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Critical Points of a Family of Complex-Valued Polynomials

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Abstract. For $k,m,n \in \mathbb{N}$, let $P(k,m,n)$ be the family of complex-valued polynomials of the form $p(z) = z^k(z-r_1)^m(z-r_2)^n$ with $|r_1| = |r_2| = 1$. The Gauss-Lucas Theorem guarantees that the critical points of $p \in P(k,m,n)$ will lie in the unit disk. This paper further explores the location and structure of these critical points. When $m = n$, the unit disk contains a *desert region*, $\{z \in \mathbb{C} : |z| < \frac{k}{k+2m}\}$, in which critical points do not occur, and a critical point almost always determines a polynomial uniquely. When $m \neq n$, the unit disk contains two desert regions, and each c is the critical point of at most two polynomials in $P(k,m,n)$.

1. Introduction

The Gauss-Lucas Theorem guarantees that the critical points of a complex-valued polynomial will lie in the convex hull of its roots [4]. For example, if p has three non-collinear roots, then its critical points will lie in the triangle with vertices located at its roots. For $k,m,n \in \mathbb{N}$ ($0 \notin \mathbb{N}$), several recent papers ([3], [2], [1]) have studied critical points of the family of polynomials

$$\mathcal{P}_{k,m,n} = \left\{p : \mathbb{C} \to \mathbb{C} \mid p(z) = (z-1)^k(z-r_1)^m(z-r_2)^n \text{ with } |r_1| = |r_2| = 1\right\}.$$

Critical points of polynomials in $\mathcal{P}_{1,1,1}$ are studied in [3]. For this family of polynomials, the unit disk contains a *desert region*, $\{z \in \mathbb{C} : |z - \frac{2}{3}| < \frac{1}{3}\}$, in which critical points do not occur, and a critical point almost always (with the exception of two points) determines a polynomial uniquely. Critical points of polynomials in $\mathcal{P}_{1,1,2}$ are characterized in [2]. Due to the loss of symmetry in the multiplicities of r_1 and r_2, the unit disk contains two desert regions in which critical points do not occur: $\{z \in \mathbb{C} : |z - \frac{3}{4}| < \frac{1}{4}\}$ and the interior of $2x^4 - 3x^3 + x + 4x^2y^2 - 3xy^2 + 2y^4 = 0$. Furthermore, each c in the unit disk and outside the closure of the desert regions, is the critical point of exactly two polynomials in $\mathcal{P}_{1,1,2}$.

For $k,m,n \in \mathbb{N}$, [1] extends the results of [3] and [2] by characterizing the critical points of polynomials in $\mathcal{P}_{k,m,n}$. When $m = n$, similar to [3], the unit disk contains a single desert region, and a critical point almost always determines a polynomial uniquely. When $m \neq n$,

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similar to [2], the unit disk contains two desert regions, and each c is the critical point of at most two polynomials in $\mathcal{P}_{k,m,n}$.

For $k,m,n \in \mathbb{N}$, this paper investigates a variation of [1] by characterizing the critical points of the family of polynomials

$$P(k,m,n) = \left\{p : \mathbb{C} \to \mathbb{C} \mid p(z) = z^k(z-r_1)^m(z-r_2)^n \text{ with } |r_1| = |r_2| = 1\right\}.$$

As motivating examples, we investigate $P(1,1,1)$ and $P(1,1,2)$, and then use those results to characterize the critical points of polynomials in $P(k,m,m)$ and $P(k,m,n)$.

2. Critical Points

We begin our discussion by introducing some notation. For $\alpha > 0$, we let $\mathcal{O}(\alpha)$ represent the circle centered at the origin with radius α. That is,

$$\mathcal{O}(\alpha) = \{z \in \mathbb{C} \,\big|\, |z| = \alpha\}.$$

A polynomial of the form

$$p(z) = z^k(z-r_1)^m(z-r_2)^n$$

with $|r_1| = |r_2| = 1$ and $k,m,n \in \mathbb{N}$ has $k+m+n-1$ critical points. Differentiation gives

$$p'(z) = z^{k-1}(z-r_1)^{m-1}(z-r_2)^{n-1}q(z)$$

with

$$q(z) = (k+m+n)z^2 - (k(r_1+r_2) + nr_1 + mr_2)z + kr_1r_2.$$

There are $k-1$ critical points at $z=0$, $m-1$ critical points at r_1, $n-1$ critical points at r_2, and two additional critical points in the unit disk.

Definition 2.1. Given $p \in P(k,m,n)$ we say c is a *nontrivial* critical points of p if $q(c) = 0$. We call the remaining critical points *trivial*.

This paper will characterize the nontrivial critical points of polynomials in $P(k,m,n)$. We begin with an example.

Example 2.2. Let $p \in P(k,m,n)$ have a nontrivial critical point at $c \in \mathcal{O}(1)$. By the Gauss Lucas Theorem, c must lie in the convex hull of the roots of $p(z) = z(z-r_1)^n(z-r_2)^m$. That is, $c \in \mathcal{O}(1)$ is in the convex hull of a set containing two points on the unit circle and the origin. This can only happen if $c = r_1$ or $c = r_2$. Furthermore, as c is a nontrivial critical point, we have $c = r_1 = r_2$.

