The Minnesota Journal of Undergraduate Mathematics

Minimal Noise-Induced Stabilization of One-Dimensional Diffusions

Tony Allen[1], Emily Gebhardt[2], Adam Kluball[3], and Tiffany Kolba[4]

[1]West Virginia University
[2]Mercyhurst University
[3]Bethany Lutheran College
[4]Valparaiso University

The Minnesota Journal of Undergraduate Mathematics

Volume 3 (2017-2018 Academic Year)

Sponsored by
School of Mathematics
University of Minnesota
Minneapolis, MN 55455

The Minnesota Journal of Undergraduate Mathematics

Volume 3 (2017-2018 Academic Year)

Minimal Noise-Induced Stabilization of One-Dimensional Diffusions

Tony Allen* [1], Emily Gebhardt[2], Adam Kluball[3], and Tiffany Kolba[4]

[1]West Virginia University
[2]Mercyhurst University
[3]Bethany Lutheran College
[4]Valparaiso University

Abstract. The phenomenon of noise-induced stabilization occurs when an unstable deterministic system of ordinary differential equations is stabilized by the addition of randomness into the system. In this paper, we investigate under what conditions one-dimensional, autonomous stochastic differential equations are stable, where we take the notion of stability to be that of global stochastic boundedness. Specifically, we find the minimum amount of noise necessary for noise-induced stabilization to occur when the drift and noise coefficients are power, polynomial, exponential, or logarithmic functions.

1. Introduction

Noise-induced stabilization occurs when the addition of randomness to an unstable deterministic system of ordinary differential equations (ODEs) results in a stable system of stochastic differential equations (SDEs). Noise-induced stabilization is quite an intriguing and surprising phenomenon as one's first intuition is often that noise will only serve to further destabilize the system. Moreover, it is typically the case that the more noise present, the stronger the stabilizing effect.

We are particularly focused on the minimum amount of noise required for noise-induced stabilization to occur for one-dimensional, autonomous SDEs of the form

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t). \tag{1}$$

Here $b(x)$ is the drift coefficient, which pushes the solution deterministically in some direction, $\sigma(x)$ is the noise coefficient, which controls the strength of the noise, and $B(t)$ is standard one-dimensional Brownian motion. We restrict our attention to the case where $b(x)$ and $\sigma(x)$ are continuous functions. We also assume that $b(x)$ and $\sigma(x)$ are locally Lipschitz in order to ensure that there exists a unique solution to (1) up until the possible time of explosion [1, 4].

* *Corresponding author*

There are many different notions of stability in the literature, but our sense of stability comes from that of global stochastic boundedness, where there exists a bound such that $X(t)$ is bounded with arbitrarily high probability for all time. The notion of stability is defined more formally below [3].

Definition 1.1. $X(t)$ is *stable* if for all initial conditions and all $\epsilon > 0$, there exists some bound R such that

$$P(|X(t)| \leq R) > 1 - \epsilon$$

for all $t \geq 0$.

In the deterministic setting when $\sigma(x) = 0$ and $X(t)$ is a solution to an ODE, the definition of stability reduces to that of boundedness. Hence, the unstable deterministic systems that we consider either blow up in finite time or wander off to infinity, for at least some initial conditions. We say that noise-induced stabilization occurs when the addition of noise to an unstable ODE results in a stable SDE, where the definition of stable is as given in Definition 1.1.

A classic example of noise-induced stabilization involves geometric Brownian motion, which is the solution to the SDE

$$dX(t) = rX(t)dt + aX(t)dB(t)$$

where r and a are constants. While most SDEs cannot be solved explicitly, the explicit solution for geometric Brownian motion is

$$X(t) = X(0)e^{(r-\frac{a^2}{2})t+aB(t)}.$$

When $a = 0$ and the solution is deterministic, $X(t)$ is stable for $r \leq 0$ since the solution is bounded by $|X(0)|$, but is unstable for $r > 0$ since the solution converges to either plus or minus infinity as $t \to \infty$. However, when $a \neq 0$ and the solution is stochastic, $X(t)$ is stable for $a^2 > 2r$ since the solution converges to zero with probability one as $t \to \infty$. When $a^2 < 2r$, the solution converges to plus or minus infinity with probability one, and when $a^2 = 2r$, the solution fluctuates between arbitrary large and arbitrary small values [4]. Thus in the stochastic setting, $X(t)$ is unstable when $a^2 \leq 2r$. Hence, noise-induced stabilization occurs for geometric Brownian motion when $r > 0$ and $a^2 > 2r$ since this is when the deterministic solution is unstable, but the addition of the stochastic term has a stabilizing effect. Note that in order to obtain noise-induced stabilization, the noise coefficient needs to be sufficiently large in order to overcome the deterministic drift towards infinity.

Previous work by Scheutzow [5] has shown sufficient conditions for the occurrence of noise-induced stabilization in one-dimensional diffusions. In this paper, we find *necessary* and sufficient conditions for noise-induced stabilization to occur when the drift and noise coefficients are restricted to certain forms. Section 2 discusses useful background information on the techniques that we use to prove noise-induced stabilization. Sections 3, 4, 5, and 6 present and prove our results concerning noise-induced stabilization for when the drift and noise coefficients are general power functions, polynomials, exponential functions, and logarithmic functions, respectively. In particular, our theorem in Section 4 regarding polynomial stabilization encompasses the example of geometric Brownian motion described previously since the drift coefficient, $b(x) = rx$, and noise coefficient, $\sigma(x) = ax$, are polynomials for geometric Brownian motion.

2. Background

In this section, we discuss preliminary information on the methods used to find our results. In particular, our work uses a well-known result from [2] to determine the stability of SDEs, and specifically whether noise-induced stabilization occurs. For ease, we will refer to this result as the "Stochastic Stability Theorem."

Stochastic Stability Theorem. Consider a one-dimensional, autonomous SDE of the form

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$$

where $b(x)$ and $\sigma(x)$ are continuous, locally Lipschitz functions and there exists $\ell_0 \geq 0$ such that $\sigma(x) \neq 0$ for all $|x| \geq \ell_0$.

