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The Minnesota Journal of Undergraduate Mathematics

Volume 3 (2017-2018 Academic Year)

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# Chaotic Dynamics in a Family of Set-Valued Functions

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**ABSTRACT.** We explore the topological dynamics of a family of set-valued functions introduced by W. T. Ingram in 2015. We focus on three particular properties associated with chaotic behavior: Devaney chaos, the specification property, and positive topological entropy. We show that for certain parameters, the function exhibits all three forms of chaos. We also discuss interesting patterns that arise in the periodic points.

## 1. INTRODUCTION

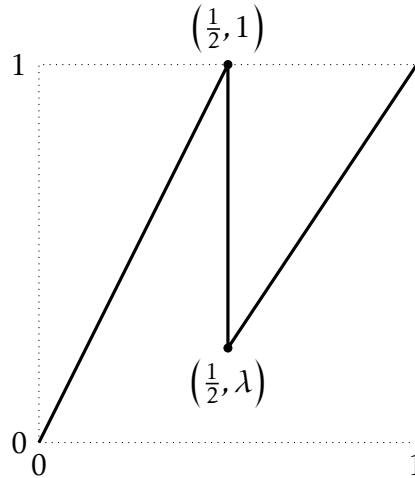
There are numerous ways to define what it means for a topological dynamical system to be chaotic. Perhaps the most widely used definition of chaos, now called Devaney chaos (see Definition 2.4), is given by Devaney in [6]. Another is positive topological entropy, a topological invariant introduced by Adler, Konheim, and McAndrew [1]. Stronger than both of these is the specification property, introduced by Bowen in [3]. We study these three properties in the context of a particular family of set-valued functions.

In 2004, Mahavier [12] began the study of inverse limits of upper semi-continuous, set-valued functions. In recent years, there has been significant research in this area, primarily focusing on the continuum theoretic properties of these inverse limits. Many of the fundamental results concerning inverse limits of set-valued functions can be found in [7]. In more recent years, there has been an increased focus on set-valued functions from the perspective of topological dynamics. Raines and Tennant [14] give the definitions of Devaney chaos, the specification property, and topological entropy for set-valued functions, and topological entropy of set-valued functions is further explored by Kelly and Tennant in [9]. Other recent work on the dynamics of set-valued functions can be found in [13, 5, 10].

In this paper, we focus on a family of set-valued functions introduced by Ingram in [8]. The functions depend on one parameter,  $\lambda$ , and are defined on  $[0, 1]$  by

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FIGURE 1. The graph of  $F_\lambda$ 

$$F_\lambda(x) = \begin{cases} \{2x\} & x \in [0, \frac{1}{2}) \\ [\lambda, 1] & x = \frac{1}{2} \\ \{2(1-\lambda)(x-1)+1\} & x \in (\frac{1}{2}, 1] \end{cases} \quad (1)$$

where  $0 \leq \lambda \leq 1$ . (The graph of  $F_\lambda$  is pictured in Figure 1.) Ingram studied the topology of the inverse limits for these functions and noticed interesting properties emerged when  $\lambda$  was of the form  $2^{-n}$  for some  $n \in \mathbb{N}$ . Our focus will be the topological dynamics of this family of functions and their dependence on  $\lambda$ .

## 2. BACKGROUND AND DEFINITIONS

We begin by defining a metric on a set.

**Definition 2.1.** Let  $X$  be a set. A *metric* on  $X$  is a function  $d : X \times X \rightarrow \mathbb{R}$  satisfying the following for all  $x, y, z \in X$ :

- (1)  $d(x, y) \geq 0$ ,
- (2)  $d(x, y) = 0$  if and only if  $x = y$ ,
- (3)  $d(x, y) = d(y, x)$ ,
- (4)  $d(x, z) \leq d(x, y) + d(y, z)$ .

A *metric space* is an ordered pair,  $(X, d)$ , where  $X$  is a set and  $d$  is a metric on  $X$ . We say that  $X$  is *compact* if every sequence in  $X$  has a subsequence converging to a point in  $X$ . Given a compact metric space,  $X$ , we denote the set of all non-empty, compact subsets of  $X$  by  $2^X$ .

If  $X$  and  $Y$  are compact metric spaces, a function  $F : X \rightarrow 2^Y$  is *upper semi-continuous* at a point  $x \in X$  if, for every open set  $U \subseteq Y$  containing  $F(x)$ , there exists an open set,  $V \subseteq X$ ,

containing  $x$  such that  $F(v) \subseteq U$  for all  $v \in V$ . If  $F$  is upper semi-continuous at every point  $x \in X$ , we simply say that  $F$  is *upper semi-continuous*.

Any single-valued function  $f : X \rightarrow Y$  can be viewed as a set-valued function by defining  $F : X \rightarrow 2^Y$  to be  $F = \{f(x)\}$ . Note that in this case,  $F$  is upper semi-continuous if and only if  $f$  is continuous.

This paper focuses on the topological dynamics of the family of set-valued functions  $F_\lambda$  defined in Section 1. A *topological dynamical system* is an ordered pair,  $(X, F)$ , where  $X$  is a compact metric space set and  $F : X \rightarrow 2^X$  is upper semi-continuous. For the sake of brevity, we will refer to a topological dynamical system as a dynamical system.

If  $X, Y$ , and  $Z$  are compact metric spaces with  $F : X \rightarrow 2^Y$  and  $G : Y \rightarrow 2^Z$  then we define the *composition of  $G$  with  $F$*  to be the function  $G \circ F : X \rightarrow 2^Z$  given by

$$G \circ F(x) = \bigcup_{y \in F(x)} G(y).$$

Note that if  $F$  and  $G$  are upper semi-continuous then  $G \circ F$  is as well.

