.

We know from the geometry of the circle that there exist exactly two lines through P that are tangent to the unit circle. If we draw a line through the points of tangency between these two lines and the unit circle, we obtain the construction in Figure 1 which we call the *chord line* of P.

Consider the family of all chord lines constructed in this manner for every point P belonging to $f(t)$. We propose that there exists some parametric function $g(t)$ that we call the *chord inverse* of $f(t)$ which is tangent to every line in this family. Consequently, the chord inverse can be thought of as the envelope of a specially constructed family of linear curves in the plane. [4]

In order to study the properties of the chord inverse, we first derive a transformation which maps an arbitrary, twice differentiable function $f(t)$ to its chord inverse $g(t)$. To start, suppose that the two lines through P are tangent to the unit circle at A and B. Our first step is to find an expression for the chord line L in terms of a and b.

We begin by noticing that since $\overleftrightarrow{AP}$ and $\overleftrightarrow{BP}$ are tangent to U, the radii $\overline{OA}$ and $\overline{OB}$ are perpendicular to these lines at A and B. Consequently, the right triangles $\triangle OAP$ and

* *Corresponding author*

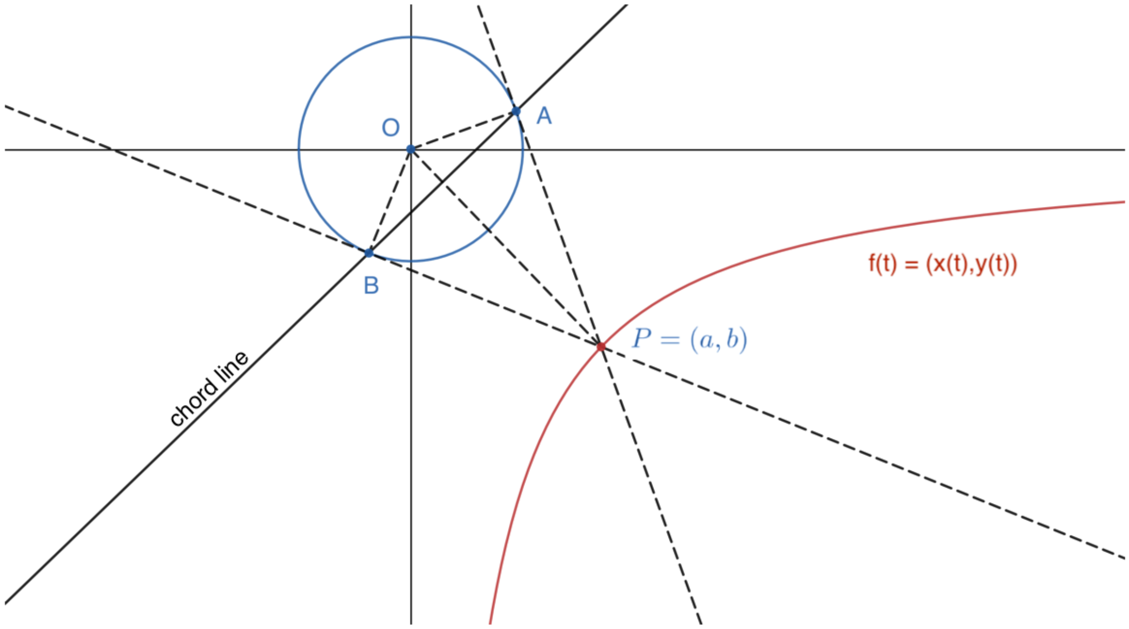


FIGURE 1. The chord line of P

$\triangle OBP$ are congruent. Since the segment $\overline{OP}$ has a length of $\sqrt{a^2+b^2}$ and the radii $\overline{OA}$ and $\overline{OB}$ have lengths of 1, then

$$\overline{PA} = \overline{PB} = \sqrt{a^2+b^2-1}$$

by the Pythagorean Theorem. Since $\overline{PA} = \overline{PB}$, there exists a circle centered at P with radius $\sqrt{a^2+b^2-1}$ which intersects U at A and B as shown in Figure 2.

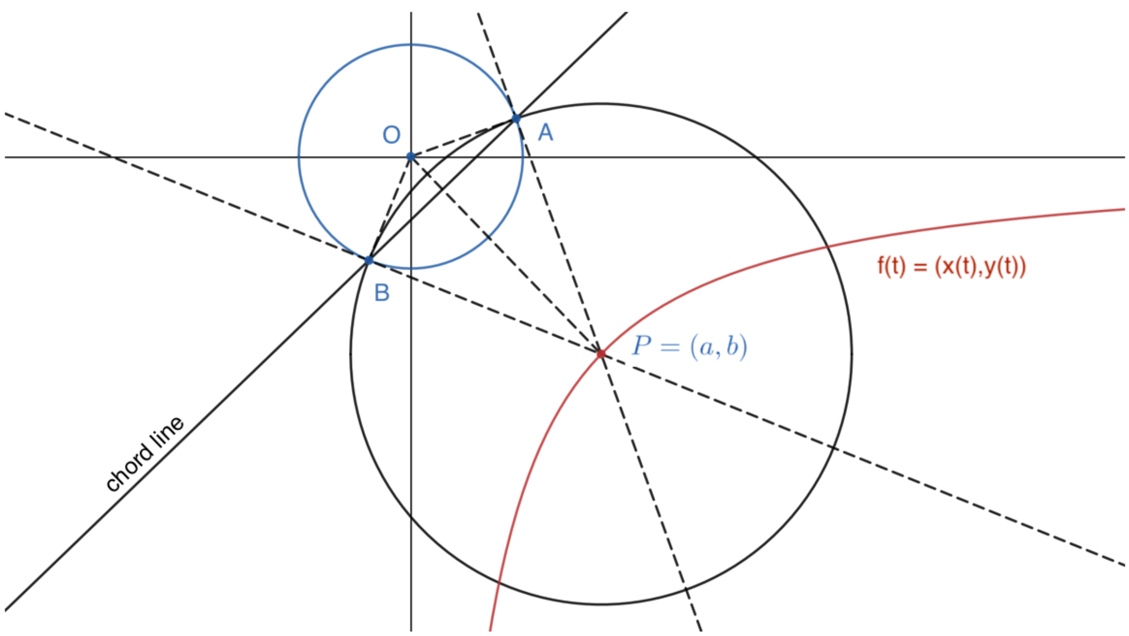


FIGURE 2. The circle centered at P that intersects U at A and B

This implies that A and B form the solution set of the simultaneous equations

$$x^2 + y^2 = 1 \tag{1}$$

$$(x-a)^2 + (y-b)^2 = \left(\sqrt{a^2+b^2-1}\right)^2. \tag{2}$$

Expanding (1) and subtracting (2) from it yields the equation $1 = ax + by$. Since A and B satisfy $1 = ax+by$, every point on the chord line containing $\overline{AB}$ must also satisfy $1 = ax+by$. That is, the equation of the chord line L in terms of a and b is

$$L: \; y = -\frac{a}{b}x + \frac{1}{b}.$$

Since $P = (a,b)$ is on $f(t)$, it follows for some t that $a = x(t)$ and $b = y(t)$. Thus, we know the equation of the chord line L in terms of $x(t)$ and $y(t)$. This allows us to use the Butler Transformation

$$\mathbf{B}(f(t)) = \mathbf{B}(x(t), y(t)) = \left(-\frac{y'(t)}{x'(t)}, y(t) - x(t)\left(\frac{y'(t)}{x'(t)}\right)\right)$$

to determine the chord inverse $g(t)$ of the parametric function $f(t)$. [1][3]

Before we continue, note that:

- The x-component of $\mathbf{B}(f(t))$ is the negative slope of $f(t)$'s tangent line at t.
- The y-component of $\mathbf{B}(f(t))$ is the y-intercept of $f(t)$'s tangent line at t.

This latter fact may be more difficult to see at a glance, so observe first that the slope of $f(t) = (x(t), y(t))$ at t is $y'(t)/x'(t)$. Hence, the equation of $f(t)$'s tangent line through $(x(t), y(t))$ is

$$y - y(t) = \frac{y'(t)}{x'(t)}(x - x(t)).$$

We find the y-intercept by setting $x = 0$ and solving for y:

$$\begin{aligned} y - y(t) &= \frac{y'(t)}{x'(t)}(0 - x(t)) \\ y &= y(t) - x(t)\left(\frac{y'(t)}{x'(t)}\right), \end{aligned}$$

which is the y-component of the Butler transformation.

The Butler Transformation is useful for our purposes because when it is applied twice, the original curve reappears, albeit reflected about the y-axis. That is,

$$\mathbf{B}\left(-\frac{y'(t)}{x'(t)}, y(t) - x(t)\left(\frac{y'(t)}{x'(t)}\right)\right) = (-x(t), y(t)).$$

This fact can be demonstrated by letting $X(t) = -\frac{y'(t)}{x'(t)}$ and letting $Y(t) = y(t) - x(t)\left(\frac{y'(t)}{x'(t)}\right)$. Then

$$\mathbf{B}(X(t), Y(t)) = \left(-\frac{Y'(t)}{X'(t)}, Y(t) - X(t)\left(\frac{Y'(t)}{X'(t)}\right)\right).$$

Since $x(t)$ and $y(t)$ are twice differentiable, taking the appropriate derivatives yields

$$X'(t) = -\frac{y''(t)x'(t) - y'(t)x''(t)}{(x'(t))^2}$$
$$Y'(t) = -x(t)\left(\frac{y''(t)x'(t) - y'(t)x''(t)}{(x'(t))^2}\right),$$

so substitution and simplification shows that

$$\mathbf{B}(\mathbf{B}(x(t), y(t))) = \mathbf{B}(X(t), Y(t)) = (-x(t), y(t)).$$

Therefore, given the slopes and y-intercepts of the tangent lines to some unknown parametric function $g(t)$, the Butler Transformation will reconstruct a reflection of $g(t)$.

Recall that each chord line L of the form

$$L: \ y = -\frac{x(t)}{y(t)}x + \frac{1}{y(t)}$$

is a tangent line to the chord inverse of $f(t) = (x(t), y(t))$. Given the Butler Transformation's reconstructive property which we have just demonstrated, we can therefore use L's slope and y-intercept to obtain a reflection of the chord inverse $g(t)$.