Define the following quantities:

$$s(x) = \begin{cases} \exp\left[\int_{\ell}^{x} \frac{-2b(z)}{\sigma^2(z)} dz\right] & \text{for } x > \ell \\ \exp\left[\int_{-\ell}^{x} \frac{-2b(z)}{\sigma^2(z)} dz\right] & \text{for } x < -\ell \end{cases} \tag{2}$$

$$S(x) = \begin{cases} \int_{\ell}^{x} s(y)dy & \text{for } x > \ell \\ \int_{-\ell}^{x} s(y)dy & \text{for } x < -\ell \end{cases} \tag{3}$$

$$m(x) = \frac{1}{s(x)\sigma^2(x)} \tag{4}$$

$$M(x) = \begin{cases} \int_{\ell}^{x} m(y)dy & \text{for } x > \ell \\ \int_{-\ell}^{x} m(y)dy & \text{for } x < -\ell. \end{cases} \tag{5}$$

The SDE is stable if and only if there exists $\ell \geq \ell_0$ such that

$$S(\infty) = \infty, \quad S(-\infty) = -\infty, \quad |M(\infty)| < \infty, \quad |M(-\infty)| < \infty.$$

Note that since the lower limit of integration does not affect the convergence or divergence of these integrals, ℓ can be any real number greater than or equal to ℓ_0.

The formula for $S(x)$ is obtained by choosing the function that enables the drift term of $dS(X(t))$ to be identically equal to zero. Hence, by Ito's formula [4], $S(x)$ is the solution to the ODE

$$\frac{dS(x)}{dx} b(x) + \frac{1}{2} \frac{d^2 S(x)}{dx^2} \sigma^2(x) = 0.$$

$S(x)$ is often referred to as the *natural scale* of the SDE and it is precisely the conditions $S(\infty) = \infty$ and $S(-\infty) = -\infty$ than ensure that the solution to the SDE returns to the interval $[-\ell_0, \ell_0]$ with probability one. $M(x)$ is often referred to as the *speed measure* of the SDE and the additional conditions $|M(\infty)| < \infty$ and $|M(-\infty)| < \infty$ ensure that the expected time to return to the interval $[-\ell_0, \ell_0]$ is finite, which is equivalent to our notion of stability [1, 2].

The conditions in the "Stochastic Stability Theorem" for an SDE to be stable all involve the convergence or divergence of deterministic integrals. Most of the integrals we encounter based upon our forms for $b(x)$ and $\sigma(x)$ cannot be evaluated explicitly; rather we employ standard calculus techniques, such as the comparison theorem or limit comparison theorem, to determine the convergence or divergence [6].

The power of the "Stochastic Stability Theorem" is that it is an if and only if statement, and hence allows us to determine precisely when noise-induced stabilization occurs, given explicit forms for the drift and noise coefficients, $b(x)$ and $\sigma(x)$. In particular, we utilize the theorem in order to determine for a given $b(x)$ corresponding to an unstable ODE and a given class of noise coefficients specified by a few free parameters, what the minimum parameter values are in order for $\sigma(x)$ to have a stabilizing effect. We seek the *minimum* values rather than the maximum since the formulas for $s(x)$ and $m(x)$ given by equations (2) and (4) demonstrate that when the noise coefficient $\sigma(x)$ grows more quickly, the conditions for stability are more easily obtained.

3. Power Function Stabilization

In this section, we consider ODEs where the drift coefficient is a general power function, i.e.

$$dX(t) = b(X(t))dt$$

where

$$b(x) = \begin{cases} r|x|^q & \text{for } |x| \geq 1 \\ r & \text{for } |x| < 1 \end{cases}$$

with r and q any real numbers. The absolute value of x is used in $b(x)$ so that the solution is well-defined for both positive and negative initial conditions, since q is not restricted here to integer values. In addition, $b(x)$ is defined piecewise with a constant value for $|x| < 1$ in order to preserve the local Lipschitz condition for x near zero for all values of q. Note that since our definition of stability is that of global stochastic boundedness, the behavior of $b(x)$ and $\sigma(x)$ for x near zero does not affect the stability of the SDE; rather, only the behavior of $b(x)$ and $\sigma(x)$ as $x \to \pm\infty$ affects the stability.

The ODEs defined above are unstable for any $r \neq 0$. In particular, the solutions to the ODEs blow up in finite time for $q > 1$ and wander off to infinity for $q \leq 1$, for at least some initial conditions. We consider perturbing the ODEs by adding a noise term that also takes the form of a general power function, i.e.

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$$

where

$$\sigma(x) = \begin{cases} a|x|^p & \text{for } |x| \geq 1 \\ a & \text{for } |x| < 1 \end{cases}$$

with a and p any real numbers.

Our goal was to determine for given values of r and q, what are necessary and sufficient conditions on the values of a and p in order for noise-induced stabilization to occur. These conditions are given in Theorem 3.1. In particular, there exist critical values for a and p such that for all values of a and p above these critical values, noise-induced stabilization occurs, and for values of a and p below these critical values, noise-induced stabilization does not occur. Hence, we view these critical values as "minimal" conditions for noise-induced stabilization.

Theorem 3.1. Consider the SDE

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$$

where

$$b(X(t)) = \begin{cases} r|X(t)|^q & \text{for } |X(t)| \geq 1 \\ r & \text{for } |X(t)| < 1. \end{cases}$$

and

$$\sigma(X(t)) = \begin{cases} a|X(t)|^p & \text{for } |X(t)| \geq 1 \\ a & \text{for } |X(t)| < 1. \end{cases}$$

with r, q, a, and p any real numbers. Noise-induced stabilization occurs if and only if $r \neq 0$ and one of the following sets of conditions is met:

- $p > \max\left\{\frac{1}{2}, \frac{q+1}{2}\right\}$ and $a \neq 0$, or
- $p = \frac{q+1}{2}$ and $\begin{cases} a^2 q > 2|r| & \text{for } q \leq 1 \\ a^2 \geq 2|r| & \text{for } q > 1. \end{cases}$