Let  $(X, F)$  be a dynamical system. We define the following for all  $x \in X$ :

$$\begin{aligned} F^0(x) &= \{x\} \\ F^1(x) &= F(x) \\ F^2(x) &= F \circ F(x) \\ &\vdots \\ F^n(x) &= F \circ F^{n-1}(x) \end{aligned}$$

for all  $n \in \mathbb{N}$ . Note that  $F^{n+m}(x) = F^n(x) \circ F^m(x)$  for all  $n, m \in \mathbb{N}$ .

An *orbit* of a point  $x$  is a sequence of points  $(x_0, x_1, x_2, \dots)$  such that  $x_{i+1} \in F(x_i)$  for all  $i \in \mathbb{N}$  and  $x_0 = x$ . Note that a point,  $x$ , may have more than one orbit. In this case we may choose any particular orbit of interest. We denote the set of all orbits by  $\text{Orb}(X, F)$ . We define the first  $n$  terms of an orbit, called an  *$n$ -orbit*, as a finite sequence  $(x_0, \dots, x_{n-1})$  in  $X$  such that for each  $i = 0, \dots, n-2$ ,  $x_{i+1} \in F(x_i)$ . We denote the set of all  $n$ -orbits by  $\text{Orb}_n(X, F)$ .

(In this paper, sequences—both finite and infinite—will be written in bold, and their terms will be written in italics.)

Let  $x \in X$  and  $(x_i)_{i=0}^\infty$  be an orbit of  $x$ . The orbit is *periodic* if there is some  $n \in \mathbb{N}$  such that  $(x_i)_{i=0}^\infty = (x_j)_{j=n}^\infty$ . The smallest such  $n$  satisfying this condition is called the *period* of the orbit. We say that a point  $x \in X$  is *periodic* if it has a periodic orbit. If  $(x_i)_{i=1}^\infty$  is an orbit and  $x_i = x_j$  for all  $i, j \geq 0$  then we call it a *fixed orbit*. A point is a *fixed point* if it has a fixed orbit. Note again that a single point may have many orbits. For a point to be periodic (or fixed), we require only that it have at least one such orbit.

In this paper we discuss our function and its potential chaotic behavior. In order to define chaos, we must first define a dense set and a transitive function.

**Definition 2.2.** Let  $X$  be a metric space. A set  $A \subseteq X$  is *dense* in  $X$  if and only if for all  $x \in X$  and  $\varepsilon > 0$  there exists  $a \in A$  such that  $d(x, a) < \varepsilon$ .

**Definition 2.3.** Let  $(X, F)$  be a dynamical system. We say  $F$  is *topologically transitive* if and only if for all  $x, y \in X$  and any  $\varepsilon > 0$  there exists  $z \in X$  and  $n \in \mathbb{N}$  such that  $d(z, y) < \varepsilon$  and  $d(p, x) < \varepsilon$  for some  $p \in F^n(z)$ .

There are a few unique definitions of chaos in mathematics. The first we introduce is known as Devaney chaos.

**Definition 2.4.** A dynamical system,  $(X, F)$ , is *Devaney chaotic* if and only if:

- (1) it is topologically transitive,
- (2) the set of periodic points is dense in  $X$ .

A way to quantify the chaos of a system is through another property, the topological entropy. A system having positive topological entropy is another definition of chaos. In order to define topological entropy we must first give a few preliminary definitions.

**Definition 2.5.** Let  $X$  be a compact metric space. A set  $S \subseteq X$  is  $\varepsilon$ -*separated* if for each  $x, y \in S$  and  $x \neq y$ ,  $d(x, y) \geq \varepsilon$ .

**Definition 2.6.** Let  $(X, F)$  be a dynamical system and  $n \in \mathbb{N}$ . We define a metric  $d_n$  on  $\text{Orb}_n(X, F)$  as follows: if  $\mathbf{x} = (x_0, \dots, x_{n-1})$  and  $\mathbf{y} = (y_0, \dots, y_{n-1})$  are  $n$ -orbits, then

$$d_n(\mathbf{x}, \mathbf{y}) = \max_{0 \leq j \leq n-1} d(x_j, y_j).$$

Given  $\varepsilon > 0$ , we define the number  $s_{n,\varepsilon}(F)$  to be the largest cardinality of an  $\varepsilon$ -separated subset of  $\text{Orb}_n(X, F)$ .

**Definition 2.7.** Let  $(X, F)$  be a dynamical system. The *topological entropy* of  $F$  is defined to be

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n} \log s_{n,\varepsilon}(F)$$

where  $\log$  represents the natural logarithm.

A third property associated with chaos is the specification property which is introduced by Bowen in [3]. For continuous, single-valued functions, Sigmund showed in [15] that every continuous function with the specification property is also Devaney chaotic, and the same is shown for set-valued functions in [14] where the specification property is first defined for set-valued functions. This property states that a single periodic orbit can be used to approximate segments of arbitrarily many other orbits in the system. In the context of continuous, single-valued functions, the specification property and its many variations has been widely studied. A thorough presentation of the various forms of this property and their implications is given in [11].

In this definition, we discuss a list of orbits  $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_s \in \text{Orb}(X, F)$ . Given  $1 \leq n \leq s$ , we write  $\mathbf{y}_n = (y_i^n)_{i=0}^{\infty}$ . (Thus  $y_i^n$  represents the  $i$ th term in the orbit  $\mathbf{y}_n$ .)

**Definition 2.8.** Let  $(X, F)$  be a dynamical system. We say  $F$  has the *specification property* if for every  $\varepsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for any  $\mathbf{y}_1, \dots, \mathbf{y}_s \in \text{Orb}(X, F)$ , integers  $0 = a_1 \leq b_1 < a_2 \leq b_2 < \dots < a_s \leq b_s$  satisfying  $a_{n+1} - b_n > N$  for  $n = 1, \dots, s-1$ , and any natural number  $P > b_s + N$ , there exists a periodic orbit  $\mathbf{x} \in \text{Orb}(X, F)$  with period  $P$  such that for all  $n = 1, \dots, s$  and  $a_n \leq i \leq b_n$ , we have  $d(x_i, y_i^n) < \varepsilon$ .