Suppose $g(t) = (u(t), v(t))$ is the chord inverse of $f(t) = (x(t), y(t))$. Let $X(t) = \frac{x(t)}{y(t)}$ denote the negative slope of L and let $Y(t) = \frac{1}{y(t)}$ denote L's y-intercept. Then

$$\mathbf{B}(X(t), Y(t)) = \left(-\frac{Y'(t)}{X'(t)}, Y(t) - X(t)\left(\frac{Y'(t)}{X'(t)}\right)\right) = (-u(t), v(t)).$$

Taking the appropriate derivatives shows that

$$X'(t) = \frac{x'(t)y(t) - x(t)y'(t)}{(y(t))^2}$$
$$Y'(t) = -\frac{y'(t)}{(y(t))^2}.$$

Subsequent substitution and simplification shows that

$$\mathbf{B}(X(t), Y(t)) = \left(-\frac{-\frac{y'(t)}{(y(t))^2}}{\frac{x'(t)y(t)-x(t)y'(t)}{(y(t))^2}}, \frac{1}{y(t)} - \frac{x(t)}{y(t)}\left(\frac{-\frac{y'(t)}{(y(t))^2}}{\frac{x'(t)y(t)-x(t)y'(t)}{(y(t))^2}}\right)\right)$$
$$= \left(\frac{y'(t)}{x'(t)y(t) - x(t)y'(t)}, \frac{x'(t)}{x'(t)y(t) - x(t)y'(t)}\right)$$

Since $\mathbf{B}(X(t), Y(t)) = (-u(t), v(t))$, we have the following result.

Proposition 1.1. *Let $f(t) = (x(t), y(t))$ be a twice differentiable function of t on some domain. Then the* chord inverse *of $f(t)$ is given by the transformation*

$$\mathbf{C}(f(t)) = \mathbf{C}(x(t), y(t)) = \left(\frac{-y'(t)}{x'(t)y(t) - x(t)y'(t)}, \frac{x'(t)}{x'(t)y(t) - x(t)y'(t)}\right).$$

Therefore, given any twice differentiable function $f(t)$, we can obtain its chord inverse $g(t)$ using the transformation $\mathbf{C}$, which was our stated goal at the beginning of this section. Note that although we could have derived the chord inverse by applying the more general methods used to determine envelopes to families of curves, we were able to avoid many of the complications introduced by such methods since the members of our family of curves are exclusively linear. [1][3]

We now turn from our initial derivation of $\mathbf{C}$ to a brief overview of the results when it is applied to several common functions. Starting with lines, circles, and parabolas, we calculate and observe both general and specific examples to better understand the behaviors and properties of chord inverses.

2. The chord inverse in action.

2.1. **Chord inverses of lines.** Suppose that the line $y = mx + b$ lies exterior to the unit circle. We can parametrize this line as

$$f(t) = (t, mt + b),$$

Applying $\mathbf{C}$, we find

$$\mathbf{C}(t, mt + b) = \left(\frac{-m}{(mt + b) - (mt)}, \frac{1}{(mt + b) - (mt)}\right) = \left(-\frac{m}{b}, \frac{1}{b}\right),$$

which is a fixed point. This implies that if $f(t)$ is a line, then the chord lines of every point P on $f(t)$ will contain a common point. As a specific example, let $f(t) = (t, -t + 2)$. Then

$$\mathbf{C}(t, -t + 2) = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Figures 3 and 4 on page 6 show some of these chord lines and their behavior as P traverses $f(t)$.

2.2. **Chord inverses of centered circles.** Consider a circle of radius $r \geq 1$ centered at the origin. We can parametrize this circle as

$$f(t) = (r\cos(t), r\sin(t)),\ r \geq 1.$$

Applying $\mathbf{C}$, we find

$$\mathbf{C}(r\cos(t), r\sin(t)) = \left(\frac{-r\cos(t)}{-r^2\sin^2(t) - r^2\cos^2(t)}, \frac{-r\sin t}{-r^2\sin^2(t) - r^2\cos^2(t)}\right) = \left(\frac{\cos(t)}{r}, \frac{\sin(t)}{r}\right)$$

which is a circle of radius $\frac{1}{r}$ centered at the origin. Notice here that $\mathbf{C}(r\cos(t), r\sin(t))$ is exactly equivalent to the unit circle inversion of $f(t)$, an important fact which will reappear later in our analysis of the chord inverse. [2] As an example, let $f(t) = (2\cos(t), 2\sin(t))$. Then

$$\mathbf{C}(2\cos(t), 2\sin(t)) = \left(\frac{\cos(t)}{2}, \frac{\sin(t)}{2}\right)$$

as shown in Figure 5 on page 7.

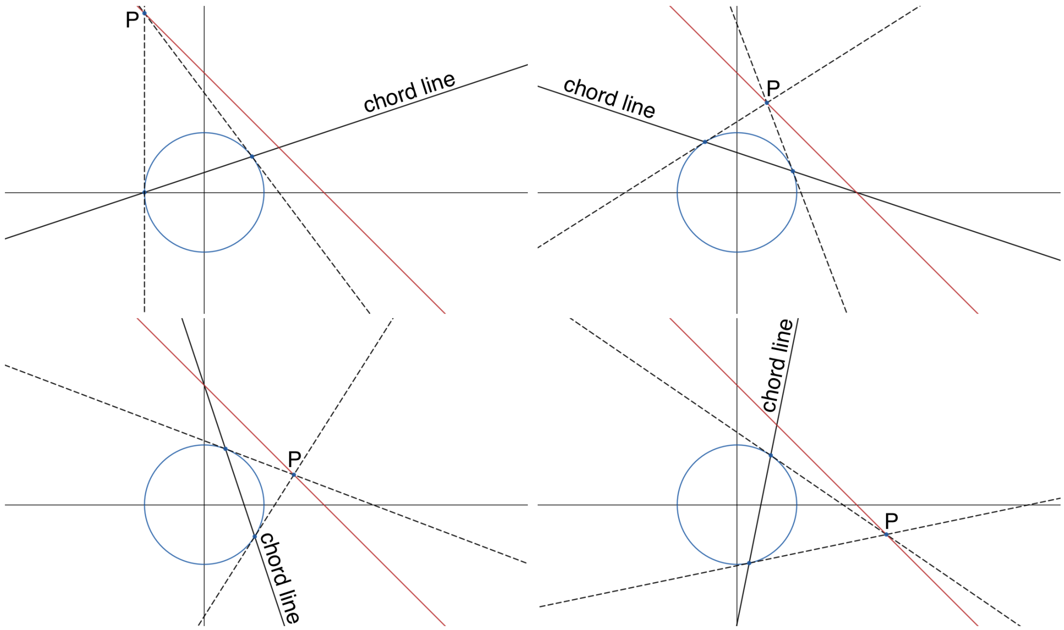


Figure 3. Sample points P on $f(t)$ with their chord lines

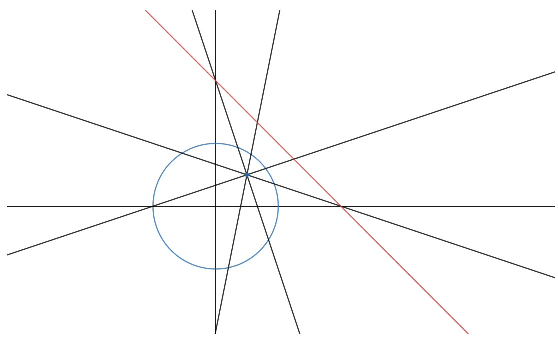


Figure 4. Four chord lines of $f(t)$ viewed simultaneously, intersecting at $(\frac{1}{2}, \frac{1}{2})$

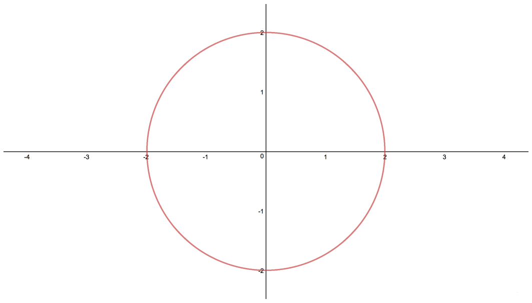
2.3. **The chord inverse of a parabola.** Let $f(t)$ be the parabola $f(t) = \left(t, t^2 + \frac{3}{2}\right)$, no point of which lies within the interior of the unit circle. Then

$$\mathbf{C}\left(t, t^2 + \frac{3}{2}\right) = \left(\frac{-2t}{\left(t^2 + \frac{3}{2}\right) - t(2t)}, \frac{1}{\left(t^2 + \frac{3}{2}\right) - t(2t)}\right) = \left(\frac{-4t}{3 - 2t^2}, \frac{2}{3 - 2t^2}\right)$$

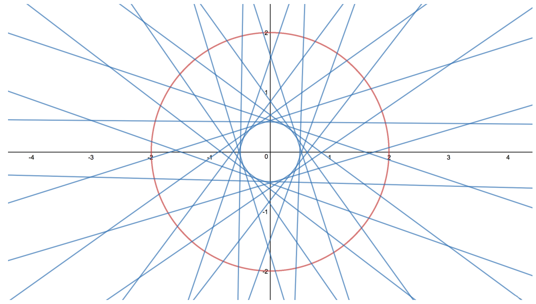
as shown in Figure 6 on page 7.