The condition of $r \neq 0$ is necessary for noise-induced stabilization because when $r = 0$, the deterministic solution is already stable. When $q > 1$, Theorem 3.1 implies that the critical values for noise-induced stabilization are

$$p = \frac{q+1}{2} \text{ and } |a| = \sqrt{2|r|}.$$

When $0 < q \leq 1$, the critical values are

$$p = \frac{q+1}{2} \text{ and } |a| = \sqrt{\frac{2|r|}{q}}.$$

When $q \leq 0$, the critical values are $p = \frac{1}{2}$ and $|a| = 0$. Note that in some of these cases, noise-induced stabilization occurs at the critical values, whereas in other cases, noise-induced stabilization only occurs above the critical values, with the strict inequalities indicated in Theorem 3.1. Stabilization is most dependent upon the powers in the general power functions, where for any p above its critical value, noise-induced stabilization occurs regardless of the magnitude of the coefficient a (as long as it is non-zero). However, magnitudes of the coefficient a above its critical value will not produce noise-induced stabilization unless the power p is at or above its critical value.

Figure 1 shows three separate graphs depicting the phenomenon of noise-induced stabilization in the case where the drift and noise coefficients are general power functions. The

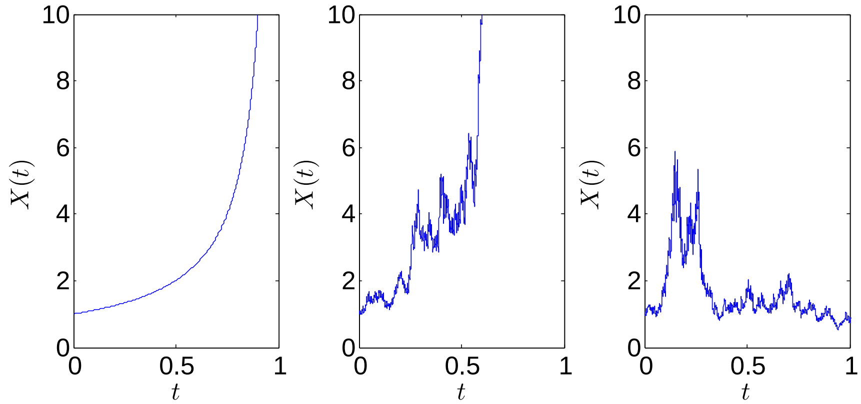

Figure 1. Solution to unstable power ODE (left), along with simulations of corresponding SDE without sufficient noise for stabilization (center) and SDE with sufficient noise for stabilization (right).

graph on the far left shows the solution to the ODE $dX(t) = |X(t)|^2 dt$ with initial condition $X(0) = 1$, which explodes in finite time and is thus unstable. The center graph depicts a simulation of the corresponding SDE where the noise term has $a = \sqrt{2}$ and $p = 1$, which is insufficient for stabilization since $1 = p < \frac{q+1}{2} = \frac{3}{2}$. The final image depicts a simulation of the SDE where the noise term has $a = \sqrt{2}$ and $p = \frac{3}{2}$. This SDE is stable and, in fact, exhibits "minimal" noise-induced stabilization since $p = \frac{q+1}{2}$ and $a^2 = 2r$.

Proof of Theorem 3.1. If $Y(t) = -X(t)$, then $Y(t)$ must have the same stability classification as $X(t)$ since they have the same magnitude and our definition of stability is that of stochastic boundedness. Now $Y(t)$ is the solution to the same SDE as $X(t)$, but with r and a replaced with $-r$ and $-a$. Thus, when proving Theorem 3.1, it suffices to prove the case with $r > 0$ and $a \neq 0$.

From the "Stochastic Stability Theorem" conditions, evaluation of the $s(x)$ term with $\ell = 1$ gives

$$s(x) = \begin{cases} \exp\left[\frac{-2r}{a^2}\int_1^x z^{q-2p}\,dz\right] & \text{for } x > 1 \\ \exp\left[\frac{-2r}{a^2}\int_{-1}^x (-z)^{q-2p}\,dz\right] & \text{for } x < -1 \end{cases}$$

where the integration depends upon the value of $q - 2p$.

Case 1: $\boxed{q - 2p + 1 \neq 0}$. Let $n = q - 2p + 1$ and $c = \frac{-2r}{a^2 n}$. Then

$$s(x) = \begin{cases} \exp\left[c(x^n - 1)\right] & \text{for } x > 1 \\ \exp\left[-c((-x)^n - 1)\right] & \text{for } x < -1 \end{cases}$$

and therefore

$$\begin{aligned}
S(\infty) &= \exp[-c]\int_1^{\infty} \exp[cx^n]\,dx \\
S(-\infty) &= \exp[c]\int_{-1}^{-\infty} \exp[-c(-x)^n]\,dx \\
&= -\exp[c]\int_1^{\infty} \exp[-cx^n]\,dx \\
M(\infty) &= \frac{1}{a^2}\exp[c]\int_1^{\infty} x^{-2p}\exp[-cx^n]\,dx \\
M(-\infty) &= \frac{1}{a^2}\exp[-c]\int_{-1}^{-\infty} (-x)^{-2p}\exp[c(-x)^n]\,dx \\
&= -\frac{1}{a^2}\exp[-c]\int_1^{\infty} x^{-2p}\exp[cx^n]\,dx.
\end{aligned}$$

Hence, we observe that $S(\infty)$ and $S(-\infty)$ both diverge if and only if $n < 0$, which corresponds to $p > \frac{q+1}{2}$. When $n < 0$, $M(\infty)$ and $M(-\infty)$ both converge if and only if $2p > 1$. Therefore, by the "Stochastic Stability Theorem," when $p \neq \frac{q+1}{2}$, $X(t)$ is stable if and only if $p > \max(\frac{1}{2}, \frac{q+1}{2})$.