Recall that we study a parameterized family of set-valued functions  $F_\lambda: [0, 1] \rightarrow 2^{[0,1]}$  given by (1). We will often consider the three pieces of this function independently. For convenience, we define  $\phi: \mathbb{R} \rightarrow \mathbb{R}$  and  $\psi: \mathbb{R} \rightarrow \mathbb{R}$  respectively by  $\phi(x) = 2x$  and  $\psi(x) = 2(1 - \lambda)(x - 1) + 1$ .

### 3. DYNAMICS WITH $\lambda \geq \frac{1}{2}$

The dynamics of  $F_\lambda$  are relatively mundane when  $\lambda \geq 1/2$ . It is obvious that there are at least two fixed points at  $x = 0$  and  $x = 1$ . However, there are two distinct results depending on if  $\lambda = 1/2$  or  $\lambda > 1/2$ . The most immediate difference being that if  $\lambda > 1/2$ , then  $x = 0$  and  $x = 1$  are the only fixed points, while for  $\lambda = 1/2$  the entire interval  $[1/2, 1]$  is fixed. The following two theorems summarize the dynamics for  $\lambda \geq 1/2$  so we can focus on  $\lambda < 1/2$ .

**Theorem 3.1.** *Suppose  $\lambda > 1/2$  and  $x \in (0, 1]$ . If  $(x_j)_{j=0}^\infty$  is any orbit of  $x$ , then  $x_j \rightarrow 1$  as  $j \rightarrow \infty$ .*

*Proof.* Let  $\{x_0, x_1, \dots\}$  be an orbit of  $x$ , and consider three cases.

Case 1: Suppose  $x > 1/2$ . Note that since  $x > 1/2$ ,  $x_{j+1} = \psi(x_j)$  for all  $j \in \mathbb{N}$ . Because  $\psi$  is continuous and  $\psi(y) > y$  for all  $y > 1/2$ , the orbit of  $x$  is also increasing and bounded above by 1, so it converges to some  $z \in (1/2, 1]$ . Thus we have that  $x_j \rightarrow z$  as  $j \rightarrow \infty$  and  $\psi(x_j) \rightarrow \psi(z)$  as  $j \rightarrow \infty$ , but we also know that  $\psi(x_j) = x_{j+1}$ . It follows that  $x_{j+1} \rightarrow \psi(z)$  and  $x_{j+1} \rightarrow z$  as  $j \rightarrow \infty$ , so  $\psi(z) = z$ . Therefore  $z$  is a fixed point, and the only fixed point of  $\psi$  is 1, so  $x_j \rightarrow 1$  as  $j \rightarrow \infty$ .

Case 2: Suppose  $x = 1/2$ . Then  $x_1 \in F_\lambda(x) = [\lambda, 1]$ , so  $x_1 \geq \lambda > 1/2$ . Note that  $\{x_1, x_2, \dots\}$  is an orbit of  $x_1 > 1/2$ , so by Case 1, we have that  $x_j \rightarrow 1$  as  $j \rightarrow \infty$ .

Case 3: Suppose  $x < 1/2$ . Then for any orbit  $(x_j)_{j=0}^\infty$  of  $x$ , we have  $x_j = 2^j x_0$  for some finite initial segment of the orbit. Then for some  $n \in \mathbb{N}$ , we will have that  $x_n \geq 1/2$ , and the sequence  $(x_j)_{j=n}^\infty$  is an orbit satisfying the conditions of either Case 1 or Case 2. Thus  $x_j \rightarrow 1$  as  $j \rightarrow \infty$ .  $\square$

**Theorem 3.2.** *Suppose  $\lambda = 1/2$  and  $x \in (0, 1]$ . If  $\{x_0, x_1, \dots\}$  is an orbit of  $x$  then both of the following hold:*

- (1) *There exists an integer  $N \geq 0$  such that  $x_j \in [1/2, 1]$  for all  $j \geq N$ .*
- (2) *There exists an integer  $M \geq 0$  such that  $x_j = x_k$  for all  $j, k \geq M$ .*

*Proof.* Let  $\{x_0, x_1, \dots\}$  be an orbit of  $x$ , and consider three cases:

Case 1: Assume  $x \in (1/2, 1]$ . Then for all  $j \in \mathbb{N}$ ,  $x_j = \psi(x_{j-1}) = x_{j-1}$ . Thus the orbit is a fixed orbit, so we may choose  $N = M = 0$ .

Case 2: Assume  $x = 1/2$ . Then it follows that  $x_j \geq 1/2$  for all  $j \geq 0$ , so we may choose  $N = 0$ . To choose  $M$  note that either  $x_j = 1/2$  for all  $j \geq 0$  in which case we may also choose  $M = 0$ ; or there exists  $M \in \mathbb{N}$  such that  $x_k > 1/2$ , so Case 1 applies to the orbit  $(x_j)_{j=M}^\infty$ .

Case 3: Assume  $x \in (0, 1/2)$ . Then just as in Theorem 3.1, we have that  $x_j = 2^j x$  until we reach an integer  $k$  satisfying  $x_k = 2^k x \geq 1/2$ . Then the orbit  $(x_j)_{j=k}^{\infty}$  satisfies the hypotheses of either Case 1 or Case 2.  $\square$

#### 4. PERIODIC POINTS

We now begin examining the dynamics of  $F_\lambda$  when  $\lambda < 1/2$ . A key component of both Devaney chaos and the specification property is the density of the periodic points. We highlight three specific forms of periodic points which exist.