$\mathbf{C}\left(t, t^2 + \frac{3}{2}\right)$ appears to be a hyperbola. We can check this by letting

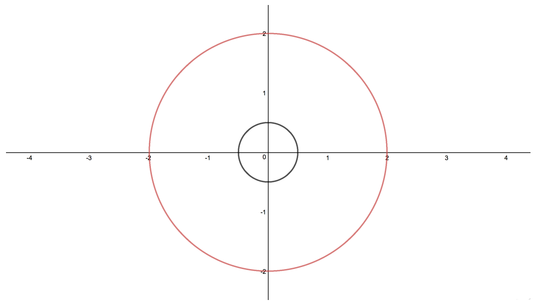
$$(x, y) = \left(\frac{-4t}{3 - 2t^2}, \frac{2}{3 - 2t^2}\right).$$



(A) $f(t) = (2\cos(t), 2\sin(t))$

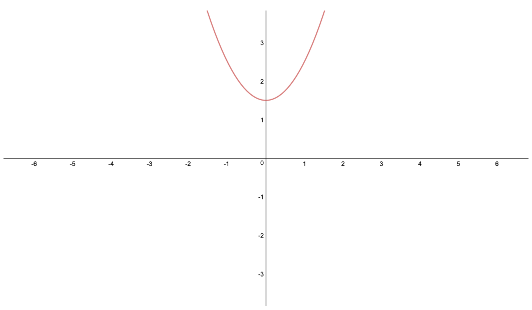


(B) $f(t)$ and its chord lines

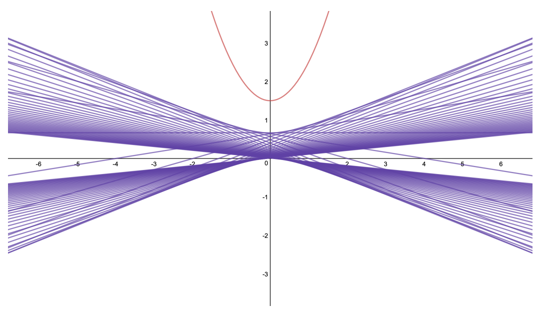


(C) $f(t)$ and its chord inverse

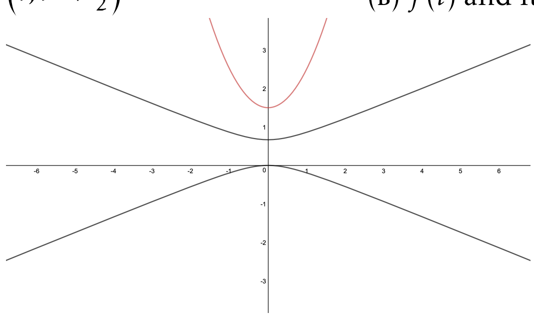
FIGURE 5. The geometry of $\mathbf{C}(2\cos(t), 2\sin(t))$



(A) $f(t) = \left(t, t^2 + \frac{3}{2}\right)$



(B) $f(t)$ and its chord lines



(C) $f(t)$ and its chord inverse

FIGURE 6. The geometry of $\mathbf{C}\left(t, t^2 + \frac{3}{2}\right)$

We therefore find that

$$\begin{aligned} y &= \frac{2}{3-2t^2} \\ 3-2t^2 &= \frac{2}{y} \\ t^2 &= \frac{3}{2} - \frac{1}{y} \\ t &= \pm\sqrt{\frac{3}{2} - \frac{1}{y}}. \end{aligned}$$

Substitution yields

$$\begin{aligned} x &= \frac{-4\left(\pm\sqrt{\frac{3}{2} - \frac{1}{y}}\right)}{3-2\left(\frac{3}{2} - \frac{1}{y}\right)} \\ x^2 &= 4y^2\left(\frac{3}{2} - \frac{1}{y}\right) \\ x^2 + \frac{2}{3} &= 6\left(y - \frac{1}{3}\right)^2, \end{aligned}$$

finally giving

$$6\left(y - \frac{1}{3}\right)^2 - x^2 = \frac{2}{3}.$$

This hyperbolic equation is equivalent to $\mathbf{C}\left(t, t^2 + \frac{3}{2}\right)$, which confirms our earlier observation.

3. Chord Inverses from a Different Point of View

Until now we have dealt strictly with functions that lie exterior to the unit circle. Our selectiveness here stems from the fact that no line tangent to the unit circle can pass through a point within its interior, by the definition of tangency and the properties of the circle. As such, our geometric motivation for the chord inverse makes no sense for points and curves which occupy this interior space.

It is with some surprise, then, that we find that most functions which pass through the unit circle have well-defined chord inverses. For example, let $f(t) = (t, t^2)$. This parabola passes through the unit circle, but its chord inverse

$$\mathbf{C}(t, t^2) = \left(\frac{-2t}{(t^2) - t(2t)}, \frac{1}{(t^2) - t(2t)}\right) = \left(\frac{2}{t}, \frac{1}{-t^2}\right)$$

clearly exists. This result implies that our initial geometric motivation for the chord inverse is merely one interpretation of a more general principle. We concern ourselves next with determining this principle.

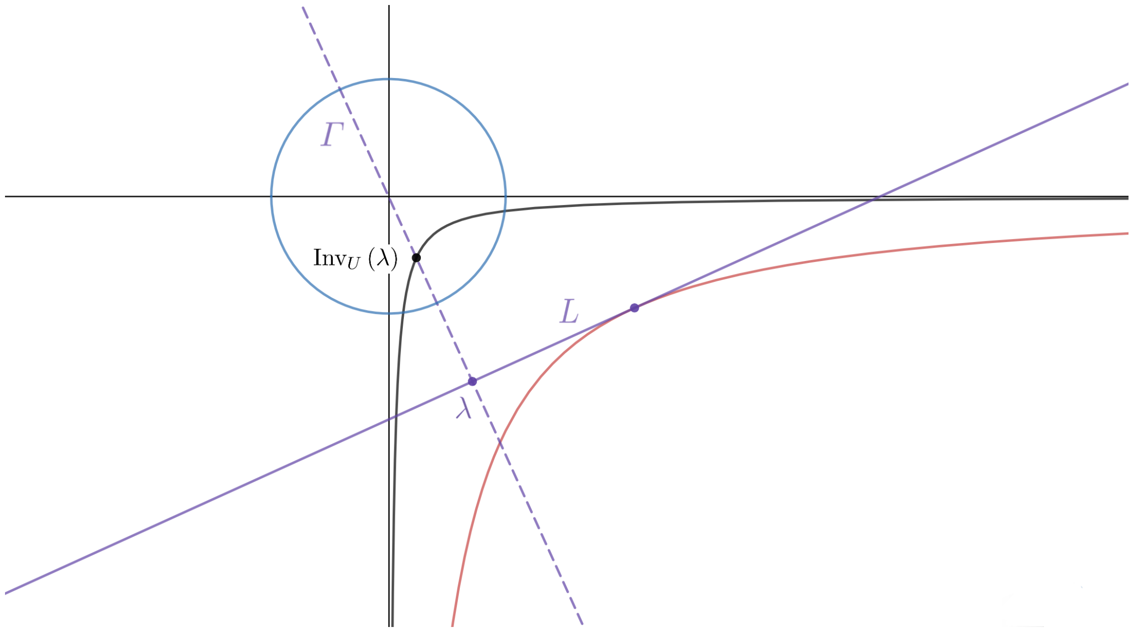


Figure 7. The geometric construction of Proposition 3.1

Recall that the chord inverse of the line $f(t) = (t, -t+2)$ is the point $(\frac{1}{2}, \frac{1}{2})$. If $(x, y) \neq (0, 0)$ is a point in the plane, then its unit circle (U) inversion $\mathrm{Inv}_U(x, y)$ is the point

$$\mathrm{Inv}_U(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right).[2]$$

Now, suppose that a line Γ is constructed which passes through the origin and the point $\lambda = (1, 1)$. Then Γ is perpendicular to $f(t)$ at λ and passes through $(\frac{1}{2}, \frac{1}{2})$. In this instance, we see that

$$\mathrm{Inv}_U(\lambda) = \mathbf{C}(t, -t+2).$$

If we view the line $f(t)$ in this example as being its own tangent line, we arrive at an alternate geometric approach to the chord inverse which is defined both inside and outside the unit circle U.

Proposition 3.1. *Let the line L be a tangent line to the function $f(t)$ at t_0 which does not intersect the origin. Let the line Γ be the line passing through the origin such that Γ is perpendicular to L, and denote the intersection point between Γ and L by $\lambda = (c, d)$. Then*

$$Inv_U(\lambda) = \mathbf{C}(f(t_0)).$$

Proof. Let the line L which does not intersect the origin be tangent to the curve $f(x(t), y(t))$ at the point (a, b). Select t_0 so that $x(t_0) = a$ and $y(t_0) = b$. Let

$$\frac{y'(t_0)}{x'(t_0)} = m.$$

Then the line L is given by

$$L: \ y = m(x-a) + b.$$

Let the line Γ be the line passing through the origin which is perpendicular to L. Then the line Γ is given by

$$\Gamma: \ y = -\frac{x}{m}.$$

Let (c,d) be the point of intersection between L and Γ. Then by substitution, we have

$$\begin{aligned} -\frac{c}{m} &= m(c-a)+b \\ c &= \frac{m(ma-b)}{1+m^2}. \end{aligned}$$

We also have, by substitution, that

$$\begin{aligned} d &= m(-md-a)+b \\ &= \frac{b-ma}{1+m^2}. \end{aligned}$$

Therefore, the point (c,d) is given by

$$(c,d) = \left(\frac{m(ma-b)}{1+m^2}, \frac{b-ma}{1+m^2}\right).$$

The inversion of any point (x,y) across the unit circle U is given by

$$\mathrm{Inv}_U(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right).$$

Therefore, the inversion of (c,d) across the unit circle U is given by

$$\begin{aligned} \mathrm{Inv}_U(c,d) &= \left(\frac{\frac{m(ma-b)}{1+m^2}}{\left(\frac{m(ma-b)}{1+m^2}\right)^2 + \left(\frac{b-ma}{1+m^2}\right)^2}, \frac{\frac{b-ma}{1+m^2}}{\left(\frac{m(ma-b)}{1+m^2}\right)^2 + \left(\frac{b-ma}{1+m^2}\right)^2}\right) \\ &= \left(\frac{-m}{b-ma}, \frac{1}{b-ma}\right). \end{aligned}$$

Since $\frac{y'(t_0)}{x'(t_0)} = m$, $x(t_0) = a$, and $y(t_0) = b$, this point can be written as

$$\begin{aligned} \left(\frac{-m}{b-ma}, \frac{1}{b-ma}\right) &= \left(\frac{-\frac{y'(t_0)}{x'(t_0)}}{y(t_0) - x(t_0)\left(\frac{y'(t_0)}{x'(t_0)}\right)}, \frac{1}{y(t_0) - x(t_0)\left(\frac{y'(t_0)}{x'(t_0)}\right)}\right) \\ &= \left(\frac{-y'(t_0)}{x'(t_0)y(t_0) - x(t_0)y'(t_0)}, \frac{x'(t_0)}{x'(t_0)y(t_0) - x(t_0)y'(t_0)}\right), \end{aligned}$$

which equals $\mathbf{C}(f(t_0))$, the chord inverse of $f(t)$ evaluated at t_0. $\square$

Not only is this definition indifferent to the location of $f(t)$ with respect to the unit circle, it also clarifies what $\mathbf{C}$ is actually doing when it transforms $f(t)$. As a result, we are now free to examine the chord inverses of functions which pass through the unit circle.