Therefore, $p \in P(k,m,n)$ has a nontrivial critical point at $c \in \mathcal{O}(1)$ if and only if $p(z) = z^k(z-c)^{m+n}$. In this case, $q(z) = (z-c)((k+m+n)z - kc)$, and one can calculate that the second nontrivial critical point is $\frac{kc}{k+m+n} \in \mathcal{O}(\frac{k}{k+m+n})$. To summarize, $c \in \mathcal{O}(1)$ is a nontrivial critical points of $p \in P(k,m,n)$ if and only if $r_1 = r_2$.

A similar argument shows that $c = 0$ will never be a nontrivial critical point of a polynomial in $P(k,m,n)$. We can say even more.

Theorem 2.3. *No polynomial in $P(k,m,n)$ has a nontrivial critical point inside $\mathcal{O}(\frac{k}{k+m+n})$.*

Proof. Suppose c_1 and c_2 are nontrivial critical point of $p(z) = z^k(z-r_1)^m(z-r_2)^n$. Then, c_1 and c_2 are roots of $q(z) = (k+m+n)z^2 - (k(r_1+r_2)+nr_1+mr_2)z + kr_1r_2$ and it follows that $c_1c_2 = \frac{kr_1r_2}{k+m+n}$. As $|r_1| = |r_2| = 1$, $|c_1| \leq 1$ and $|c_2| \leq 1$, we have

$$|c_1| \geq |c_1c_2| = \frac{k}{k+m+n}$$

and the result follows. □

To characterize the nontrivial critical points of a polynomial in $P(k,m,n)$, we investigate the relationship between its roots and nontrivial critical points. Suppose c is a nontrivial critical point of $p(z) = z^k(z-r_1)^m(z-r_2)^n \in P(k,m,n)$. Then,

$$0 = q(c) = (k+m+n)c^2 - (k(r_1+r_2)+nr_1+mr_2)c + kr_1r_2$$

and it follows that

$$r_1 = \frac{(k+m)cr_2 - (k+m+n)c^2}{kr_2 - c(k+n)} \text{ and } r_2 = \frac{(k+n)cr_1 - (k+m+n)c^2}{kr_1 - c(k+m)}.$$

Definition 2.4. Given $c \in \mathbb{C}\backslash\{0\}$, define

$$f_{c,1}(z) = \frac{(k+m)cz - (k+m+n)c^2}{kz - c(k+n)} \text{ and } f_{c,2}(z) = \frac{(k+n)cz - (k+m+n)c^2}{kz - c(k+m)},$$

and let $S_1 = f_{c,1}(\mathcal{O}(1))$ and $S_2 = f_{c,2}(\mathcal{O}(1))$.

The functions $f_{c,1}$ and $f_{c,2}$ are fractional linear transformations with $f_{c,1}(r_2) = r_1$ and $f_{c,2}(r_1) = r_2$. The above work has established the following result.

Theorem 2.5. *Suppose $c \in \mathbb{C}\backslash\{0\}$. Then, $p(z) = z^k(z-r_1)^m(z-r_2)^n \in P(k,m,n)$ has a nontrivial critical point at c if and only if $f_{c,1}(r_2) = r_1$ and $f_{c,2}(r_1) = r_2$.*

When $c \neq 0$, $f_{c,1}$ and $f_{c,2}$ are invertible with $(f_{c,1})^{-1} = f_{c,2}$. Since $f_{c,1}(z)$ and $f_{c,2}(z)$ are fractional linear transformations, they send circles and lines to circles and lines. Therefore, S_1 and S_2 are either circles or lines. An additional result concerning fractional linear transformations [2, page 491], will be of interest.

Theorem 2.6. *A fractional linear transformation T sends the unit circle to the unit circle if and only if*

$$T(z) = \frac{\overline{\alpha}z - \overline{\beta}}{\beta z - \alpha}$$

for some $\alpha, \beta \in \mathbb{C}$ with $|\frac{\alpha}{\beta}| \neq 1$.

Example 2.7. For $c \neq 0$,

$$f_{c,1}(z) = \frac{(k+m)cz - (k+m+n)c^2}{kz - c(k+n)} = \frac{(k+m)z - (k+m+n)c}{\frac{k}{c}z - (k+n)},$$

We recall that $f_{c,1}$ is a fractional linear transformation such that $f_{c,1}(\mathcal{O}(1)) = S(1)$ and Theorem 2.6 implies $S_1 = \mathcal{O}(1)$ if and only if

$$\overline{(k+n)} = u(k+m) \text{ and } \overline{k}/\,\overline{c} = u(k+m+n)c$$

for some $u \in \mathbb{C}$ with $|u| = 1$. Since $k, m, n \in \mathbb{N}$, we then get $k+n = u(k+m)$ which implies $u \in (0, \infty)$. Therefore, $u = 1$ and it follows that $m = n$. The right equation gives $c\overline{c} = \frac{k}{u(k+m+n)}$, so that $|c|^2 = \frac{k}{u(k+m+n)} = u\frac{k}{k+m+n}$. Then $|c|^2 \in \mathbb{R}$ so the right hand side is real-valued as well which implies $u \in \mathbb{R}$. This gives that $|c| = \sqrt{\frac{k}{K+m+n}}$ which is equivalent to $c \in \mathcal{O}(\frac{k}{k+m+n})$. Therefore, $S_1 = \mathcal{O}(1)$ if and only if $m = n$ and $c \in \mathcal{O}\left(\sqrt{\frac{k}{k+2m}}\right)$. Similar calculations yield, $S_2 = \mathcal{O}(1)$ if and only if $m = n$ and $c \in \mathcal{O}\left(\sqrt{\frac{k}{k+2m}}\right)$.