Recall that we define  $\phi: \mathbb{R} \rightarrow \mathbb{R}$  and  $\psi: \mathbb{R} \rightarrow \mathbb{R}$  by  $\phi(x) = 2x$  and  $\psi(x) = 2(1 - \lambda)(x - 1) + 1$ .

**Theorem 4.1.** *Suppose  $\lambda < 1/2$ , and let  $x \in (1/2, 1)$  be the solution to the equation  $\psi(x) = 1/2$ . Then for all natural numbers  $n \geq 2$ ,  $x$  has a periodic orbit of period  $n$ .*

*Proof.* Note that

$$x \in \left(\frac{1}{2}, 1\right) \subseteq [\lambda, 1] = F_\lambda\left(\frac{1}{2}\right),$$

so for any  $n \geq 2$ , we may define a periodic orbit  $(x_j)_{j=0}^{\infty}$  beginning with  $x_0 = x$ ; then for  $j = 1, \dots, n-1$ , we let  $x_j = 1/2$ , and for  $j \geq n$ , let  $x_j = x_{j-n}$ . This orbit takes the form

$$\left( x, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{n-1}, x, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{n-1}, x, \frac{1}{2}, \dots \right).$$

$\square$

We discuss topological entropy more thoroughly in Section 6; however this theorem provides a very simple argument that  $F_\lambda$  has positive topological entropy when  $\lambda < 1/2$ . According to [9, Theorem 6.2], if there is a point with two different periodic orbits, then the system has positive topological entropy. Thus the following corollary follows immediately from Theorem 4.1 and [9, Theorem 6.2].

**Corollary 4.2.** *If  $\lambda < 1/2$ , then  $h(F_\lambda) > 0$ .*

The periodic orbits outlined in Theorem 4.1 all exist so long as  $\lambda < 1/2$ . The next two results discuss periodic points whose existence depends even more on  $\lambda$ .

**Theorem 4.3.** *Let  $\lambda \leq 1/2^n$ . Then  $x = 1/2^n$  is a periodic point with period  $n$ .*

*Proof.* Let  $x_0 = 1/2^n$ . For  $j = 1, \dots, n-1$ , let  $x_j = 1/2^{n-j}$ , so in particular  $x_{n-1} = 1/2$ . Since  $\lambda \leq 1/2^n$ , we have that  $1/2^n \in F_\lambda(1/2)$ , so we may set  $x_n = 1/2^n$ . Likewise, for  $j > n$ , let  $x_j = x_{j-n}$ . The resulting orbit  $(x_j)_{j=0}^{\infty}$  is periodic with period  $n$ .  $\square$

**Theorem 4.4.** *Let  $\lambda < 1/2^n$ . Then there exists a periodic point with period  $n$  of the form*

$$x_\lambda = \frac{-1 + 2\lambda}{2^n(\lambda - 1) + 1}.$$

*Proof.* First, if  $n = 1$ , then  $x_\lambda = 1$ , and  $F_\lambda(1) = \{1\}$ , so 1 is a periodic point of period 1. For the remainder of the proof we suppose that  $n \geq 2$ .

By computation, we have that  $\psi \circ \phi^{n-1}(x_\lambda) = x_\lambda$ . We must show that  $x_\lambda \in F_\lambda^n(x_\lambda)$ . To do this, we show that  $1/2^n < x_\lambda < 1/2^{n-1}$ , so that it will follow that  $F_\lambda^n(x_\lambda) = \{\psi \circ \phi^{n-1}(x_\lambda)\}$ .

To do this, we will think of  $x_\lambda$  as a function of  $\lambda$ . Observe that if we plug in  $\lambda = 1/2^n$ , then  $x_\lambda = 1/2^n$ . Also, if  $\lambda = 0$ , then  $x_\lambda = 1/(2^n - 1) < 1/2^{n-1}$ . Thus to establish that  $1/2^n < x_\lambda < 1/2^{n-1}$  when  $0 < \lambda < 1/2^n$ , it is enough to show that  $x_\lambda$  is a continuous, decreasing function of  $\lambda$  when  $0 < \lambda < 1/2^n$ .

The only point of discontinuity for  $x_\lambda$  is  $\lambda = (2^n + 1)/2^n$  which is greater than 1, so  $x_\lambda$  is continuous on  $0 < \lambda < 1/2^n$ . To see that it is decreasing, observe that

$$\frac{d}{d\lambda} x_\lambda = \frac{2 - 2^n}{(1 - 2^n + 2^n \lambda)^2}.$$

The denominator is positive, and the numerator is negative so long as  $n > 1$ . Thus we have that  $x_\lambda$  is decreasing on the interval  $0 < \lambda < 1/2^n$ . It follows that if  $\lambda < 1/2^n$ , then  $1/2^n < x_\lambda < 1/2^{n-1}$ . Therefore  $x_\lambda \in F_\lambda^n(x_\lambda)$ .  $\square$

## 5. DEVANEY CHAOS

Even when  $\lambda < 1/2$ , the function  $F_\lambda$  is not Devaney chaotic on its entire domain, but we show in this section that when  $F_\lambda$  is restricted to the domain  $[\lambda, 1]$ , it is chaotic. To prove the function is Devaney chaotic on that interval we must first show that the set of periodic points is dense in  $[\lambda, 1]$ , and that  $F_\lambda$  restricted to  $[\lambda, 1]$  is topologically transitive. Our first step in doing this is to prove that the set of pre-images of  $x = 1/2$  is dense in the interval.

In this lemma we discuss the diameter of an interval. If  $I \subseteq \mathbb{R}$  is an interval, we define the *diameter* of  $I$  to be  $\text{diam}(I) = \sup(I) - \inf(I)$ .