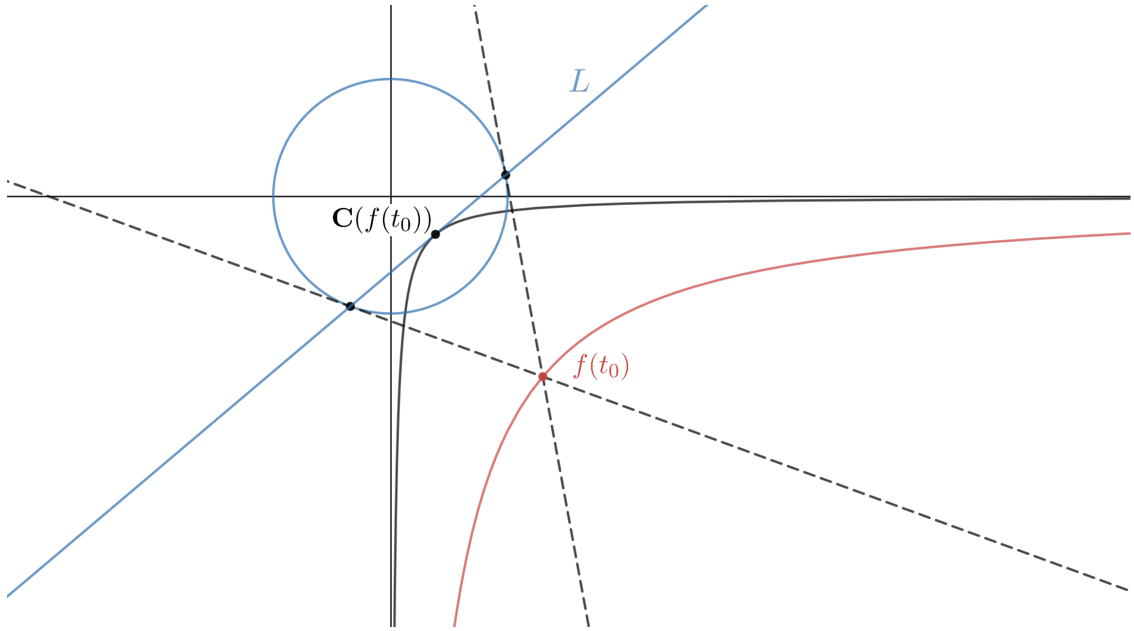


Figure 8. The geometric construction of Proposition 3.2

In addition to removing restrictions on which functions can be chosen, Proposition 3.1 implies that every tangent line of $f(t)$ corresponds to a single point $\mathbf{C}(f(t_0))$ on $\mathbf{C}(f(t))$. A similar relationship exists between $\mathbf{C}(f(t_0))$ and the chord line of $f(t_0)$.

Proposition 3.2. *Let $f(t)$ have chord inverse $\mathbf{C}(f(t))$. If the line L is the chord line of the point $f(t_0)$, then L is tangent to $\mathbf{C}(f(t))$ at the point $\mathbf{C}(f(t_0))$.*

Proof. Suppose $\mathbf{C}(f(t)) = (u(t), v(t))$. Let $L = L_1$ be the chord line corresponding to the point $f(t_0) = (x(t_0), y(t_0))$. Then L_1 is the line

$$L_1: \ y = -\frac{x(t_0)}{y(t_0)}x + \frac{1}{y(t_0)}.$$

Let L_2 be a tangent line to $\mathbf{C}(f(t))$ at $(u(t), v(t))$. Then L_2 is the line

$$L_2: \ y = \frac{v'(t)}{u'(t)}(x - u(t)) + v(t).$$

By the definition of $\mathbf{C}$,

$$u'(t) = \frac{d}{dt}\left(\frac{-y'(t)}{x'(t)y(t) - x(t)y'(t)}\right)$$
$$v'(t) = \frac{d}{dt}\left(\frac{x'(t)}{x'(t)y(t) - x(t)y'(t)}\right).$$

So

$$\begin{aligned} u'(t) &= \frac{y(t)\,(x''(t)y'(t)-x'(t)y''(t))}{(x'(t)y(t)-x(t)y'(t))^2} \\ v'(t) &= \frac{x(t)\,(y''(t)x'(t)-y'(t)x''(t))}{(x'(t)y(t)-x(t)y'(t))^2} \end{aligned}$$

and

$$\frac{v'(t)}{u'(t)} = \frac{x(t)\,(y''(t)x'(t)-y'(t)x''(t))}{y(t)\,(x''(t)y'(t)-x'(t)y''(t))} = -\frac{x(t)}{y(t)}.$$

Therefore L_1 and L_2 have the same slope $m = -\frac{x(t_0)}{y(t_0)}$ at $t = t_0$.

We can find the y-intercept of L_2 when $t = t_0$ by setting $x = 0$. This gives

$$\begin{aligned} y &= \frac{v'(t_0)}{u'(t_0)}(x-u(t_0))+v(t_0) \\ &= -\frac{x(t_0)}{y(t_0)}\left(0-\frac{-y'(t_0)}{x'(t_0)y(t_0)-x(t_0)y'(t_0)}\right)+\left(\frac{x'(t_0)}{x'(t_0)y(t_0)-x(t_0)y'(t_0)}\right) \\ &= \frac{-x(t_0)y'(t_0)}{y(t_0)\,(x'(t_0)y(t_0)-x(t_0)y'(t_0))}+\frac{x'(t_0)y(t_0)}{y(t_0)\,(x'(t_0)y(t_0)-x(t_0)y'(t_0))} \\ &= \frac{1}{y(t_0)}. \end{aligned}$$

Since the y-intercept and the slope of L_1 are the same as the y-intercept and slope of L_2 when $t = t_0$, they are the same line. Since $L_1 = L_2$, then the chord line $L_{t_0} = L_1$ is tangent to $\mathbf{C}(f(t))$ at $(u(t_0), v(t_0)) = \mathbf{C}(f(t_0))$. □

Recall that Proposition 3.1 implies that every tangent line to $f(t)$ corresponds to a single point on $\mathbf{C}(f(t))$. In a similar fashion, Proposition 3.2 implies that every tangent line to $\mathbf{C}(f(t))$ corresponds to a single point on $f(t)$. Together, these two propositions indicate a dichotomous relationship between $f(t)$ and its chord inverse which we formalize in Proposition 3.3.

Proposition 3.3. *Suppose that $\mathbf{C}(f(t))$ is the chord inverse of a twice differentiable curve defined on some domain of t. Then*

$$\mathbf{C}(\mathbf{C}(f(t))) = f(t).$$

Proof. Let $f(x(t), y(t))$ be a twice differentiable curve defined on some domain of t. Then by the definition of $\mathbf{C}$,

$$\mathbf{C}(f(t)) = \left(\frac{-y'(t)}{y(t)x'(t)-x(t)y'(t)}, \frac{x'(t)}{y(t)x'(t)-x(t)y'(t)}\right) = (X(t), Y(t)).$$

Then

$$\mathbf{C}(\mathbf{C}(f(t))) = \left(\frac{-Y'(t)}{Y(t)X'(t)-X(t)Y'(t)}, \frac{X'(t)}{Y(t)X'(t)-X(t)Y'(t)}\right).$$

By the quotient and product rules for derivatives,

$$X'(t) = y(t)\left(\frac{y'(t)x''(t) - x'(t)y''(t)}{(y(t)x'(t) - x(t)y'(t))^2}\right)$$
$$Y'(t) = x(t)\left(\frac{y''(t)x'(t) - x''(t)y'(t)}{(y(t)x'(t) - x(t)y'(t))^2}\right).$$

Substituting, we find

$$Y(t)X'(t) - X(t)Y'(t) = \frac{y'(t)x''(t) - x'(t)y''(t)}{(y(t)x'(t) - x(t)y'(t))^2}.$$

It follows that

$$\frac{-Y'(t)}{Y(t)X'(t) - X(t)Y'(t)} = \frac{-x(t)\left(\frac{y''(t)x'(t)-x''(t)y'(t)}{(y(t)x'(t)-x(t)y'(t))^2}\right)}{\frac{y'(t)x''(t)-x'(t)y''(t)}{(y(t)x'(t)-x(t)y'(t))^2}} = x(t)$$
$$\frac{X'(t)}{Y(t)X'(t) - X(t)Y'(t)} = \frac{y(t)\left(\frac{y'(t)x''(t)-x'(t)y''(t)}{(y(t)x'(t)-x(t)y'(t))^2}\right)}{\frac{y'(t)x''(t)-x'(t)y''(t)}{(y(t)x'(t)-x(t)y'(t))^2}} = y(t).$$

Therefore $\mathbf{C}(\mathbf{C}(f(t))) = f(t)$. □

4. Characteristics of Chord Inverses

One of the direct implications of Proposition 3.1 was that the chord inverse of a tangent line L to $f(t)$ at t_0 yields a point on $\mathbf{C}(f(t))$ provided that L does not contain the origin. Suppose now that L *does* contain the origin. Then L can be generally parameterized as $L = (t, mt)$ and its chord inverse is

$$\mathbf{C}(t, mt) = \left(\frac{-m}{(mt) - (t)(m)}, \frac{1}{(mt) - (t)(m)}\right),$$

which is clearly undefined. Hence, Proposition 3.1 implies that there exists an infinite discontinuity in $\mathbf{C}(f(t))$ at t_0.