Case 2: $\boxed{q - 2p + 1 = 0}$. In this case, the $s(x)$ term takes the form

$$s(x) = \begin{cases} \exp\left[\frac{-2r}{a^2}\int_1^x z^{-1}\,dz\right] = x^{\frac{-2r}{a^2}} & \text{for } x > 1 \\ \exp\left[\frac{-2r}{a^2}\int_{-1}^x (-z)^{-1}\,dz\right] = (-x)^{\frac{2r}{a^2}} & \text{for } x < -1 \end{cases}$$

Evaluation of the other terms in the "Stochastic Stability Theorem" gives

$$\begin{aligned}
S(\infty) &= \int_1^{\infty} x^{\frac{-2r}{a^2}}\,dx \\
S(-\infty) &= \int_{-1}^{-\infty} (-x)^{\frac{2r}{a^2}}\,dx = -\int_1^{\infty} x^{\frac{2r}{a^2}}\,dx \\
M(\infty) &= \frac{1}{a^2}\int_1^{\infty} x^{\frac{2r}{a^2}-2p}\,dx \\
M(-\infty) &= \frac{1}{a^2}\int_{-1}^{-\infty} (-x)^{\frac{-2r}{a^2}-2p}\,dx = -\frac{1}{a^2}\int_1^{\infty} x^{\frac{-2r}{a^2}-2p}\,dx.
\end{aligned}$$

Now $S(\infty)$ and $S(-\infty)$ both diverge if and only if $\frac{2r}{a^2} \leq 1$, which is equivalent to $a^2 \geq 2r$. Similarly, $M(\infty)$ and $M(-\infty)$ both converge if and only if $\frac{2r}{a^2} - 2p < -1$, which is equivalent to $a^2 q > 2r$. Hence when $p = \frac{q+1}{2}$, in order for $X(t)$ to be stable, we must have $a^2 \geq 2r$ and $a^2 q > 2r$. When $q > 1$, the stricter condition is $a^2 \geq 2r$ and when $q \leq 1$, the stricter condition is $a^2 q > 2r$. $\square$

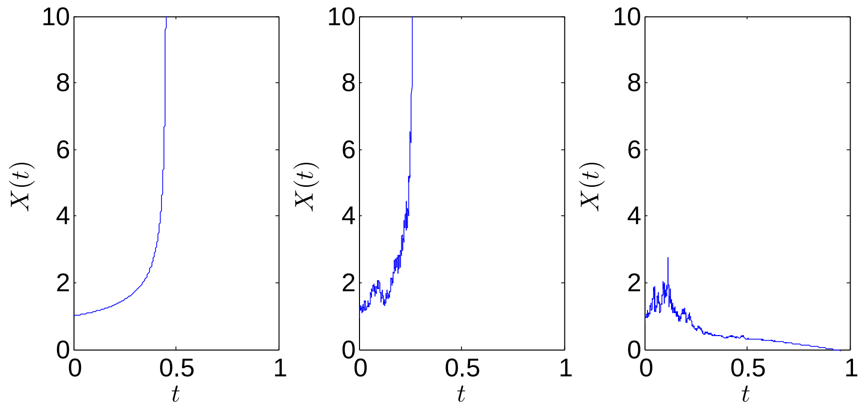

Figure 2. Solution to unstable polynomial ODE (left), along with simulations of corresponding SDE without sufficient noise for stabilization (center) and SDE with sufficient noise for stabilization (right).

4. Polynomial Stabilization

In this section, we investigate stabilizing ODEs of the form

$$dX(t) = b(X(t))dt,$$

where $b(x)$ is any polynomial of degree q. If r is the coefficient of the highest degree term of $b(x)$, this ODE is unstable when q is even for any $r \neq 0$ and when q is odd for any $r > 0$. The unstable solution wanders off to infinity when $q = 0$ or $q = 1$ and blows up in finite time when $q \geq 2$. We again consider adding a noise coefficient that takes the same general form as the drift coefficient.

Theorem 4.1. Consider the SDE

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$$

where $b(x)$ is any polynomial of degree q, where the coefficient of the highest degree term is r, and $\sigma(x)$ is any polynomial of degree p, where the coefficient of the highest degree term is $a \neq 0$. If q is even, assume $r \neq 0$, and if q is odd, assume $r > 0$. Then noise-induced stabilization occurs if and only if one of the following sets of conditions is met:

- $p > \frac{q+1}{2}$ or
- $p = \frac{q+1}{2}$ and $\begin{cases} a^2 > 2r & \text{for } q = 1 \\ a^2 \geq 2r & \text{for } q \geq 3 \end{cases}$.

Note that the conditions for noise-induced stabilization to occur in this case where the drift and noise coefficients are polynomials are exactly equivalent to the conditions when $b(x)$ and $\sigma(x)$ are general power functions.

Figure 2 shows three separate graphs depicting the phenomenon of noise-induced stabilization in the case where the drift and noise coefficients are polynomials. The graph on the far left shows the solution to the ODE $dX(t) = (X(t)^3 + X(t) - 1)dt$ with initial condition

$X(0) = 1$, which diverges off to infinity and is thus unstable. The center graph depicts a simulation of the corresponding SDE with noise coefficient $\sigma(X(t)) = \sqrt{2}X(t)$, which is insufficient for stabilization since $1 = p < \frac{q+1}{2} = 2$. The final image depicts a simulation of the SDE where the noise term $\sqrt{2}X(t)^2 dB(t)$ is added to the original unstable ODE. This SDE is stable and, in fact, exhibits "minimal" noise-induced stabilization since $p = \frac{q+1}{2}$ and $a^2 = 2r$.

Proof of Theorem 4.1. Without loss of generality, we assume $r > 0$.