Given $c \in \mathbb{C}\backslash\{0\}$, there are associated circles (or lines) $S_1 = f_{c,1}(\mathcal{O}(1))$ and $S_2 = f_{c,2}(\mathcal{O}(1))$. Since $r_1, r_2 \in \mathcal{O}(1)$, $f_{c,1}(r_2) = r_1$ and $f_{c,2}(r_1) = r_2$, it follows that $r_1 \in S_1 \cap \mathcal{O}(1)$ and $r_2 \in S_2 \cap \mathcal{O}(1)$. The sets $S_1 \cap \mathcal{O}(1)$ and $S_2 \cap \mathcal{O}(1)$ give candidates for the roots of polynomials in $P(k,m,n)$ with nontrivial critical points at c. When $m = n$, $f_{c,1} = f_{c,2}$ so that $S_1 = S_2$. In this case, Lemma 2.8 provides a relationship between $|S_1 \cap \mathcal{O}(1)|$ and the number of polynomials in $P(k,m,m)$ with a nontrivial critical point at c.

Lemma 2.8. *Suppose $m = n$ and $c \in \mathbb{C}\backslash\{0\}$.*

(1) *If $S_1 \cap \mathcal{O}(1) = \emptyset$, then no polynomial in $P(k,m,m)$ has a nontrivial critical point at c.*

(2) *If $S_1 = \mathcal{O}(1)$, then $p(z) = z^k(z-r)^m(z-f_{c,1}(r))^m \in P(k,m,m)$ has a nontrivial critical point at c for each $r \in \mathcal{O}(1)$.*

(3) *If $|S_1 \cap \mathcal{O}(1)| \in \{1,2\}$, then c is the nontrivial critical point of a unique polynomial in $P(k,m,m)$.*

Proof. Let $c \in \mathbb{C}\backslash\{0\}$.

(1) If $S_1 \cap \mathcal{O}(1) = \emptyset$, then no point in $\mathbb{C}$ is eligible to be r_1 or r_2. Therefore, no polynomial in $P(k,m,m)$ has a nontrivial critical point at c.

(2) If $S_1 = \mathcal{O}(1)$, Theorem 2.5 implies that c is a nontrivial critical point of $p(z) = z^k(z-r)^m(z-f_{c,1}(r))^m \in P(k,m,m)$ has a nontrivial critical point at c for each $r \in \mathcal{O}(1)$.

(3) If $S_1 \cap \mathcal{O}(1) = \{a\}$, then $f_{c,1}(a) = a$ and Theorem 2.5 implies $p(z) = z^k(z-a)^{2m}$ is the only polynomial in $P(k,m,m)$ with a nontrivial critical point at c.

   Suppose $S_1 \cap \mathcal{O}(1) = \{a,b\}$ with $a \neq b$. If $f_{c,1}(a) = a$, the definitions of $f_{c,1}$ and S_1 imply $f_{c,1}(b) = b$. Then, by Theorem 2.5, c is a nontrivial critical point of $p(z) = z^k(z-a)^{2m}$ and $p(z) = z^k(z-b)^{2m}$, which contradicts the Gauss-Lucas Theorem. Therefore, $f_{c,1}(a) = b$ so that $f_{c,1}(b) = a$ and Theorem 2.5 implies $p(z) = z^k(z-a)^m(z-b)^m \in P(k,m,m)$ has a nontrivial critical point at c. Furthermore, as $S_1 \cap \mathcal{O}(1) = \{a,b\}$, no other polynomial in $P(k,m,m)$ has a nontrivial critical point at c.

□

When $m \neq n$, $S_1 \neq S_2$. Extensions of results in [2] (Lemmas 10 and 11 on page 493) imply that $|S_1 \cap \mathcal{O}(1)| = |S_2 \cap \mathcal{O}(1)|$ (our Lemma 2.9) and that $|S_1 \cap \mathcal{O}(1)|$ is the number of polynomials in $P(k,m,n)$ having a nontrivial critical point at c (our Lemma 2.10).

Lemma 2.9. *Let $c \in \mathbb{C}\backslash\{0\}$ and $m \neq n$. Then $|S_1 \cap \mathcal{O}(1)| = |S_2 \cap \mathcal{O}(1)| \in \{0,1,2\}$.*

Lemma 2.10. *Let $c \in \mathbb{C}\backslash\{0\}$ and $m \neq n$.*

(1) *If $S_1 \cap \mathcal{O}(1) = \emptyset$, then no polynomial in $P(k,m,n)$ has a nontrivial critical point at c.*

(2) *If $|S_1 \cap \mathcal{O}(1)| = 1$, then c is the nontrivial critical point of exactly one polynomial in $P(k,m,n)$.*

(3) *If $|S_1 \cap \mathcal{O}(1)| = 2$, then c is the nontrivial critical point of exactly two polynomials in $P(k,m,n)$.*

2.1. **Center and Radius of S_1.** To characterize the nontrivial critical points of polynomials in $P(k,m,n)$, Lemmas 2.8, 2.9 and 2.10 suggest that we need to further understand S_1.