Proposition 4.1. *Suppose that the tangent line to some twice differentiable nonlinear function $f(t) = (t, y(t))$ at $f(t_0)$ intersects the origin. If $f(t_0) \neq (0,0)$, then $\mathbf{C}(f(t))$ is asymptotic to least one line.*

Proof. Let $f(t) = (t, y(t))$ be some twice differentiable nonlinear function that does not pass through the origin. Choose t_0 such that the tangent line to $f(t)$ at $f(t_0)$ contains the origin. Then the equation of the tangent line is

$$\begin{aligned} y &= y'(t_0)(x - t_0) + y(t_0) \\ &= y'(t_0)x + (y(t_0) - t_0(y'(t_0))) \end{aligned}$$

Since this tangent line contains the origin,

$$y(t_0) - t_0(y'(t_0)) = 0.$$

Recall that the chord inverse of $f(t)$ is

$$\mathbf{C}(f(t)) = \left(\frac{-y'(t)}{x'(t)y(t) - x(t)y'(t)}, \frac{x'(t)}{x'(t)y(t) - x(t)y'(t)} \right).$$

Since $x(t) = t$ and $x'(t) = 1$,

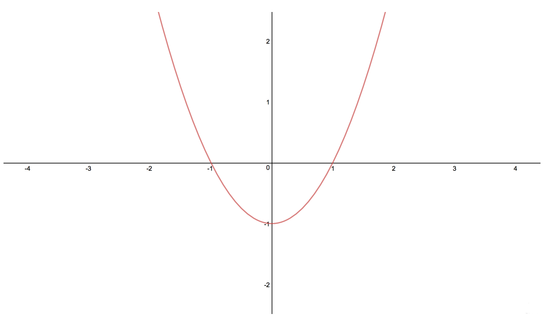
$$\lim_{t \to t_0} \left(\frac{-y'(t)}{x'(t)y(t) - x(t)y'(t)}, \frac{x'(t)}{x'(t)y(t) - x(t)y'(t)} \right) = \left(\lim_{t \to t_0} \frac{-y'(t)}{y(t) - t(y'(t))}, \lim_{t \to t_0} \frac{1}{y(t) - t(y'(t))} \right).$$

Notice here that the y-component increases or decreases without bound as $t \to t_0$ since $y(t_0) - t_0(y'(t_0)) = 0$. Consequently, if the limit of the x-coordinate is equal to a finite constant c, then the chord inverse will be asymptotic to the vertical line $x = c$ at $t = t_0$. Otherwise, if the limit of the x-coordinate is undefined, then the chord inverse will be asymptotic to a non-vertical line at $t = t_0$. □

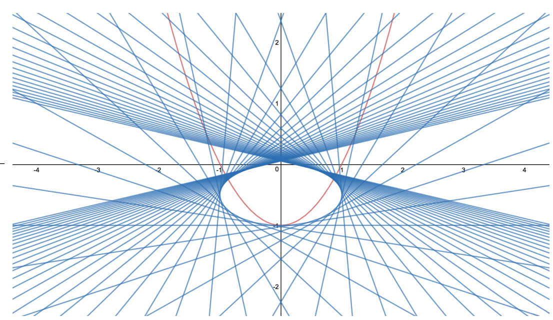
Consider the following examples. First, let $f(t) = (t, t^2 - 1)$. Then

$$\mathbf{C}(t, t^2 - 1) = \left(\frac{-2t}{(t^2 - 1) - (t)(2t)}, \frac{1}{(t^2 - 1) - (t)(2t)} \right) = \left(\frac{2t}{t^2 + 1}, \frac{-1}{t^2 + 1} \right).$$

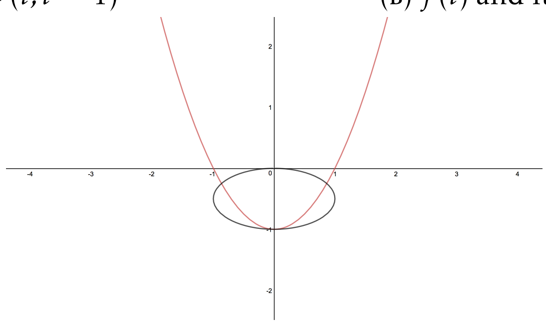
Note that no tangent lines to $f(t)$ intersect the origin. Consequently, $\mathbf{C}(t, t^2 - 1)$ is closed and has no asymptotes as shown in Figure 9 below.



(A) $f(t) = (t, t^2 - 1)$



(B) $f(t)$ and its chord lines



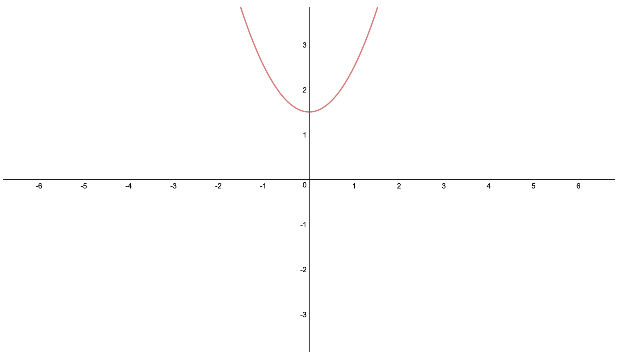
(C) $f(t)$ and its chord inverse

FIGURE 9. The geometry of $\mathbf{C}(t, t^2 - 1)$ and its properties

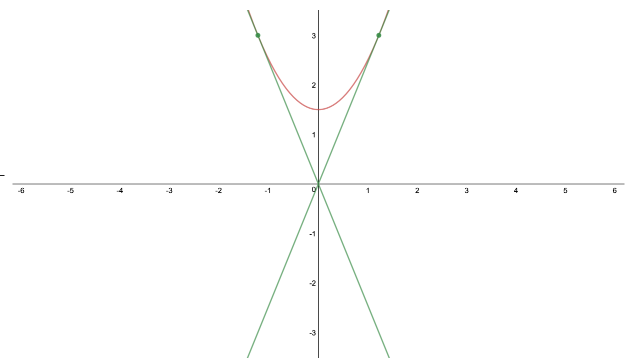
Second, let $f(t) = \left(t, t^2 + \frac{3}{2}\right)$. Recall from earlier that

$$\mathbf{C}\left(t, t^2 + \frac{3}{2}\right) = \left(\frac{-2t}{\left(t^2 + \frac{3}{2}\right) - t(2t)}, \frac{1}{\left(t^2 + \frac{3}{2}\right) - t(2t)}\right) = \left(\frac{-4t}{3 - 2t^2}, \frac{2}{3 - 2t^2}\right)$$

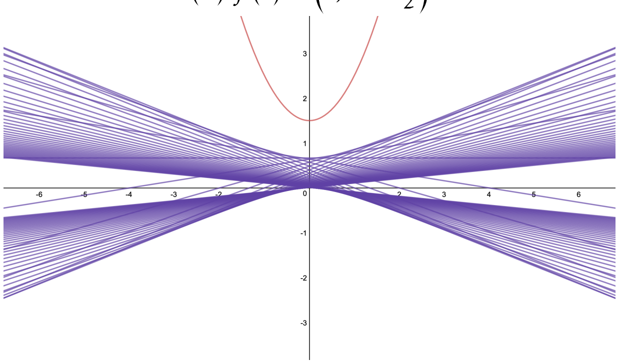
Note that there exist two tangent lines to $f(t)$ which intersect the origin. Consequently, $\mathbf{C}\left(t, t^2 + \frac{3}{2}\right)$ has two oblique asymptotes as shown in Figure 10 below.



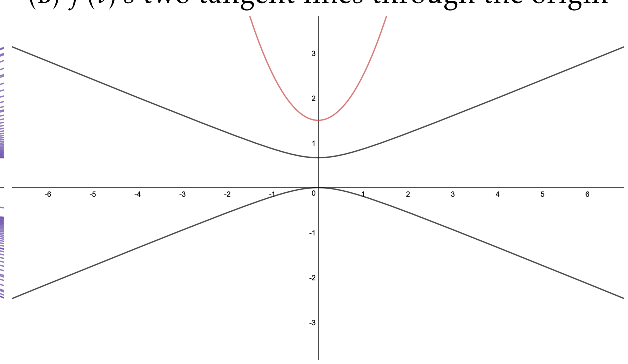
(A) $f(t) = \left(t, t^2 + \frac{3}{2}\right)$



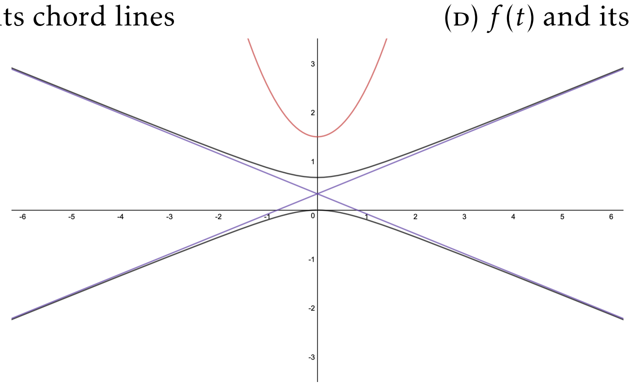
(B) $f(t)$'s two tangent lines through the origin