**Lemma 5.1.** *Suppose  $\lambda < 1/2$ , and let  $\varepsilon > 0$ . There exists  $N \in \mathbb{N}$  such that for any interval  $I \subset [\lambda, 1]$  with  $\text{diam}(I) \geq \varepsilon$ , we have  $1/2 \in f^N(I)$ .*

*Proof.* Let  $I \subseteq [\lambda, 1]$  be an interval with  $\text{diam}(I) \geq \varepsilon$ . Recall that the slope of  $\phi$  is 2, and the slope of  $\psi$  is  $2(1 - \lambda)$ . Since  $\lambda < 1/2$ , we have that  $2(1 - \lambda) > 1$ . Thus, we may choose  $N \in \mathbb{N}$  such that

$$[2(1 - \lambda)]^N \varepsilon \geq (1 - \lambda) = \text{diam}([\lambda, 1]).$$

Note that since  $1/2 \in F_\lambda(1/2)$ , it suffices to show that  $1/2 \in F^n(I)$  for some  $n \leq N$ .

Now  $I$  is an interval, so if  $1/2 \notin I$ , then we have that either  $I \subseteq [\lambda, 1/2)$  or  $I \subseteq (1/2, 1]$ . Thus either  $F(I) = \phi(I)$  or  $F(I) = \psi(I)$ . Both  $\phi$  and  $\psi$  are continuous functions, so  $F(I)$  is an interval. Moreover, the diameter of  $F(I)$  is greater than the diameter of  $I$  by a factor of either 2 or  $2(1 - \lambda)$  (depending on whether  $F(I) = \phi(I)$  or  $F(I) = \psi(I)$ ). In either case we have that

$$\text{diam}[F(I)] \geq 2(1 - \lambda) \text{diam}(I).$$



It follows that

$$\begin{aligned} \text{diam}(F^N(I)) &\geq [2(1-\lambda)]^N \text{diam}(I) \\ &\geq [2(1-\lambda)]^N \varepsilon \\ &\geq \text{diam}([\lambda, 1]), \end{aligned}$$

so  $1/2 \in F^N(I)$ . □

**Corollary 5.2.** *Suppose  $\lambda < 1/2$ , and let*

$$A = \{x \in [\lambda, 1] : \frac{1}{2} \in F_\lambda^n(x) \text{ for some } n \in \mathbb{N}\}.$$

*Then  $A$  is dense in  $[\lambda, 1]$ .*

*Proof.* Let  $x \in [\lambda, 1]$ , and let  $\varepsilon > 0$ . Choose  $N$  according to Lemma 5.1. Consider the interval  $I = (x-\varepsilon, x+\varepsilon) \cap [\lambda, 1]$ , and note that  $\text{diam}(I) \geq \varepsilon$ . Then by Lemma 5.1,  $1/2 \in F^N(I)$ , so there exists  $a \in I$  such that  $1/2 \in F^N(a)$ . Since  $a \in I$ , we have that  $d(x, a) < \varepsilon$ , and since  $1/2 \in F^N(a)$ , we have that  $a \in A$ . Therefore  $A$  is dense in  $[\lambda, 1]$ . □

Now we have the necessary prerequisite information to prove  $F_\lambda$  is Devaney chaotic on  $[\lambda, 1]$ .

**Theorem 5.3.** *If  $\lambda < 1/2$ , then  $F_\lambda$  is Devaney Chaotic on  $[\lambda, 1]$ .*

*Proof.* By Corollary 5.2 we know that the set of points that contain  $1/2$  in their orbit is dense in  $[\lambda, 1]$ . These are also periodic points, because  $F(1/2) = [1/2, 1/2]$ , so if  $x \in [1/2, 1]$ , and  $1/2 \in F^n(x)$ , then  $x \in F^{n+1}(x)$ , so  $x$  has a periodic orbit. Adding the periodic points not already contained in the set will make the already dense set even more dense. Therefore the set of periodic points is dense in  $[\lambda, 1]$ .

Assume  $x, y \in [\lambda, 1]$ . By Corollary 5.2, for any  $\varepsilon > 0$  there exists  $z$  within  $\varepsilon$  of  $x$  such that  $1/2 \in F_\lambda^n(z)$  for some  $n \in \mathbb{N}$ . Then  $[\lambda, 1] \subseteq F_\lambda^{n+1}(z)$ . In particular,  $y \in F_\lambda^{n+1}(z)$ , so  $F_\lambda$  restricted to  $[\lambda, 1]$  is topologically transitive.

Combining these facts we have that  $F_\lambda$  is Devaney chaotic on  $[\lambda, 1]$ . □

## 6. THE SPECIFICATION PROPERTY

The Specification Property is a strong result that implies a function is Devaney chaotic and has positive topological entropy. Here we prove that  $F_\lambda$  has this property when restricted to the domain  $[\lambda, 1]$ .

**Theorem 6.1.**  *$F_\lambda$  has the Specification Property on  $[\lambda, 1]$ .*

*Proof.* Let  $\varepsilon > 0$ . Choose  $N \in \mathbb{N}$  according to Lemma 5.1. Let  $\mathbf{y}_1, \dots, \mathbf{y}_s \in \text{Orb}([\lambda, 1], F_\lambda)$ , and let  $0 = a_1 \leq b_1 < \dots < a_s \leq b_s$  such that  $a_{n+1} - b_n > N$  for all  $n = 1, \dots, s-1$ . We now begin the process of defining the periodic orbit,  $\mathbf{x}$ . We will define the orbit first in the  $b_n$  coordinate for each  $n \in \mathbb{N}$ .