Case 1: $\boxed{q - 2p + 1 \neq 0}$. Since $b(x)$ is a polynomial of degree q with leading coefficient $r > 0$ and $\sigma^2(x)$ is a polynomial of degree $2p$ with leading coefficient $a^2 > 0$, there exist positive constants k_1, k_2, and ℓ such that

$$k_1 x^{q-2p} \leq \frac{b(x)}{\sigma^2(x)} \leq k_2 x^{q-2p}$$

for all $x \geq \ell$. Let $n = q - 2p + 1$. Then for $x \geq \ell$,

$$\exp\left[\frac{-2k_2}{n}(x^n - \ell^n)\right] \leq s(x) \leq \exp\left[\frac{-2k_1}{n}(x^n - \ell^n)\right].$$

Hence, we observe that $S(\infty)$ diverges if and only if $n < 0$, which corresponds to $p > \frac{q+1}{2}$. In addition, for $x \geq \ell$,

$$m(x) \leq \frac{1}{\sigma^2(x)} \exp\left[\frac{2k_2}{n}(x^n - \ell^n)\right].$$

Hence, when $n < 0$, $M(\infty)$ converges as long as the degree of $\sigma^2(x)$ is strictly greater than one. This condition of $2p > 1$ is guaranteed since $n < 0$ implies $2p > q + 1$ and the smallest possible value of q is zero. The proofs for $S(-\infty)$ and $M(-\infty)$ follow similarly. Therefore, by the "Stochastic Stability Theorem," when $p \neq \frac{q+1}{2}$, $X(t)$ is stable if and only if $p > \frac{q+1}{2}$.

Case 2: $\boxed{q - 2p + 1 = 0}$. Since $b(x)$ is a polynomial of degree q with leading coefficient $r > 0$ and $\sigma^2(x)$ is a polynomial of degree $2p$ with leading coefficient $a^2 > 0$, there exist positive constants c_1, c_2, k, and ℓ such that

$$\frac{r}{a^2}x^{-1} - kx^{-2} \leq \frac{b(x)}{\sigma^2(x)} \leq \frac{r}{a^2}x^{-1} + kx^{-2}$$

and

$$c_1 x^{2p} \leq \sigma^2(x) \leq c_2 x^{2p}$$

for all $x \geq \ell$. Then for $x \geq \ell$,

$$\left(\frac{x}{\ell}\right)^{\frac{-2r}{a^2}} \exp\left[2k(x^{-1} - \ell^{-1})\right] \leq s(x) \leq \left(\frac{x}{\ell}\right)^{\frac{-2r}{a^2}} \exp\left[-2k(x^{-1} - \ell^{-1})\right].$$

Hence, $S(\infty)$ diverges if and only if $\frac{2r}{a^2} \leq 1$, which is equivalent to $a^2 \geq 2r$. Because of the bounds on $s(x)$ and $\sigma^2(x)$, we obtain that

$$\frac{x^{\frac{2r}{a^2}-2p} \exp\left[2k(x^{-1} - \ell^{-1})\right]}{c_2 \ell^{\frac{2r}{a^2}}} \leq m(x) \leq \frac{x^{\frac{2r}{a^2}-2p} \exp\left[-2k(x^{-1} - \ell^{-1})\right]}{c_1 \ell^{\frac{2r}{a^2}}}$$

for $x \geq \ell$. Thus, $M(\infty)$ converges if and only if $\frac{2r}{a^2} - 2p < -1$, which is equivalent to $a^2 q > 2r$. The proofs for $S(-\infty)$ and $M(-\infty)$ follow similarly. Therefore, by the "Stochastic Stability Theorem," $X(t)$ is stable when $p = \frac{q+1}{2}$ if and only if $a^2 \geq 2r$ and $a^2 q > 2r$. When $q = 1$, the stricter condition is $a^2 > 2r$, and when $q \geq 3$, the stricter condition is $a^2 \geq 2r$. □

While the stability classification in the case where $b(x)$ and $\sigma(x)$ are polynomials is identical to the case where $b(x)$ and $\sigma(x)$ are general power functions, it is not true that any continuous functions $b(x)$ and $\sigma(x)$ with asymptotic behavior equal to rx^q and ax^p, respectively, will have the same stability classification. For example, consider the SDE with $b(x) = rx^3$ and $\sigma(x) = \sqrt{2r}x^2$ where $r > 0$. Then by Theorem 4.1, the SDE exhibits noise-induced stabilization since $a^2 = 2r$ and $p = \frac{q+1}{2}$. However, consider instead the SDE with the same $b(x)$, but with

$$\sigma(x) = \sqrt{2r}x^2 \left(1 + \frac{2}{\ln(|x|)}\right)^{-\frac{1}{2}} \tag{6}$$

for $|x| \geq 2$ ($\sigma(x)$ can be anything nonzero for $|x| < 2$ such that the function is continuous and locally Lipschitz). Note that the asymptotic behavior of $\sigma(x)$ is indeed $\sqrt{2r}x^2$. Setting $\ell = 2$, for $x \geq \ell$,

$$\begin{aligned} s(x) &= \exp\left[\int_2^x \frac{-1}{z} + \frac{-2}{z\ln(z)} dz\right] \\ &= \exp\left[-\ln(x) + \ln(2) - 2\ln(\ln(x)) + 2\ln(\ln(2))\right] \\ &= \frac{2(\ln(2))^2}{x(\ln(x))^2}. \end{aligned}$$

Hence,

$$S(\infty) = \int_2^\infty \frac{2(\ln(2))^2}{x(\ln(x))^2} dx = 2\ln(2) < \infty$$

and the SDE is unstable by the "Stochastic Stability Theorem." Thus, the reason that an SDE where $b(x)$ and $\sigma(x)$ are any polynomials has the same stability classification as the corresponding SDE where $b(x)$ and $\sigma(x)$ are replaced with just their leading terms is because the difference between a polynomial and its leading term is bounded by a function whose leading term is one degree less. This fact was essential to the proof of Theorem 4.1 in the critical case where $p = \frac{q+1}{2}$, but this property did not hold for the example $\sigma(x)$ defined by (6).

5. Exponential Function Stabilization

This section considers ODEs where the drift coefficient is an exponential function, i.e.

$$dX(t) = r(\exp[X(t)])^q dt$$

where r and q are any real numbers. These ODEs are unstable for any $r \neq 0$. We consider the addition of a noise coefficient that is also an exponential function and determine under which conditions the resulting SDEs are stable.