(C) $f(t)$ and its chord lines



(D) $f(t)$ and its chord inverse



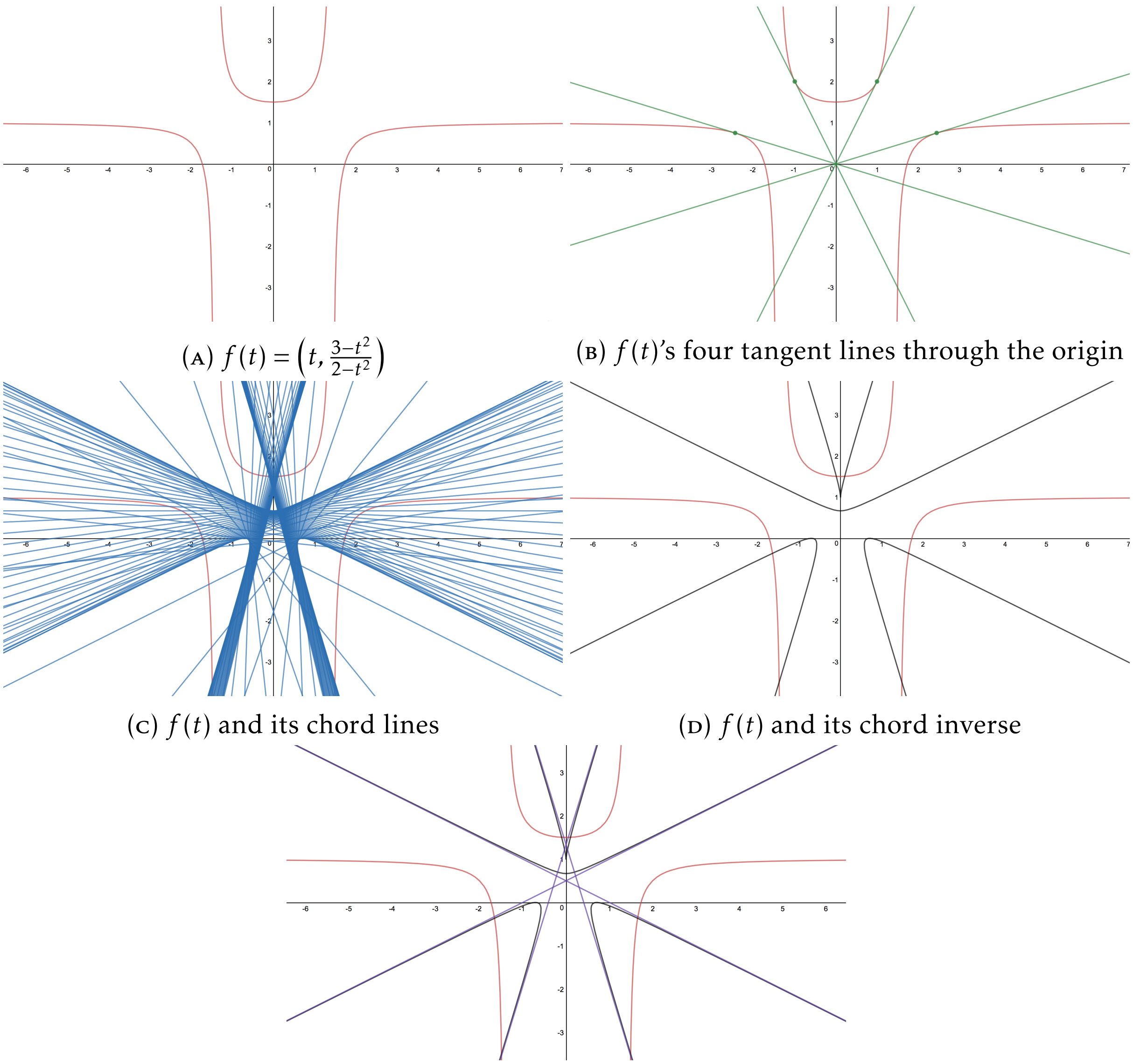
(E) $f(t)$, its chord inverse, and the chord inverse's two asymptotes

FIGURE 10. The geometry of $\mathbf{C}\left(t, t^2 + \frac{3}{2}\right)$ and its properties

Finally, let $f(t) = \left(t, \frac{3-t^2}{2-t^2}\right)$. Then

$$\mathbf{C}\left(t, \frac{3-t^2}{2-t^2}\right) = \left(\frac{-\frac{2t}{(2-t^2)^2}}{\frac{3-t^2}{2-t^2} - t\left(\frac{2t}{(2-t^2)^2}\right)}, \frac{1}{\frac{3-t^2}{2-t^2} - t\left(\frac{2t}{(2-t^2)^2}\right)}\right)$$
$$= \left(\frac{-2t}{(t^2-6)(t^2-1)}, \frac{(2-t^2)^2}{(t^2-6)(t^2-1)}\right)$$

In this case, there exist four tangent lines to $f(t)$ which intersect the origin. As a result, $\mathbf{C}\left(t, \frac{3-t^2}{2-t^2}\right)$ has four oblique asymptotes as shown in Figure 11 below.



(A) $f(t) = \left(t, \frac{3-t^2}{2-t^2}\right)$

(B) $f(t)$'s four tangent lines through the origin

(C) $f(t)$ and its chord lines

(D) $f(t)$ and its chord inverse

(E) $f(t)$, its chord inverse, and the chord inverse's four asymptotes

FIGURE 11. The geometry of $\mathbf{C}\left(t, \frac{3-t^2}{2-t^2}\right)$ and its properties

4.1. **Terminal Points.** Suppose that $f(t)$ is the hyperbola $f(t) = \left(t, \frac{t+1}{t-1}\right)$. Then

$$\begin{aligned}
\mathbf{C}\left(t, \frac{t+1}{t-1}\right) &= \left(\frac{-\left(\frac{-2}{(t-1)^2}\right)}{\frac{t+1}{t-1} - t\left(\frac{-2}{(t-1)^2}\right)}, \frac{1}{\frac{t+1}{t-1} - t\left(\frac{-2}{(t-1)^2}\right)} \right) \\
&= \left(\frac{2}{(t+1)(t-1)+2t}, \frac{(t-1)^2}{(t+1)(t-1)+2t} \right) \\
&= \left(\frac{2}{t^2+2t-1}, \frac{(t-1)^2}{t^2+2t-1} \right)
\end{aligned}$$

Note that the domain of this chord inverse is $\{t \mid t \neq -1 \pm \sqrt{2}\}$. Therefore, the "outer" boundaries of this domain are $\pm\infty$. We evaluate the limit as t approaches the outer boundaries of this domain and find

$$\lim_{t\to\pm\infty} \left(\frac{2}{(t+1)(t-1)+2t}, \frac{(t-1)^2}{(t+1)(t-1)+2t} \right) = (0,1).$$

It appears that $\mathbf{C}\left(t, \frac{t+1}{t-1}\right)$ approaches the point $(0,1)$ from two directions as $t \to \pm\infty$. We call this the chord inverse's *singular terminal point*, a feature which arises from the end behavior of $f(t)$ as it approaches the asymptote $y = 1$ as $t \to \pm\infty$. (see Figure 12 on page 18)

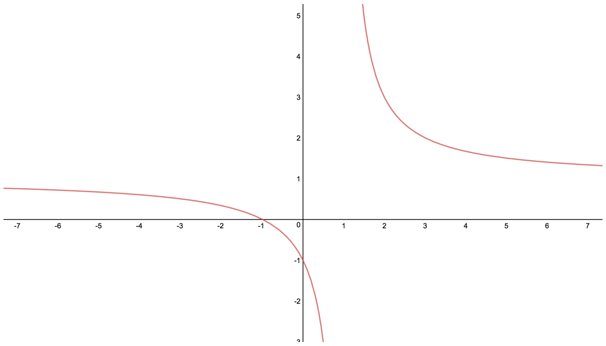
Now, suppose $f(t) = \left(t, \frac{1-\sqrt{t+1}}{\sqrt{t+1}}\right)$. Then

$$\begin{aligned}
\mathbf{C}\left(t, \frac{1-\sqrt{t+1}}{\sqrt{t+1}}\right) &= \left(\frac{-\left(\frac{-1}{2(t+1)^{3/2}}\right)}{\frac{1-\sqrt{t+1}}{\sqrt{t+1}} - t\left(\frac{-1}{2(t+1)^{3/2}}\right)}, \frac{1}{\frac{1-\sqrt{t+1}}{\sqrt{t+1}} - t\left(\frac{-1}{2(t+1)^{3/2}}\right)} \right) \\
&= \left(\frac{1}{2+3t-2(t+1)^{3/2}}, \frac{2(t+1)^{3/2}}{2+3t-2(t+1)^{3/2}} \right).
\end{aligned}$$

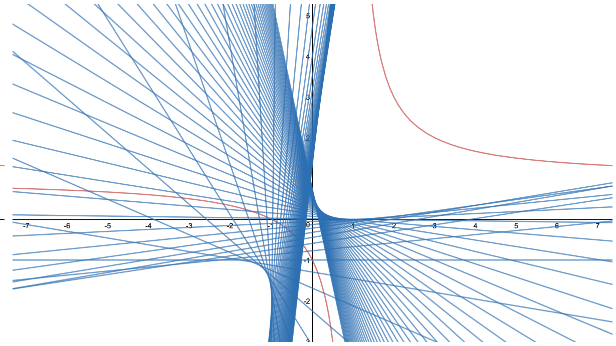
This time, the domain of the chord inverse is $\{t \mid t \geq -1\}$. Evaluating the limits as $\mathbf{C}(f(t))$ approaches the "outer" boundaries -1 and ∞ of its domain yields

$$\begin{aligned}
\lim_{t\to-1^+} \left(\frac{1}{2+3t-2(t+1)^{3/2}}, \frac{2(t+1)^{3/2}}{2+3t-2(t+1)^{3/2}} \right) &= (-1,0) \\
\lim_{t\to\infty} \left(\frac{1}{2+3t-2(t+1)^{3/2}}, \frac{2(t+1)^{3/2}}{2+3t-2(t+1)^{3/2}} \right) &= (0,-1).
\end{aligned}$$

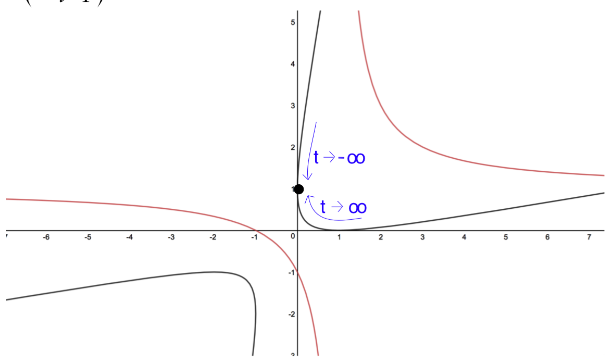
So $\mathbf{C}\left(t, \frac{1-\sqrt{t+1}}{\sqrt{t+1}}\right)$ approaches the points $(-1,0)$ and $(0,-1)$ as $t \to -1^+$ and $t \to \infty$ respectively. We call these points the chord inverse's *twin terminal points*. Since $f(t)$ does not approach the same asymptote on both boundaries of its domain like in our previous example, we obtain two terminal points instead of one. (see Figure 13 on page 19)