Example 2.11. For $c \neq 0$, $S_1 = f_{c,1}(\mathcal{O}(1))$ is either a circle or a line. Thinking of a line as a circle with a point at infinity, we observe that $S_1 = f_{c,1}(\mathcal{O}(1))$ is a line whenever there exists a $z_o \in \mathcal{O}(1)$ that makes the denominator of $f_{c,1}(z_o)$ equal to zero. This occurs whenever $kz_o - (n+k)c = 0$. In this case, $\frac{c}{z_o} = \frac{k}{n+k}$. Take the modulus of both sides and note that $|z_o| = 1$ and it follows that $|c| = \left|\frac{k}{n+k}\right|$. Therefore, S_1 is a line if and only if $c \in \mathcal{O}(\frac{k}{k+n})$.

For $c \in \mathcal{O}(\alpha)$ with $\alpha \neq \frac{k}{k+n}$, S_1 is a circle. We use methods from [2] to determine the center and radius of S_1. Since $S_1 = f_{c,1}(\mathcal{O}(1))$, $z \in S_1$ if and only if there exists some $w \in \mathcal{O}(1)$ with $f_{c,1}(w) = z$. Furthermore, as $(f_{c,1})^{-1} = f_{c,2}$, $z \in S_1$ if and only if $|f_{c,2}(z)| = 1$. That is,

$$\left|\frac{-c(k+n)z + (k+m+n)c^2}{-kz + (k+m)c}\right| = 1.$$

Therefore, $z \in S_1$ if and only if

$$\left|\frac{c(k+n)}{k}\right|\left|z - \frac{(k+m+n)c}{k+n}\right| = \left|z - \frac{(k+m)c}{k}\right|.$$

From introductory complex analysis, when $d \neq 1$, the solution set of

$$d|z-v| = |z-u|$$

is a circle with center $C = \frac{d^2v-u}{d^2-1}$ and radius $R = |v-u||\frac{d}{d^2-1}|$. Direct calculations establish the following result.

Lemma 2.12. *Suppose $c \in \mathcal{O}(\alpha)$ with $\alpha \neq \frac{k}{k+n}$. Then, S_1 is a circle with center C and radius R given by*

$$C = \frac{\Big[(k+n)(k+m+n)\alpha^2 - (k+m)k\Big]c}{(k+n)^2\alpha^2 - k^2} \quad \text{and} \quad R = \frac{mn\alpha^2}{|\alpha^2(k+n)^2 - k^2|}. \tag{1}$$

3. Determining the Desert Regions

When $S_1 \cap \mathcal{O}(1) = \emptyset$, Lemmas 2.8 and 2.10 imply that c lies in a desert region. As S_1 varies continuously with c, c will lie on the boundary of a desert region when S_1 is tangent to $\mathcal{O}(1)$. This proves to be an interesting case!

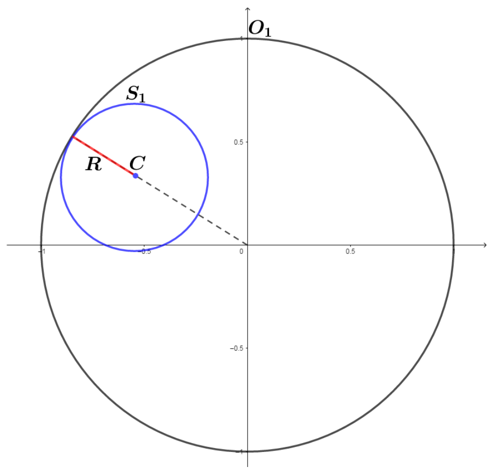

Figure 1. If S_1 is internally tangent to $\mathcal{O}(1)$, then $|C| + R = 1$.

We begin by determining when S_1 is internally tangent to $\mathcal{O}(1)$. For $c \neq 0$, if S_1 is internally tangent to $\mathcal{O}(1)$, then

$$|C| + R = 1. \tag{2}$$

See Figure 1. Substituting C and R from (1) into (2) and setting $c = x + iy$ gives

$$\begin{aligned} x^2 + y^2 &= \left(\frac{(k+n)^2\alpha^2 - k^2}{(k+n)(k+n+m)\alpha^2 - (k+m)k}\right)^2 \left(1 - \frac{mn\alpha^2}{|\alpha^2(k+n)^2 - k^2|}\right)^2 \\ &= \frac{\left(|\alpha^2(k+n)^2 - k^2| - mn\alpha^2\right)^2}{((k+n)(k+n+m)\alpha^2 - (k+m)k)^2} \end{aligned}$$

For $c \in \mathcal{O}(\alpha)$, S_1 is internally tangent to $\mathcal{O}(1)$ if and only if $|c| = \alpha$ satisfies

$$\alpha^2 = \left(\frac{(k+n)^2\alpha^2 - k^2}{(k+n)(k+n+m)\alpha^2 - (k+m)k}\right)^2 \left(1 - \frac{mn\alpha^2}{|\alpha^2(k+n)^2 - k^2|}\right)^2. \tag{3}$$

As $\mathcal{O}(\alpha)$ and (3) are circles centered at the origin, we have established the following result.

Lemma 3.1. *Let $c \in \mathcal{O}(\alpha)$ with $\alpha \in (0,1]$. Then, S_1 is internally tangent to $\mathcal{O}(1)$ if and only if α satisfies (3).*

In order to proceed, we investigate the $m = n$ and $m \neq n$ cases separately.