Case 1: Suppose  $(y_{b_n}^n, y_{b_n}^n + \varepsilon) \in [\lambda, 1]$ . According to Lemma 5.1, we have  $1/2 \in F_\lambda^N((y_{b_n}^n, y_{b_n}^n + \varepsilon))$ . Therefore we may choose  $x_{b_n} \in (y_{b_n}^n, y_{b_n}^n + \varepsilon)$  such that  $1/2 \in F_\lambda^N(x_{b_n})$ . We define the remaining terms of  $\mathbf{x}$  between the  $a_n$  and  $b_n$  coordinates as follows:

Suppose  $x_j$  is defined for some  $a_n < j \leq b_n$  so that  $|x_j - y_j^n| < \varepsilon$ . Since  $\mathbf{y}_n$  is an orbit,  $y_j^n = 2y_{j-1}^n$ ,  $y_j^n = 2(1 - \lambda)(y_{j-1}^n - 1) + 1$ , or  $y_{j-1}^n = 1/2$ . Consider three sub-cases:

Sub-case (a): Suppose  $y_{j-1}^n = 1/2$ . Let  $x_{j-1} = 1/2$ . Clearly,  $|x_j - y_j^n| < \varepsilon$ .

Sub-case (b): Suppose  $y_{j-1}^n \neq 1/2$  and  $y_j^n = 2(1 - \lambda)(y_{j-1}^n - 1) + 1$ . Define

$$x_{j-1} = \frac{1}{2(1 - \lambda)}(x_j - 1) + 1.$$

Then

$$\begin{aligned} |x_{j-1} - y_{j-1}^n| &= \left| \frac{(x_j - 1)}{2(1 - \lambda)} + 1 - \left( \frac{y_j^n - 1}{2(1 - \lambda)} + 1 \right) \right| \\ &= \frac{1}{2(1 - \lambda)} |x_j - y_j^n| \\ &< \frac{1}{2(1 - \lambda)} \varepsilon. \end{aligned}$$

Note that because  $\lambda < 1/2$ ,  $1/[2(1 - \lambda)] < 1$ , so  $|x_{j-1} - y_{j-1}^n| < \varepsilon$ .

Sub-case (c): Suppose  $y_{j-1}^n \neq 1/2$  and  $y_{j-1}^n = 1/2 y_j^n$ . Define  $x_{j-1} = 1/2 x_j$ . Then  $|x_j - y_{j-1}^n| = 1/2 |x_j - y_j^n| < 1/2 \varepsilon < \varepsilon$ .

Note that each  $x_j$  must be chosen from  $[\lambda, 1]$ . In this sub-case, it is important that  $x_{b_n} \in (y_{b_n}^n, y_{b_n}^n + \varepsilon)$ , so for all  $a_n \leq j \leq b_n$ ,  $x_j \geq y_j^n$ . Thus, if  $y_j^n/2 \geq \lambda$ , then  $x_j/2 \geq \lambda$  as well.

The next case deals with the possibility that  $x_{b_n}$  cannot be chosen to be greater than  $y_{b_n}^n$ .

Case 2: Suppose  $y_{b_n}^n + \varepsilon > 1$ .

Subcase (a): Suppose  $y_{j-1}^n = 1/2$ . Let  $x_{j-1} = 1/2$ . Clearly,  $|x_j - y_j^n| < \varepsilon$ .

Subcase (b): Suppose  $y_{j-1}^n \neq 1/2$  and  $y_j^n = 2(1 - \lambda)(y_{j-1}^n - 1) + 1$ . Define

$$x_{j-1} = \frac{1}{2(1 - \lambda)}(x_j - 1) + 1.$$

Then just as in Sub-case (b) of Case 1, it follows that  $|x_{j-1} - y_{j-1}^n| < \varepsilon$ .

Sub-case (c): Suppose  $y_{j-1}^n \neq 1/2$  and  $y_{j-1}^n = 1/2 y_j^n$ . If  $y_{k-1}^n = \psi^{-1}(y_k^n)$  for all  $k = j+1, \dots, b_n$ , then define  $x_{j-1} = 1/2$ . Otherwise define  $x_{j-1} = x_j/2$ .

Since  $\lambda < 1/2$ , we know that  $\psi(x) \leq x$  so in the case that  $y_{k-1}^n = \psi^{-1}(y_k^n)$  for all  $k = j+1, \dots, b_n$  we know that  $y_j^n \geq y_{j+1}^n \geq \dots \geq y_{b_n}^n$  so in particular  $|y_j^n - 1| < \varepsilon$ . In this case we define  $x_{j-1} = 1/2$  so

$$|x_{j-1} - y_{j-1}^n| = \frac{1}{2} |1 - y_{j-1}^n| < \frac{1}{2} |1 - y_j^n| < \frac{1}{2} \varepsilon < \varepsilon.$$

Otherwise define  $x_{j-1} = x_j/2$ , so

$$\left| x_{j-1} - y_{j-1}^n \right| = \frac{1}{2} \left| x_j - y_j^n \right| < \frac{1}{2} \varepsilon < \varepsilon$$

Note that once  $x_{j-1} = 1/2$ ,  $x_{j-1} > y_{j-1}^n$ , so we may revert to the processes of Case 1.

To connect each of these orbit fragments, for each  $n = 1, \dots, s-1$ , recall  $1/2 \in F_\lambda^N(x_{b_n})$ , so for  $b_n < j \leq b_n + N$ , choose  $x_j$  so that  $x_{b_n+N} = 1/2$ . If there are any integers  $j$  with  $b_n + N < j < a_{n+1}$ , let  $x_j = 1/2$ . This method is valid because  $F_\lambda(1/2) = [\lambda, 1]$ , so we have that  $x_{a_{n+1}} = F_\lambda(x_{a_{n+1}-1}) = F_\lambda(1/2)$ .

Similarly,  $x_{b_s}$  was chosen so that  $1/2 \in F_\lambda^N(x_{b_s})$ , so we may define  $x_j$  for all  $b_s < j \leq b_s + N$  so that  $x_{b_s+N} = 1/2$ . If there are any integers  $j$  such that  $b_s + N < j < P$ , let  $x_j = 1/2$ .