(A) $f(t) = \left(t, \frac{t+1}{t-1}\right)$



(B) $f(t)$ and its chord lines



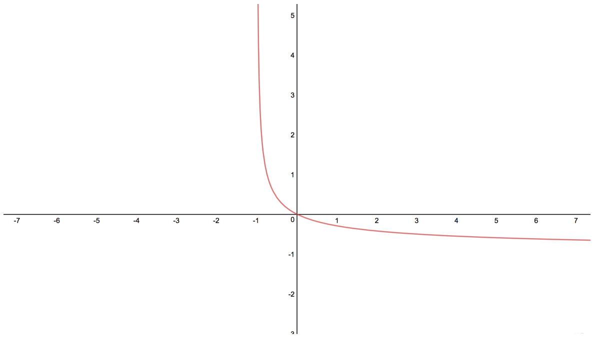
(C) $f(t)$ and its chord inverse (with a singular terminal point at $(0,1)$)

Figure 12. The geometry of $\mathbf{C}\left(t, \frac{t+1}{t-1}\right)$ and its properties

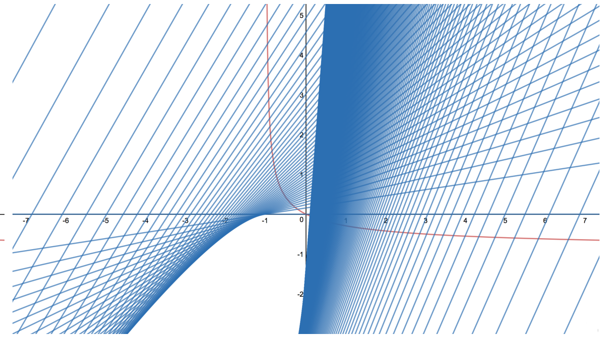
Definition 4.2. Suppose the curve $f(t)$ is defined on some domain of t. Let α and ω denote the outer boundaries of $\mathbf{C}(f(t))$'s domain, and suppose $\lim_{t\to\alpha} \mathbf{C}(f(t))$ and $\lim_{t\to\omega} \mathbf{C}(f(t))$ both exist.

- If $\lim_{t\to\alpha} \mathbf{C}(f(t)) = \lim_{t\to\omega} \mathbf{C}(f(t))$, then $\mathbf{C}(f(t))$ has a *singular terminal point* at $\lim_{t\to\alpha} \mathbf{C}(f(t))$.
- If $\lim_{t\to\alpha} \mathbf{C}(f(t)) \neq \lim_{t\to\omega} \mathbf{C}(f(t))$, then $\mathbf{C}(f(t))$ has *twin terminal points* at $\lim_{t\to\alpha} \mathbf{C}(f(t))$ and $\lim_{t\to\omega} \mathbf{C}(f(t))$.

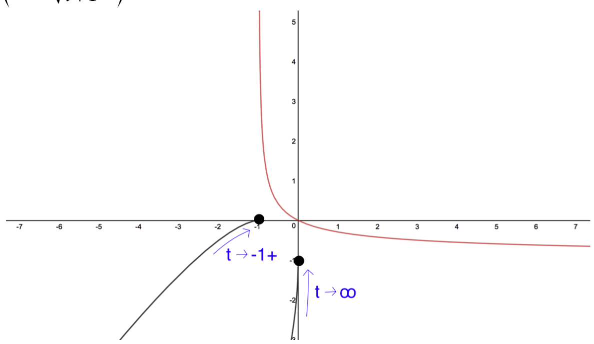
We can interpret this definition geometrically by observing the behavior of tangent lines to $f(t)$ near these boundaries. If $f(t)$ has end behavior governed by an asymptote, then tangent lines to $f(t)$ will approach this asymptote as t approaches the boundary. If this limiting asymptote does not pass through the origin, then the point on $\mathbf{C}(f(t))$ which corresponds to the tangent line for each value of t will approach the chord inverse point of the limiting asymptote. If $f(t)$ approaches the same asymptote on both boundaries of its domain, then $\mathbf{C}(f(t))$ will naturally approach the same point – the singular terminal point of $\mathbf{C}(f(t))$ – from both directions. The geometric intuition for twin terminal points follows naturally from this observation if a separate asymptote governs the end behavior on each outer boundary of $f(t)$'s domain.



(A) $f(t) = \left(t, \frac{1-\sqrt{t+1}}{\sqrt{t+1}}\right)$



(B) $f(t)$ and its chord lines

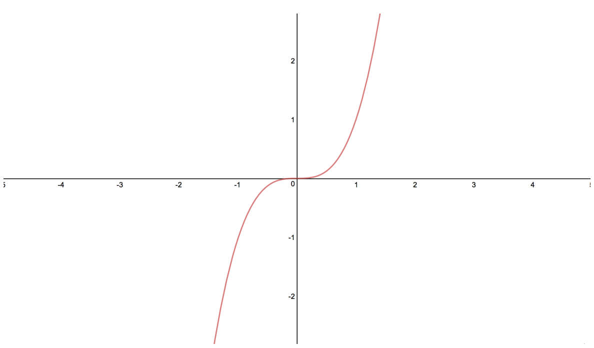


(C) $f(t)$ and its chord inverse (with twin terminal points at $(-1,0)$ and $(0,-1)$)

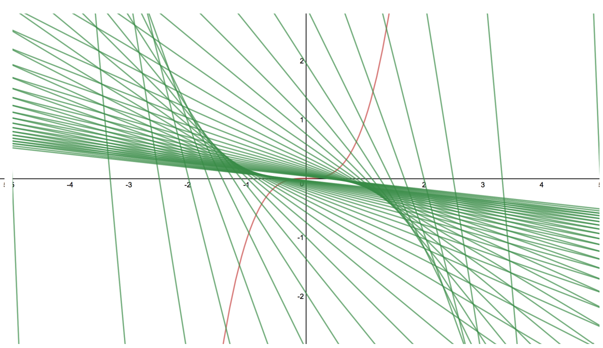
Figure 13. The geometry of $\mathbf{C}\left(t, \frac{1-\sqrt{t+1}}{\sqrt{t+1}}\right)$ and its properties

As a further example of terminal points and their behavior, suppose $f(t)$ is the parametrized cubic polynomial $f(t) = (t, t^3)$. Then

$$\mathbf{C}(t,t^3) = \left(\frac{-3t^2}{t^3 - t(3t^2)}, \frac{1}{t^3 - t(3t^2)}\right) = \left(\frac{3}{2t}, \frac{-1}{2t^3}\right).$$



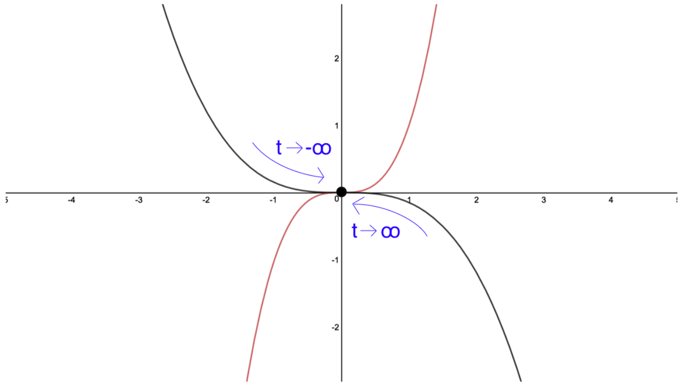
(A) $f(t) = (t, t^3)$



(B) $f(t)$ and its chord lines

Take special note of the fact that the origin is the singular terminal point of $\mathbf{C}(t,t^3)$, since

$$\lim_{t\to\pm\infty}\left(\frac{3}{2t}, \frac{-1}{2t^3}\right) = (0,0).$$



(c) $f(t)$ and its chord inverse (with a singular terminal point at $(0,0)$)

Figure 15. The geometry of $\mathbf{C}(t, t^3)$ and its properties

This leads to the following generalization and proof.

Proposition 4.3. *Let $f(t)$ be the parameterization of a polynomial function with degree $n > 1$. Then $\mathbf{C}(f(t))$ has a singular terminal point at the origin.*

Proof. Let $f(t) = (t, m_n t^n + m_{n-1} t^{n-1} + \cdots + m_1 t + m_0)$. Since $f(t)$ is a polynomial curve, its domain is $\{t \mid t \in \mathbb{R}\}$. Then the components of $\mathbf{C}(f(t))$ are

$$\mathbf{C}_x(f(t)) = \frac{-nm_n t^{n-1} - \cdots - m_1}{(m_n t^n + \cdots + m_0) - (nm_n t^n + \cdots + m_1 t)}$$
$$\mathbf{C}_y(f(t)) = \frac{1}{(m_n t^n + \cdots + m_0) - (nm_n t^n + \cdots + m_1 t)}$$

In both cases the degree of the component's denominator is greater than the degree of the numerator, so therefore the limits of both components are equal to 0 as $t \to \pm\infty$. Therefore, $\mathbf{C}(f(t))$ must have a singular terminal point at the origin. □

4.2. **Terminal Asymptotes.** As seen previously, domain-bounding asymptotes of $f(t)$ that do not pass through the origin determine the terminal points of $\mathbf{C}(f(t))$. Suppose instead that one or more domain-bounding asymptotes to $f(t)$ *do* pass through the origin. In this scenario, one or more of the chord inverse's terminal points are replaced by what we call *terminal asymptotes*.