3.1. $P(k,m,m)$. We begin the $m = n$ discussion by analyzing the $P(1,1,1)$ case. When $k = m = n = 1$, (3) implies

$$x^2 + y^2 = \frac{1}{4(3\alpha^2-1)^2}\left((4\alpha^2-1) - \frac{\alpha^2(4\alpha^2-1)}{|4\alpha^2-1|}\right)^2. \tag{4}$$

According to Lemma 3.1, S_1 is internally tangent to $\mathcal{O}(1)$ whenever (3), and therefore (4), is satisfied. Because of the $|4\alpha^2-1|$ in (4), we consider 3 cases.

(1) If $\alpha \in (0, \frac{1}{2})$, then $\frac{4\alpha^2-1}{|4\alpha^2-1|} = -1$ and (4) becomes

$$x^2 + y^2 = \frac{1}{4(3\alpha^2-1)^2}\left((4\alpha^2-1) - \alpha^2(-1)\right)^2 = \frac{(5\alpha^2-1)^2}{4(3\alpha^2-1)^2}.$$

Therefore, (3) is satisfied when

$$\alpha^2 = \frac{(5\alpha^2-1)^2}{4(3\alpha^2-1)^2}.$$

To find α that satisfy this equation, we observe that $\alpha = \pm 1$ are solutions. Then, algebraic manipulation and polynomial long division lead to the solutions $\alpha = \pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1$. For $\alpha \in (0, \frac{1}{2})$, Lemma 3.1 implies S_1 is internally tangent to $\mathcal{O}(1)$ only when $\alpha = \frac{1}{3}$.

(2) If $\alpha \in (\frac{1}{2}, 1]$, then $\frac{4\alpha^2-1}{|4\alpha^2-1|} = 1$ and (4) becomes

$$x^2 + y^2 = \frac{1}{4(3\alpha^2-1)^2}(4\alpha^2 - 1 - \alpha^2)^2 = \frac{1}{4}.$$

Therefore, for $\alpha \in (\frac{1}{2}, 1]$, (3) is not satisfied and Lemma 3.1 implies S_1 is not internally tangent to $\mathcal{O}(1)$.

(3) If $\alpha = \frac{1}{2}$, then S_1 is a line. Furthermore, as $m = n$, $S_1 = S_2$ and it follows that $\{r_1, r_2\} \subseteq S_1 \cap \mathcal{O}_1$. Therefore, S_1 is tangent to $\mathcal{O}(1)$ if and only if $r_1 = r_2$. However, according to Example 2.2, when $r_1 = r_2$, $c \notin \mathcal{O}(\frac{1}{2})$, and S_1 is not tangent to $\mathcal{O}(1)$.

Our analysis of Lemma 3.1 has established the following result.

Lemma 3.2. *The circle S_1 is internally tangent to $\mathcal{O}(1)$ if and only if $c \in \mathcal{O}(\frac{1}{3})$.*

Furthermore, S_1 will be externally tangent to $\mathcal{O}(1)$ if and only if $|C| - R = 1$. A similar analysis leads to the following result.

Lemma 3.3. *The circle S_1 is externally tangent to $\mathcal{O}(1)$ if and only if $c \in \mathcal{O}(1)$.*

The following lemma will be needed to finish the characterization of critical points of polynomials in $P(1,1,1)$.

Lemma 3.4. *If $c \in \mathcal{O}(\alpha)$ with $\alpha \in (\frac{1}{3}, 1) \backslash \left\{\sqrt{\frac{1}{3}}\right\}$, then $|S_1 \cap \mathcal{O}(1)| = 2$.*

Proof. For $0 \neq c \in \mathcal{O}(\alpha)$, define $L = \{tc \mid t \in \mathbb{R}\}$ and let v, w be the points of intersection of L and $\mathcal{O}(1)$. See Figure 2. Direct calculations give

$$v = \frac{1}{\alpha}c \text{ and } w = \frac{-1}{\alpha}c.$$

Since $v, w \in \mathcal{O}(1)$, $\{f_{c,1}(v), f_{c,1}(w)\} \subseteq S_1$. As $f_{c,1}$ is a fractional linear transformation, $f_{c,1}(L)$ is a circle or line. For $z_o \in L$, $z_o = t_o c$ for some $t_o \in \mathbb{R}$ and

$$f_{c,1}(z_o) = \frac{2c(t_o c) - 3c^2}{t_o c - 2,} = \frac{2t_o - 3}{t_o - 2}c \in L.$$

Therefore, $f_{c,1}(L) = L$. Letting $t_o = \pm\frac{1}{\alpha}$ gives

$$f_{c,1}(w) = \frac{\frac{2}{\alpha} - 3}{\frac{1}{\alpha} - 2}c = \frac{2 - 3\alpha}{1 - 2\alpha}c \in \mathcal{O}(\left|\frac{2\alpha - 3\alpha^2}{1 - 2\alpha}\right|)$$

and

$$f_{c,1}(v) = \frac{\frac{-2}{\alpha} - 3}{\frac{-1}{\alpha} - 2}c = \frac{2 + 3\alpha}{1 + 2\alpha}c \in \mathcal{O}(\frac{2\alpha + 3\alpha^2}{1 + 2\alpha}).$$

Comparing the graphs of $f(\alpha) = \left|\frac{2\alpha - 3\alpha^2}{1 - 2\alpha}\right|$ and $g(\alpha) = \frac{2\alpha + 3\alpha^2}{1 + 2\alpha}$ in Figure 2, shows that when $\alpha \in \left(\frac{1}{3}, \sqrt{\frac{1}{3}}\right) \cup \left(\sqrt{\frac{1}{3}}, 1\right)$, exactly one of $f_{c,1}(w)$ and $f_{c,1}(v)$ lies inside the unit circle. Therefore, $|S_1 \cap \mathcal{O}(1))| = 2$. □