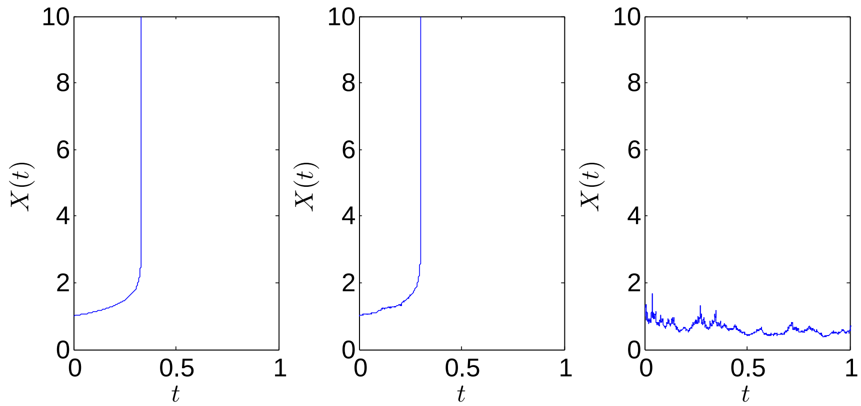

FIGURE 3. Solution to unstable exponential ODE (left), along with simulations of corresponding SDE without sufficient noise for stabilization (center) and SDE with sufficient noise for stabilization (right).

Theorem 5.1. Consider the SDE

$$dX(t) = r(\exp[X(t)])^q dt + a(\exp[X(t)])^p dB(t)$$

where r, q, a,and p are real numbers. Noise-induced stabilization occurs if and only if $r \neq 0$, $a \neq 0$ and one of the following sets of conditions is met:

- $r > 0$ and $p > \max\{0, \frac{q}{2}\}$ or
- $r < 0$ and $-p > \max\{0, \frac{-q}{2}\}$.

Figure 3 shows three separate graphs depicting the phenomenon of noise-induced stabilization in the case where the drift and noise coefficients are exponential functions. The graph on the far left shows the solution to the ODE $dX(t) = 0.05\exp(3X(t))dt$ with initial condition $X(0) = 1$, which diverges off to infinity and is thus unstable. The center graph depicts a simulation of the corresponding SDE with noise coefficient $\sigma(X(t)) = 0.05\exp(X(t))$, which is insufficient for stabilization since $1 = p < \frac{q}{2} = \frac{3}{2}$. The final image depicts a simulation of the SDE where the noise term $0.05\exp(4X(t))dB(t)$ is added to the original unstable ODE. This SDE is stable since $4 = p > \frac{q}{2} = \frac{3}{2}$.

Proof of Theorem 5.1. If $Y(t) = -X(t)$, then $Y(t)$ must have the same stability classification as $X(t)$ since they have the same magnitude. Now

$$\begin{aligned} dY(t) &= -dX(t) \\ &= -r(\exp[X(t)])^q dt - a(\exp[X(t)])^p dB(t) \\ &= -r(\exp[-Y(t)])^q dt - a(\exp[-Y(t)])^p dB(t) \\ &= -r(\exp[Y(t)])^{-q} dt - a(\exp[Y(t)])^{-p} dB(t). \end{aligned}$$

Hence, the stability of $X(t)$ with $-r$ must be equivalent to the stability with r, but with $-q$, $-p$, and $-a$ substituted for q, p, and a, respectively. Thus, when proving Theorem 5.1, it suffices to prove the case with $r > 0$.

With $\ell = 1$, the $s(x)$ term from the "Stochastic Stability Theorem" takes the form

$$s(x) = \begin{cases} \exp\left[\frac{-2r}{a^2}\int_1^x \exp[(q-2p)z]dz\right] & \text{for } x > 1 \\ \exp\left[\frac{-2r}{a^2}\int_{-1}^x \exp[(q-2p)z]dz\right] & \text{for } x < -1. \end{cases}$$

Case 1: $\boxed{q - 2p = 0}$. Integrating yields

$$s(x) = \begin{cases} \exp\left[\frac{-2r}{a^2}(x-1)\right] & \text{for } x > 1 \\ \exp\left[\frac{-2r}{a^2}(x+1)\right] & \text{for } x < -1. \end{cases}$$

Hence, $S(\infty)$ converges and $X(t)$ is unstable when $p = \frac{q}{2}$.

Case 2: $\boxed{q - 2p \neq 0}$. Integrating yields

$$s(x) = \begin{cases} c_1 \exp\left[\frac{-2r}{a^2(q-2p)}\exp[(q-2p)x]\right] & \text{for } x > 1 \\ c_2 \exp\left[\frac{-2r}{a^2(q-2p)}\exp[(q-2p)x]\right] & \text{for } x < -1 \end{cases}$$

where $c_1 = \exp\left[\frac{2r}{a^2(q-2p)}\exp[q-2p]\right]$ and $c_2 = \exp\left[\frac{2r}{a^2(q-2p)}\exp[-q+2p]\right]$. Hence, $S(\infty)$ and $S(-\infty)$ both diverge if and only if $q - 2p < 0$. Plugging in our expression for $s(x)$ gives

$$m(x) = \begin{cases} c_1 \exp[-2px]\exp\left[\frac{2r}{a^2(q-2p)}\exp[(q-2p)x]\right] & \text{for } x > 1 \\ c_2 \exp[-2px]\exp\left[\frac{2r}{a^2(q-2p)}\exp[(q-2p)x]\right] & \text{for } x < -1. \end{cases}$$

If $q - 2p < 0$, then $M(\infty)$ and $M(-\infty)$ both converge if and only if $p > 0$. Hence, by the "Stochastic Stability Theorem," $X(t)$ is stable when $p > \max(0, \frac{q}{2})$ and unstable otherwise. □

6. Logarithmic Function Stabilization

In this section we investigate the stabilization of ODEs where the drift coefficient is a logarithmic function and we perturb the systems by adding a noise coefficient that is also logarithmic.