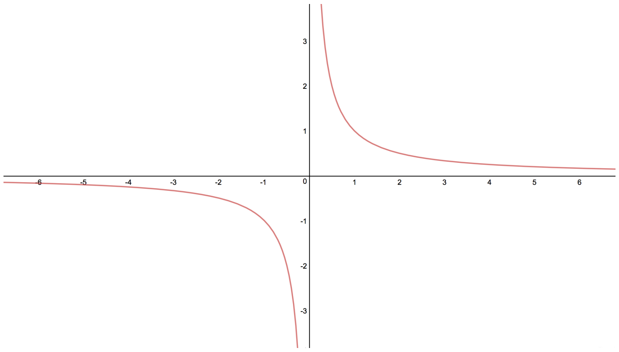
To demonstrate, let $f(t) = \left(t, \frac{1}{t}\right)$. Then

$$\mathbf{C}\left(t, \frac{1}{t}\right) = \left(\frac{-\left(\frac{-1}{t^2}\right)}{\frac{1}{t} - t\left(\frac{-1}{t^2}\right)}, \frac{1}{\frac{1}{t} - t\left(\frac{-1}{t^2}\right)}\right) = \left(\frac{1}{2t}, \frac{t}{2}\right).$$

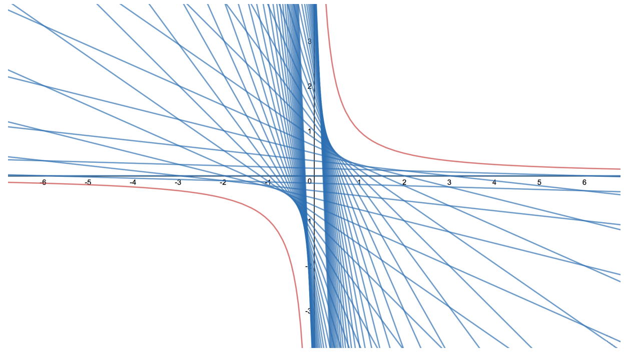
The domain of $\mathbf{C}\left(t, \frac{1}{t}\right)$ is $\{t \mid t \neq 0\}$. When we evaluate the limits at the outer boundaries of the domain, we find

$$\lim_{t \to \pm\infty} \left(\frac{1}{2t}, \frac{t}{2}\right) = \left(0, \lim_{t \to \pm\infty} \frac{t}{2}\right).$$

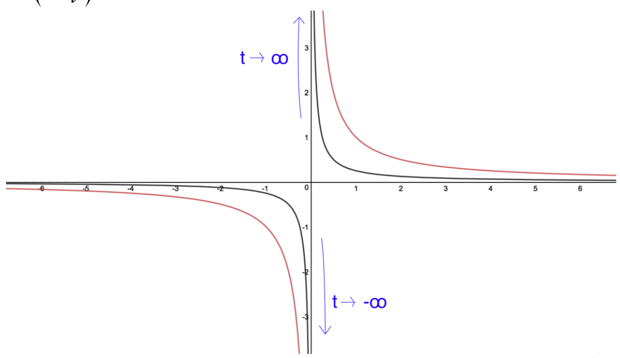
Since the y-component of the limit increases without bound as $t \to \pm\infty$, $\mathbf{C}\left(t, \frac{1}{t}\right)$ approaches the asymptote $x = 0$ on the outer boundaries of its domain. We call this the chord inverse's *singular terminal asymptote*. (see Figure 16 below)



(A) $f(t) = \left(t, \frac{1}{t}\right)$



(B) $f(t)$ and its chord lines



(C) $f(t)$ and its chord inverse (with a singular terminal asymptote at $x = 0$)

FIGURE 16. The geometry of $\mathbf{C}\left(t, \frac{1}{t}\right)$ and its properties

Now, suppose $f(t) = \left(t, \frac{1}{\sqrt{t}}\right)$. Then

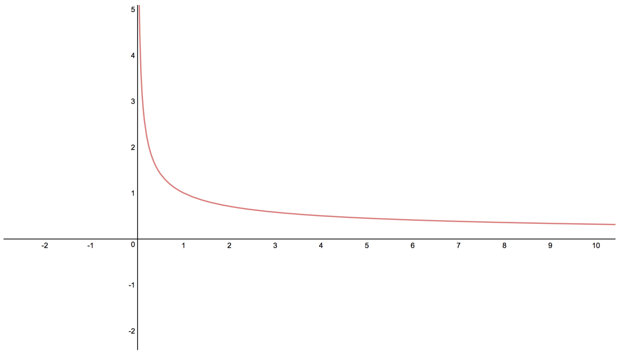
$$\mathbf{C}\left(t, \frac{1}{\sqrt{t}}\right) = \left(\frac{-\left(\frac{-1}{2t^{3/2}}\right)}{\frac{1}{\sqrt{t}} - t\left(\frac{-1}{2t^{3/2}}\right)}, \frac{1}{\frac{1}{\sqrt{t}} - t\left(\frac{-1}{2t^{3/2}}\right)} \right) = \left(\frac{1}{3t}, \frac{2\sqrt{t}}{3}\right).$$

The domain of $\mathbf{C}\left(t, \frac{1}{\sqrt{t}}\right)$ is $\{t \mid t > 0\}$. We evaluate the limits of the chord inverse at the outer boundaries of its domain and find

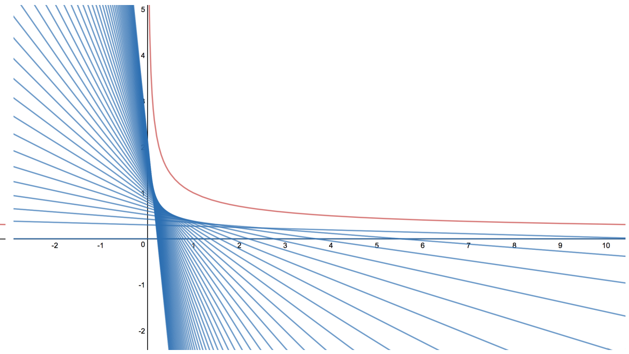
$$\lim_{t\to\infty} \left(\frac{1}{3t}, \frac{2\sqrt{t}}{3}\right) = \left(0, \lim_{t\to\infty} \frac{2\sqrt{t}}{3}\right)$$

$$\lim_{t\to 0^+} \left(\frac{1}{3t}, \frac{2\sqrt{t}}{3}\right) = \left(\lim_{t\to 0^+} \frac{1}{3t}, 0\right).$$

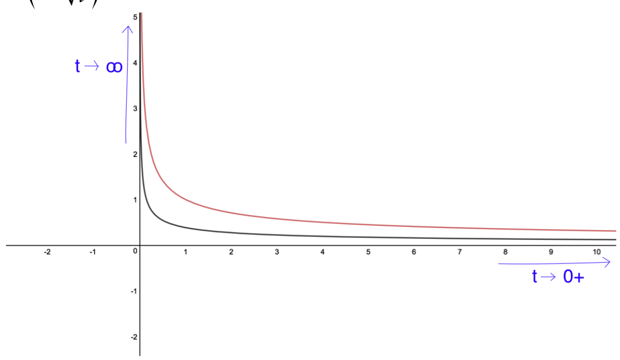
Since the y-component of the first limit increases without bound as $t \to \infty$ and the x-component of second limit increases without bound as $t \to 0^+$, $\mathbf{C}\left(t, \frac{1}{\sqrt{t}}\right)$ approaches the asymptotes $x = 0$ and $y = 0$ on the outer boundaries of its domain. We call these the chord inverse's *twin terminal asymptotes*. (see Figure 17 below)



(A) $f(t) = \left(t, \frac{1}{\sqrt{t}}\right)$



(B) $f(t)$ and its chord lines



(C) $f(t)$ and its chord inverse (with twin terminal asymptotes at $x = 0$ and $y = 0$)

FIGURE 17. The geometry of $\mathbf{C}\left(t, \frac{1}{\sqrt{t}}\right)$ and its properties

Definition 4.4. Suppose that the curve $f(t)$ approaches an origin-intersecting asymptote L_1 as t approaches an outer boundary of $f(t)$'s domain. Then $\mathbf{C}(f(t))$ contains a *terminal asymptote* M.

- If $f(t)$ approaches L_1 on both outer boundaries, then M is a *singular terminal asymptote*.
- If $f(t)$ approaches an origin-containing asymptote $L_2 \neq L_1$ on the opposite boundary of its domain, then $\mathbf{C}(f(t))$ contains two distinct *twin terminal asymptotes* M and N.

5. Conclusion

This introduction to chord inverses barely scratches the surface of a topic filled with possibilities. Many questions remain to be answered – for instance:

- We conjecture that all chord inverse asymptotes are perpendicular to the origin-intersecting tangents or asymptotes with which they are associated.
- How do horizontal, vertical, or oblique asymptotes which are not at the boundaries of $f(t)$'s domain affect $\mathbf{C}(f(t))$?
- If $f(t)$ contains inflection points, then $\mathbf{C}(f(t))$ always contains cusps. What is the precise relationship between these inflection points and cusps?
- What do chord inverses of unbounded periodic functions look like and how do they behave?
- We know that if $f(t)$ is the unit circle $(\cos t, \sin t)$ or the hyperbola $\left(t, \frac{1}{2t}\right)$, then $\mathbf{C}(f(t))$ and $f(t)$ have identical graphs. Furthermore, we observe that this property is invariant under rotation about the origin. Are there any other functions $f(t)$ such that $f(t)$ and $\mathbf{C}(f(t))$ are identical?

6. Acknowledgments

I would like to thank my research advisor, Dr. Michael McClendon, for his help and guidance throughout this project. I would also like to extend my thanks to the anonymous referees of this paper for their insightful and detailed comments concerning this paper's content and presentation. Finally, I would like to acknowledge the team behind `www.desmos.com`, a superb online graphing calculator that assisted greatly with the geometric visualization and analysis of chord inverses.

References

[1] Butler, S. *Tangent Line Transformations*, Coll Math J 34(2), 2003, 105 - 106.
[2] M. Greenberg, *Euclidean and Non-Euclidean Geometries*, 3rd ed., 1994.
[3] Horowitz, A. *Reconstructing a Function from its Set of Tangent Lines*, Am Math Monthly 96(9), 1989, 807 - 813.
[4] Weisstein, Eric W, "*Envelope*", From MathWorld – A Wolfram Web Resource, http://mathworld.wolfram.com/Envelope.html.

Student biographies

Lawrence Dongilli: (*Corresponding author:* ldongilli@uco.edu) Lawrence Dongilli is currently an undergraduate mathematics student at the University of Central Oklahoma. After his projected graduation in 2020, he plans to pursue a master's degree in pure mathematics with research interests in geometry, analysis, artificial intelligence, artificial societies, and social simulation.