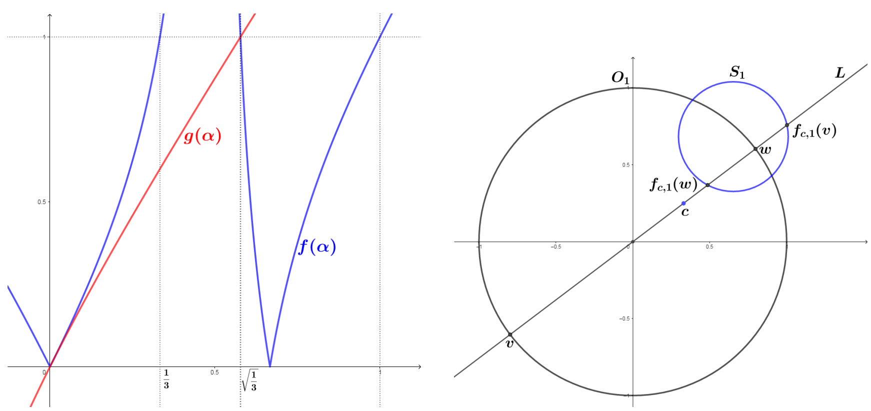

FIGURE 2. Left: $f_{c,1}(w) \in \mathcal{O}(f(\alpha))$ and $f_{c,1}(v) \in \mathcal{O}(g(\alpha))$. Right: $L = \{tc \mid t \in \mathbb{R}\}$.

We are now able to describe the critical points of polynomials in $P(1,1,1)$. See Figure 3.

Theorem 3.5. *Let $c \in \mathcal{O}(\alpha)$.*

(1) *If $\alpha \in [0, \frac{1}{3}) \cup (1, \infty)$, then no polynomial in $P(1,1,1)$ has a nontrivial critical point at c.*

(2) *If $\alpha = \sqrt{\frac{1}{3}}$, then $p(z) = z(z - r)(z - f_{c,1}(r)) \in P(1,1,1)$ has a nontrivial critical point at c for each $r \in \mathcal{O}(1)$.*

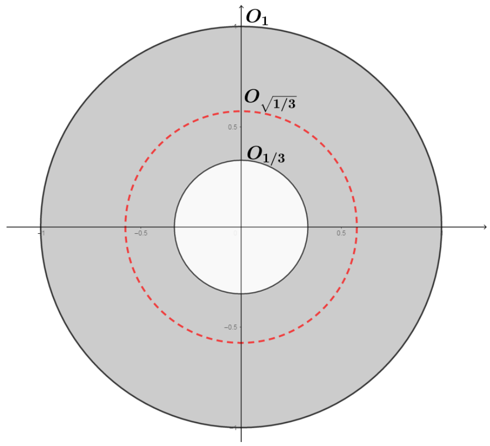

Figure 3. Critical points of polynomials in $P(1,1,1)$ cannot occur in the white regions.

(3) *If* $\alpha \in [\frac{1}{3}, 1]\backslash\left\{\sqrt{\frac{1}{3}}\right\}$*, then* c *is the nontrivial critical point of a unique polynomial in* $P(1,1,1)$.

Proof. Let $c \in \mathcal{O}(\alpha)$.

(1) If $\alpha \in (1, \infty)$, the Gauss-Lucas Theorem implies that c is not the critical point of a polynomial in $P(1,1,1)$. If $\alpha \in [0, \frac{1}{3})$, Theorem 2.3 implies no polynomial in $P(k,m,n)$ has a nontrivial critical point at c.

(2) If $\alpha = \sqrt{\frac{1}{3}}$, then Example 2.7 implies $S_1 = \mathcal{O}(1)$. By Lemma 2.8, $p(z) = z(z-r)(z-f_{c,1}(r)) \in P(1,1,1)$ has a nontrivial critical point at c for each $r \in \mathcal{O}(1)$.

(3) Supposed $\alpha \in \left(\frac{1}{3}, 1\right)\backslash\left\{\sqrt{\frac{1}{3}}\right\}$. Then, by Lemma 3.4, $|S_1 \cap \mathcal{O}(1)| = 2$ and Lemma 2.8 implies that a unique polynomial in $P(1,1,1)$ has a nontrivial critical point at c. If $\alpha \in \{\frac{1}{3}, 1\}$, Lemmas 3.2 and 3.3 imply that S_1 is tangent to $\mathcal{O}(1)$, and Lemma 2.8 implies that c is the nontrivial critical point of a unique polynomial in $P(1,1,1)$.

□

The analysis of $P(1,1,1)$ generalizes to $P(k,m,m)$.

When $m = n$, equation (3) becomes $\alpha^2 = \frac{\left((k+m)^2\alpha^2 - k^2 + m^2\alpha^2\right)^2}{(k+m)^2\left((k+2m)\alpha^2 - k\right)^2}$ and has roots $\pm 1, \pm\frac{k}{k+m}, \pm\frac{k}{k+2m}$.

Theorem 3.5 restated for $P(k,m,m)$ is as follows.

Theorem 3.6. *Let* $c \in \mathcal{O}(\alpha)$.