Finally, to ensure  $\mathbf{x}$  is periodic with period  $P$ , for all  $j \geq P$ , define  $x_j = x_{j-P}$ . In doing this, we have defined  $x_j$  for all  $j \geq 0$ , so that  $\mathbf{x} = (x_j)_{j=0}^\infty$  is a periodic orbit with period  $P$ , and for each  $n = 1, \dots, s$ , and each  $a_n \leq j \leq b_n$ , we have  $|x_j - y_j^n| < \varepsilon$ .  $\square$

## 7. PRE-IMAGES OF $\frac{1}{2}$

In the previous two sections, we relied heavily on the pre-images of  $1/2$  to prove that  $F_\lambda$  is Devaney chaotic and has the specification property on  $[\lambda, 1]$  when  $\lambda < 1/2$ . In studying these pre-images of  $1/2$  we notice an interesting property concerning the number of them. If we count the number of pre-images of  $1/2$  of order 1, order 2, and so on, the resulting sequence satisfies a basic Fibonacci-like recurrence relation.

In Example 7.1 and Theorem 7.2, we use absolute value bars to denote the cardinality of a finite set.

**Example 7.1.** Let  $\lambda = 1/4$ . Let  $A_0 = \{1/2\}$ , and for each  $n \in \mathbb{N}$ , define

$$A_n = \left\{ x \in \left( \frac{1}{4}, 1 \right] : \frac{1}{2} \in F_{1/4}^n(x) \text{ and } \frac{1}{2} \notin F_{1/4}^{n-1}(x) \right\}.$$

If for each  $n \geq 0$  we let  $a_n = |A_n|$ , then the sequence  $(a_n)_{n=0}^\infty$  is the standard Fibonacci sequence with  $a_0 = a_1 = 1$ , and  $a_{n+2} = a_{n+1} + a_n$  for all  $n \geq 0$ .

(Note that the collection  $\{A_n : n \geq 0\}$  forms a partition of all the pre-images of  $1/2$  in the interval  $(1/4, 1]$ . Some texts would describe the elements of  $A_n$  as those pre-images for which the first hitting time is  $n$ .)

*Proof.* Note that the elements of  $A_n$  are pre-images of the elements of  $A_{n-1}$ . More specifically, let  $n \in \mathbb{N}$ , and let  $x \in A_n$ . Since  $1/2 \in F^n(x)$ , there is an element  $y \in F(x)$  such that  $1/2 \in F^{n-1}(y)$ , so  $y \in A_{n-1}$ . Since  $x \neq 1/2$ , this element  $y$  is either  $\phi(x) = 2x$  or  $\psi(x)$ , so we know that either  $x \in \psi^{-1}(y)$  or  $x = y/2$ . This means we can inductively construct the sets  $A_n$  from the preceding set  $A_{n-1}$  using two types of elements.

For each  $n \geq 1$ , define

$$B_n = \left\{ x \in A_n : \text{there exists } y \in A_{n-1} \text{ where } x = \psi^{-1}(y) \right\}$$

$$C_n = \left\{ x \in A_n : \text{there exists } y \in A_{n-1} \text{ where } x = \frac{1}{2}y \right\}.$$

For each  $n \in \mathbb{N}$ , let  $b_n = |B_n|$  and  $c_n = |C_n|$ .

By definition  $A_0 = \{1/2\}$ , so  $a_0 = 1$ . The two pre-images of  $1/2$  (besides  $1/2$  itself) are  $1/4$  and  $\psi^{-1}(1/2) = 2/3$ . Since the sets  $A_n$  are restricted to the interval  $(1/4, 1]$ , the number  $1/4$  is not included in  $A_1$ . That means  $B_1 = \{2/3\}$ , and  $C_1 = \emptyset$ , so  $A_1 = B_1 \cup C_1 = \{2/3\}$ . Thus  $a_1 = 1$ .

To construct the set  $A_2$ , we look at the pre-images of  $2/3$ . We have  $7/9 = \psi^{-1}(2/3) \in B_2$ , and  $1/3 = 1/2(2/3) \in C_2$ , so  $A_2 = B_2 \cup C_2 = \{1/3, 7/9\}$ . Hence we have that  $a_2 = 2$ . Next we construct  $A_3$  by looking at the pre-images of  $1/3$  and  $7/9$ . Recall that each set  $A_n$  is restricted to the interval  $(1/4, 1]$ , so we cannot include  $1/6 = 1/2(1/3)$  in  $C_3$ , but we can include  $7/18 = 1/2(7/9)$  in  $C_3$ . In addition, we have  $5/9 = \psi^{-1}(1/3) \in B_3$  and  $23/27 = \psi^{-1}(7/9) \in B_3$ . Thus  $A_3 = B_3 \cup C_3 = \{7/18, 5/9, 23/27\}$ , so  $a_3 = 3$ .

Now to prove this pattern holds, note that for  $n \geq 1$ , we have

$$a_n = b_n + c_n. \quad (2)$$

Observe that  $\psi^{-1}((1/4, 1]) = (1/2, 1] \subseteq (1/4, 1]$ . This means that for any element  $x \in A_n$ , we have that  $\psi^{-1}(x) \in B_{n+1}$ , so

$$a_n = b_{n+1}. \quad (3)$$

However, as we observed in the construction of  $A_1, A_2$ , and  $A_3$ , only some elements of  $A_n$  may be divided by two and remain in the desired interval  $(1/4, 1]$ . Specifically,  $x/2 \in C_{n+1}$  if and only if  $x \in B_n$ , so

$$b_n = c_{n+1}. \quad (4)$$

Putting Equations (2), (3), and (4) together yields

$$\begin{aligned} a_{n+2} &= b_{n+2} + c_{n+2} \\ &= b_{n+2} + b_{n+1} \\ &= a_{n+1} + a_n. \end{aligned}$$

Therefore, the sequence  $(a_n)_{n=0}^{\infty}$  is the standard Fibonacci sequence.  $\square$

A visual description of how the elements of  $A_n$  are constructed from elements of  $A_{n-1}$  is given in Figure 2.