Theorem 6.1. Consider the SDE

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t)$$

where

$$b(x) = \begin{cases} r(\ln|x|)^q & \text{for } |x| \geq 2 \\ r(\ln 2)^q & \text{for } |x| < 2 \end{cases}$$

and

$$\sigma(x) = \begin{cases} a|x|^m(\ln|x|)^p & \text{for } |x| \geq 2 \\ a2^m(\ln 2)^p & \text{for } |x| < 2 \end{cases}$$

with r q, a, p, and m any real numbers. Noise-induced stabilization occurs if and only if $r \neq 0$, $a \neq 0$, and one of the following conditions is met:

- $m > \frac{1}{2}$,

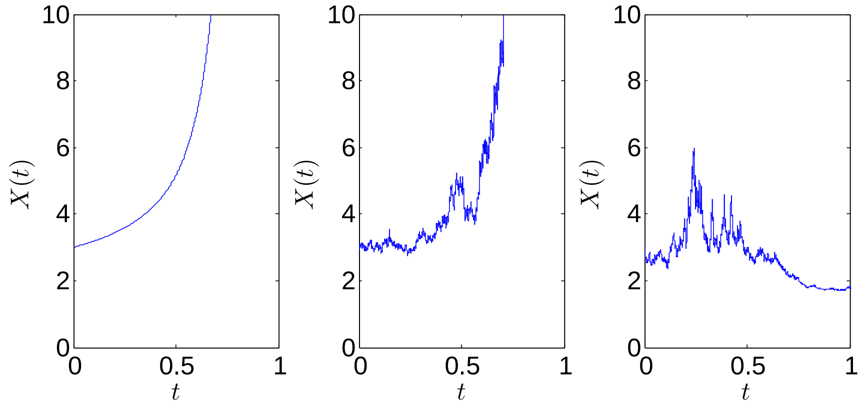

FIGURE 4. Solution to unstable logarithmic ODE (left), along with simulations of corresponding SDE without sufficient noise for stabilization (center) and SDE with sufficient noise for stabilization (right).

- $m = \frac{1}{2}$ and $p > \max(\frac{1}{2}, \frac{q+1}{2})$, or
- $m = \frac{1}{2}$ and $p = \frac{q+1}{2}$ and $a^2 q > 2|r|$.

Note that $r \neq 0$ is necessary since the ODE is already stable if $r = 0$. When $m = 0$, the noise coefficient has the same exact form as the drift coefficient, but it is impossible for the SDE to be stable in this case. Hence, when considering drift coefficients that have a logarithmic form, we allow the noise coefficient to have a slightly more general form in order to obtain noise-induced stabilization. The noise and drift coefficients are defined piecewise with a constant value for $|x| < 2$ simply to avoid a discontinuity at $x = 0$; the precise behavior for x near 0 does not affect the noise-induced stabilization as the stabilization depends on the behavior of the noise and drift coefficients as x approaches infinity. The "minimum" conditions necessary for noise-induced stabilization are $m = \frac{1}{2}$, $p = \frac{q+1}{2}$, and $a^2 q > 2|r|$, with stabilization being the most sensitive to the value of m, which is the power of $|x|$, followed by the value of p, which is the power of $\ln|x|$, and least sensitive to the value of a.

Figure 4 shows three separate graphs depicting the phenomenon of noise-induced stabilization in the case where the drift and noise coefficients are logarithmic functions. The graph on the far left shows the solution to the ODE $dX(t) = (\ln|X(t)|)^5 dt$ with initial condition $X(0) = 3$, which diverges off to infinity and is thus unstable. The center graph depicts a simulation of the corresponding SDE with noise coefficient $\sigma(X(t)) = (\ln|X(t)|)^3$, which is insufficient for stabilization since $m = 0$. The final image depicts a simulation of the SDE where the noise term $|X(t)|^{\frac{1}{2}}(\ln|X(t)|)^3 dB(t)$ is added to the original unstable ODE. This SDE is stable since $m = \frac{1}{2}$, $p = \frac{q+1}{2}$ and $a^2 q > 2r$.

Proof of Theorem 6.1. If $Y(t) = -X(t)$, then $Y(t)$ must have the same stability classification as $X(t)$ since they have the same magnitude. Now $Y(t)$ is the solution to the same SDE as $X(t)$, but with r and a replaced with $-r$ and $-a$. Thus, when proving Theorem 6.1, it suffices to prove the case with $r > 0$ and $a \neq 0$. With $\ell = 2$, the $s(x)$ term from the "Stochastic Stability Theorem," is

$$s(x) = \begin{cases} \exp\left[\frac{-2r}{a^2}\int_2^x \frac{(\ln|z|)^{q-2p}}{|z|^{2m}}dz\right] & \text{for } x > 2 \\ \exp\left[\frac{-2r}{a^2}\int_{-2}^x \frac{(\ln|z|)^{q-2p}}{|z|^{2m}}dz\right] & \text{for } x < -2. \end{cases}$$

Case 1: $\boxed{m > \frac{1}{2}}$. Then there exists a positive constant c_1 such that $(\ln|x|)^{q-2p} \leq c_1|x|^{m-\frac{1}{2}}$ for all $x \geq 2$. Hence, for $x > 2$,

$$s(x) \geq \exp\left[\frac{-2rc_1}{a^2}\int_2^x |z|^{-m-\frac{1}{2}}dz\right] = \exp\left[\frac{-2rc_1(|x|^{-m+\frac{1}{2}} - 2^{-m+\frac{1}{2}})}{a^2(-m+\frac{1}{2})}\right].$$

Letting $k_1 = \exp\left[\frac{2rc_1 2^{-m+\frac{1}{2}}}{a^2(-m+\frac{1}{2})}\right]$, we see that $s(x) \geq k_1$ for all $x > 2$ and hence, $S(\infty) = \infty$. By our bound for $s(x)$, we obtain that for $x > 2$,

$$m(x) \leq \frac{1}{k_1 a^2 |x|^{2m}(\ln|x|)^{2p}}.$$

Since $2m > 1$, $M(\infty) < \infty$. The results for $S(-\infty)$ and $M(-\infty)$ follow similarly, and hence $X(t)$ is stable whenever $m > \frac{1}{2}$.