(1) *If* $\alpha \in [0, \frac{k}{k+2m}) \cup (1, \infty)$*, then no polynomial in* $P(k,m,m)$ *has a nontrivial critical point at* c.

(2) *If* $\alpha = \sqrt{\frac{k}{k+2m}}$, *then* $p(z) = z^k(z-r)^m(z-f_{c,1}(r))^m \in P(k,m,m)$ *has a nontrivial critical point at* c *for each* $r \in \mathcal{O}(1)$.

(3) *If* $\alpha \in \left[\frac{k}{k+2m}, 1\right] \setminus \left\{\sqrt{\frac{k}{k+2m}}\right\}$, *then* c *is a nontrivial critical point of a unique polynomial in* $P(k,m,m)$.

4. Polynomials in $P(k,m,n)$ with $m \neq n$

To begin the $m \neq n$ case, we analyze the critical points of polynomials in $P(1,1,2)$. For $k = m = 1$ and $n = 2$, (3) becomes

$$x^2 + y^2 = \left(\frac{9\alpha^2 - 1}{12\alpha^2 - 2}\right)^2 \left(1 - \frac{2\alpha^2}{|9\alpha^2 - 1|}\right)^2. \tag{5}$$

According to Lemma 3.1, S_1 in internally tangent to $\mathcal{O}(1)$ whenever (3) and therefore (5) is satisfied. Because of the $|9\alpha^2 - 1|$ in (5), we consider 3 cases.

(1) If $\alpha \in (0, \frac{1}{3})$, then $\frac{9\alpha^2-1}{|9\alpha^2-1|} = -1$, and (5) becomes

$$x^2 + y^2 = \left(\frac{1 - 11\alpha^2}{12\alpha^2 - 2}\right)^2.$$

Therefore, (3) is satisfied whenever

$$\alpha^2 = \left(\frac{1 - 11\alpha^2}{12\alpha^2 - 2}\right)^2.$$

Observing that $\alpha = \pm 1$ are solutions and using polynomial division leads to $\alpha = \pm\frac{1}{4}, \pm\frac{1}{3}, \pm 1$. Therefore, for $\alpha \in (0, \frac{1}{3})$, S_1 is internally tangent to $\mathcal{O}(1)$ when $\alpha = \frac{1}{4}$.

(2) If $\alpha \in (\frac{1}{3}, 1]$, then $\frac{9\alpha^2-1}{|9\alpha^2-1|} = 1$, and (5) becomes

$$x^2 + y^2 = \left(\frac{7\alpha^2 - 1}{12\alpha^2 - 2}\right)^2. \tag{6}$$

Therefore, (3) is satisfied whenever

$$\alpha^2 = \left(\frac{7\alpha^2 - 1}{12\alpha^2 - 2}\right)^2.$$

In order to solve for α we manipulate algebraically and use the rational roots test to find that $\alpha = \pm\frac{1}{3}$ are solutions. Polynomial division leads to the remaining solutions $\alpha = \frac{\pm\sqrt{17}\pm 1}{8}$. Therefore, for $\alpha \in (\frac{1}{3}, 1]$, S_1 is internally tangent to $\mathcal{O}(1)$ when $\alpha = \frac{\sqrt{17}\pm 1}{8}$.

(3) If $c \in \mathcal{O}(\frac{1}{3})$, then S_1 is a line. Similar to the $P(1,1,1)$ case, S_1 is not tangent to $\mathcal{O}(1)$.

The analysis of (3) and $\mathcal{O}(\alpha)$ has established the following result.

Lemma 4.1. *Let $c \in \mathcal{O}(\alpha)$. Then, S_1 is internally tangent to $\mathcal{O}(1)$ if and only if $\alpha \in \left\{\frac{1}{4}, \frac{\sqrt{17}\pm 1}{8}\right\}$.*

A similar analysis determines when S_1 is externally tangent to $\mathcal{O}(1)$.

Lemma 4.2. *Let $c \in \mathcal{O}(\alpha)$. Then, S_1 is externally tangent to $\mathcal{O}(1)$ if and only if $\alpha = 1$.*

We are now able to describe the second desert region.

Theorem 4.3. *No polynomial in $P(1,1,2)$ has a nontrivial critical point on $\mathcal{O}(\alpha)$ with $\alpha \in \left(\frac{\sqrt{17}-1}{8}, \frac{\sqrt{17}+1}{8}\right)$.*

Proof. Let $c \in \mathcal{O}(\alpha)$ with $\alpha \in \left(\frac{\sqrt{17}-1}{8}, \frac{\sqrt{17}+1}{8}\right)$. Then,

$$\alpha^2 < \left(\frac{7\alpha^2 - 1}{12\alpha^2 - 2}\right)^2$$

and (2) and (6) imply $|C| + |R| < 1$. Therefore, for $\alpha \in \left(\frac{\sqrt{17}-1}{8}, \frac{\sqrt{17}+1}{8}\right)$, $S_1 \cap \mathcal{O}(1) = \emptyset$ and Lemma 2.10 implies that no polynomial in $P(1,1,2)$ has a nontrivial critical point on $\mathcal{O}(\alpha)$. □

The following lemma will be needed to characterize the nontrivial critical points of polynomials in $P(1,1,2)$. The proof is similar to that of Lemma 3.4. For the line $L = \{tc \mid t \in \mathbb{R}\}$ and v, w as defined in Lemma 3.4, it can be shown that $f_{c,1}(L) = L$ and that exactly one of $f_{c,1}(w) \in S_1$ and $f_{c,1}(v) \in S_1$ lie inside the unit circle. This leads to the following result.