Case 2: $\boxed{m < \frac{1}{2}}$. Then there exists a positive constant c_2 such that $(\ln|x|)^{q-2p} \geq c_2|x|^{m-\frac{1}{2}}$ for all $x \geq 2$. Hence, for $x > 2$,

$$s(x) \leq \exp\left[\frac{-2rc_2}{a^2}\int_2^x |z|^{-m-\frac{1}{2}}dz\right] = \exp\left[\frac{-2rc_2(|x|^{-m+\frac{1}{2}} - 2^{-m+\frac{1}{2}})}{a^2(-m+\frac{1}{2})}\right].$$

Since $-m + \frac{1}{2} > 0$, $S(\infty) < \infty$, and thus $X(t)$ is unstable.

Case 3: $\boxed{m = \frac{1}{2}}$. Then $s(x)$ takes the form

$$s(x) = \begin{cases} \exp\left[\frac{-2r}{a^2}\int_2^x \frac{(\ln|z|)^{q-2p}}{|z|}dz\right] & \text{for } x > 2 \\ \exp\left[\frac{-2r}{a^2}\int_{-2}^x \frac{(\ln|z|)^{q-2p}}{|z|}dz\right] & \text{for } x < -2. \end{cases}$$

Case 3.1: $\boxed{q - 2p + 1 \neq 0}$. Integrating with a u-substitution where $u = \ln|z|$ and $du = \frac{dz}{z}$, we obtain

$$s(x) = \begin{cases} c_1 \exp\left[\frac{-2r(\ln|x|)^{q-2p+1}}{a^2(q-2p+1)}\right] & \text{for } x > 2 \\ c_2 \exp\left[\frac{2r(\ln|x|)^{q-2p+1}}{a^2(q-2p+1)}\right] & \text{for } x < -2 \end{cases}$$

where c_1 and c_2 are positive constants. When $q - 2p + 1 < 0$, $S(\infty) = \infty$. In addition,

$$M(\infty) = \int_2^\infty \frac{\exp\left[\frac{2r(\ln|x|)^{q-2p+1}}{a^2(q-2p+1)}\right]}{c_1 a^2 |x|(\ln|x|)^{2p}}dx = \frac{1}{c_1 a^2}\int_{\ln 2}^\infty \frac{\exp\left[\frac{2r}{a^2}\frac{u^{q-2p+1}}{q-2p+1}\right]}{u^{2p}}du.$$

This integral converges if and only if $q-2p+1<0$ and $2p>1$. The results for $S(-\infty)$ and $M(-\infty)$ are analogous, and thus when $m=\frac{1}{2}$ and $q-2p+1\neq 0$, $X(t)$ is stable if and only if $p>\max(\frac{1}{2},\frac{q+1}{2})$.

Case 3.2: $\boxed{q-2p+1=0}$. In this case,

$$s(x)=\begin{cases} c(\ln|x|)^{\frac{-2r}{a^2}} & \text{for } x>2 \\ c^{-1}(\ln|x|)^{\frac{2r}{a^2}} & \text{for } x<-2 \end{cases}$$

where $c=(\ln 2)^{\frac{2r}{a^2}}$. Hence, $S(\infty)=\infty$ for any value of $a\neq 0$. In addition,

$$M(\infty)=\int_2^\infty \frac{(\ln|x|)^{\frac{2r}{a^2}-2p}}{ca^2|x|}dx=\frac{1}{ca^2}\int_{\ln 2}^\infty u^{\frac{2r}{a^2}-2p}.$$

This integral converges if and only if $\frac{2r}{a^2}-2p<-1$. The results for $S(-\infty)$ and $M(-\infty)$ are analogous, and thus when $m=\frac{1}{2}$ and $p=\frac{q+1}{2}$, $X(t)$ is stable if and only if $a^2q>2r$. □

7. Conclusion

In this paper we have investigated the phenomenon of noise-induced stabilization, in which the addition of randomness to an unstable system of ODEs creates a system of stable SDEs, where our notion of stability is that of global stochastic boundedness. In particular, we have proven the precise minimum amount of noise necessary for stabilization of one-dimensional diffusions when the drift and noise coefficients are general power functions, polynomials, exponential functions, or logarithmic functions.

Future work could investigate when noise-induced stabilization occurs for other particular forms of the drift and noise coefficients. In addition, we could explore for a given drift coefficient, what is the minimum amount of noise required for stabilization when the noise coefficient is not restricted to a particular form. Other work could explore minimum noise requirements for noise-induced stabilization in two-dimensional or higher systems, where proving stabilization is much more complex.

8. Acknowledgments

We would like to thank the National Science Foundation Grant No. DMS-1262852 for funding to support this research.

References

[1] Richard Durrett. *Stochastic calculus*. Probability and Stochastics Series. CRC Press, Boca Raton, FL, 1996. A practical introduction.

[2] Samuel Karlin and Howard M. Taylor. *A second course in stochastic processes*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.

[3] Rafail Khasminskii. *Stochastic stability of differential equations*, volume 66 of *Stochastic Modelling and Applied Probability*. Springer, Heidelberg, second edition, 2012. With contributions by G. N. Milstein and M. B. Nevelson.

[4] Bernt Øksendal. *Stochastic differential equations*. Universitext. Springer-Verlag, Berlin, sixth edition, 2003. An introduction with applications.

[5] M. Scheutzow. Noise-induced transitions for one-dimensional diffusions. *Stochastic Anal. Appl.*, 14(5):535–563, 1996.

[6] James Stewart. *Calculus: Early Transcendentals*. Thompson Brooks/Cole, 6th edition, 2008.

Student biographies

Tony Allen (*Corresponding author:* tgallen@mix.wvu.edu) completed an REU at Valparaiso University in the summer of 2015. Since then, he has graduated with a B.S. in Mathematics from West Virginia University in May 2017, and will begin graduate school at Purdue University in the fall.

Emily Gebhardt graduated from Mercyhurst University in May 2016 with a B.A. in Mathematics and Physics. In the summer of 2015, she completed an REU at Valparaiso University. Currently, she is completing her master's degree in Biostatistics at Emory University.

Adam Kluball graduated from Bethany Lutheran College in May 2016 with a B.A. in Mathematics and Chemistry. In the summer of 2015, he completed an REU at Valparaiso University. Currently, he is completing his master's degree in Mathematics at Montana State University.