Lemma 4.4. *If $c \in \mathcal{O}(\alpha)$ with $\alpha \in (\frac{1}{4}, \frac{\sqrt{17}-1}{8}) \cup (\frac{\sqrt{17}+1}{8}, 1)$, then $|S_1 \cap \mathcal{O}(1)| = 2$.*

We are now able to characterize the nontrivial critical points of polynomials in $P(1,1,2)$. See Figure 4.

Theorem 4.5. *Let $c \in \mathcal{O}(\alpha)$.*

(1) *If $\alpha \in [0, \frac{1}{4}) \cup \left(\frac{\sqrt{17}-1}{8}, \frac{\sqrt{17}+1}{8}\right) \cup (1, \infty)$, then no polynomial in $P(1,1,2)$ has a nontrivial critical point at c.*

(2) *If $\alpha \in \left\{\frac{1}{4}, \frac{\sqrt{17}\pm 1}{8}, 1\right\}$, then c is the nontrivial critical point of a unique polynomial in $P(1,1,2)$.*

(3) *If $\alpha \in \left(\frac{1}{4}, \frac{\sqrt{17}-1}{8}\right) \cup \left(\frac{\sqrt{17}+1}{8}, 1\right)$, then c is the nontrivial critical point of exactly two polynomials in $P(1,1,2)$.*

Proof. Let $c \in \mathcal{O}(\alpha)$

(1) If $\alpha > 1$, the Gauss-Lucas Theorem implies c is not the critical point of a polynomial in $P(1,1,2)$. If $\alpha \in (0, \frac{1}{4}) \cup \left(\frac{\sqrt{17}-1}{8}, \frac{\sqrt{17}+1}{8}\right)$, Theorems 2.3 and 4.3 imply no polynomial in $P(1,1,2)$ has a nontrivial critical point at c.

(2) If $\alpha \in \left\{\frac{1}{4}, \frac{\sqrt{17}\pm 1}{8}, 1\right\}$, Lemmas 2.10, 4.1, and 4.2 imply that c is the nontrivial critical point of a unique polynomial in $P(1,1,2)$.

(3) Suppose $\alpha \in \left(\frac{1}{4}, \frac{\sqrt{17}-1}{8}\right) \cup \left(\frac{\sqrt{17}+1}{8}, 1\right)$. Lemma 4.4 implies $|S_1 \cap \mathcal{O}(1)| = 2$, and by Lemma 2.10, c is a nontrivial critical point of exactly two polynomials in $P(1,1,2)$.

□

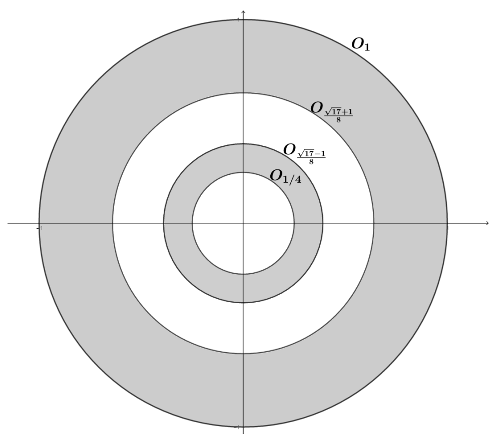

Figure 4. Critical points of polynomials in $P(1,1,2)$ do not occur in the white regions.

The analysis of $P(1,1,2)$ can be extended to $P(k,m,n)$ with $m \neq n$. The analysis includes finding the values of α that satisfy (3) and extending the lemmas used in the $P(1,1,2)$ case.

To conveniently state the main result we set

$$\alpha_{\pm} = \frac{\sqrt{(m-n)^2 + 4k(k+m+n)} \pm |m-n|}{2(k+m+n)}.$$

Theorem 4.6. *Let $c \in \mathcal{O}(\alpha)$.*

(1) *If $\alpha \in \left[0, \frac{k}{k+m+n}\right) \cup (\alpha_-, \alpha_+) \cup (1, \infty)$, then no polynomial in $P(k,m,n)$ has a nontrivial critical point at c.*

(2) *If $\alpha \in \left\{\frac{k}{k+m+n}, \alpha_{\pm}, 1\right\}$, then c is the nontrivial critical point of a unique polynomial in $P(k,m,n)$.*

(3) *If $\alpha \in \left(\frac{k}{k+m+n}, \alpha_-\right) \cup (\alpha_+, 1)$, then c is the nontrivial critical point of exactly two polynomials in $P(k,m,n)$.*

5. Conclusion

This concludes our characterization of the nontrivial critical point of polynomials in $P(k,m,n)$. However, there is still more to be discovered. For example, as a consequence of Theorem 2.3, if $p \in P(k,m,n)$ has nontrivial critical points $c_1 \in \mathcal{O}(\alpha)$ and $c_2 \in \mathcal{O}(\beta)$, then $\mathcal{O}(\alpha)$ is the inversion of $\mathcal{O}(\beta)$ across the circle $\mathcal{O}(\sqrt{\frac{k}{k+m+n}})$. What other structure is associated with the nontrivial critical points of polynomials in $P(k,m,n)$? Additionally, for c in the unit disk, is it possible to determine the polynomial(s) in $P(k,m,n)$ with a nontrivial critical point at c? Many interesting and open questions remain.

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Student biographies

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