There is a natural generalization of this result for all values  $\lambda$  of the form  $1/2^k$ .

**Theorem 7.2.** Assume  $\lambda = \frac{1}{2^k}$  for some  $k \in \mathbb{N}$ . Let  $A_0 = \{1/2\}$ , and for each  $n \in \mathbb{N}$  define

$$A_n = \left\{ x \in (\lambda, 1] : \frac{1}{2} \in F_{\lambda}^n(x) \text{ and } \frac{1}{2} \notin F_{\lambda}^{n-1}(x) \right\}.$$

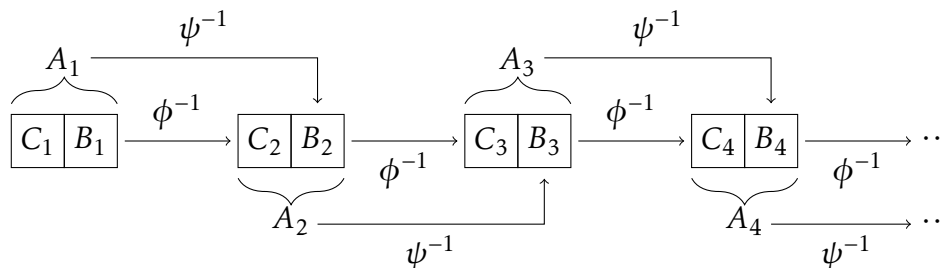


FIGURE 2. Construction of  $A_n$

If for each  $n \geq 0$ , we let  $a_n = |A_n|$ , then the sequence  $(a_n)_{n=0}^\infty$  satisfies the recurrence relation

$$a_{n+k} = \sum_{j=0}^{k-1} a_{n+j}.$$

*Proof.* For each  $1 \leq n < k - 1$  and each  $j = 0, \dots, n - 1$ , define

$$B_{n,j} = \left\{ x \in A_n : \text{there exists } y \in A_{n-j-1} \text{ where } x = \frac{1}{2^j} \psi^{-1}(y) \right\}.$$

For  $1 \leq n < k - 1$ , we also define

$$B_{n,n} = \left\{ \frac{1}{2^{n+1}} \right\}.$$

For each  $n \geq k - 1$  and each  $j = 0, \dots, k - 2$ , we define

$$B_{n,j} = \left\{ x \in A_n : \text{there exists } y \in A_{n-j-1} \text{ where } x = \frac{1}{2^j} \psi^{-1}(y) \right\},$$

and for  $n \geq k - 1$ , we also define

$$B_{n,k-1} = \left\{ x \in A_n : \text{there exists } y \in A_{n-k+1} \text{ where } x = \frac{1}{2^{k-1}} y \right\}.$$

We denote the cardinality of each set  $B_{n,j}$  by  $b_{n,j}$ .

For any  $n \geq 1$ , if  $x \in B_{n,0}$  then  $x/2 \in B_{n+1,1}$ ,  $x/4 \in B_{n+2,2}, \dots, x/2^j \in B_{n+j,j}$ . This holds for all  $j = 1, \dots, k - 1$ , so we have that for all  $n \geq 1$  and all  $j = 1, \dots, k - 1$ ,  $b_{n,0} = b_{n+j,j}$ . We also

know that for all  $n \geq 0$ ,  $a_n = b_{n+1,0}$ . Putting these together yields that for all  $n \geq 0$ ,

$$\begin{aligned} a_{n+k} &= \sum_{j=0}^{k-1} b_{n+k,j} \\ &= \sum_{j=0}^{k-1} b_{n+k-j,0} \\ &= \sum_{i=1}^k b_{n+i,0} \\ &= \sum_{i=1}^k a_{n+i-1} \\ &= \sum_{l=0}^{k-1} a_{n+l} \end{aligned}$$

which is our intended result. □

## 8. TOPOLOGICAL ENTROPY

Now that we know  $F_\lambda$  has the specification property on  $[\lambda, 1]$  when  $\lambda < 1/2$ , it follows from a theorem due to Raines and Tennant [14], that the system also has positive topological entropy. We will demonstrate a method of calculating a more precise lower bound for the topological entropy of the system for specific values of  $\lambda$ .

First, we will consider the entropy when  $\lambda \geq 1/2$ . Topological entropy is a measure of how spread out orbits can get. We showed in Theorem 3.1 that if  $\lambda > 1/2$ , then every orbit (other than the 0 orbit) is limiting to 1. It follows that the topological entropy of  $F_\lambda$  is zero when  $\lambda > 1/2$ . Similarly, if  $\lambda = 1/2$ , then by Theorem 3.2, every orbit is eventually fixed, so the entropy must be zero.

For the rest of this section, we will consider the case that  $\lambda < 1/2$ . By Theorem 6.1,  $F_\lambda$  has the specification property on  $[\lambda, 1]$ , so we know that the topological entropy of the system is positive. (We also showed this in Corollary 4.2.) When  $\lambda$  is of the form  $1/2^n$  for some  $n \in \mathbb{N}$ , we may be more precise and calculate a lower bound for the entropy. This is because, in this case,  $F_\lambda$  is a generalized Markov function.

For traditional, single-valued functions, we say that a function  $f: [0, 1] \rightarrow [0, 1]$  is a *Markov map* if there exists a partition  $0 = a_0 < a_1 < \dots < a_n = 1$  such that for each  $j = 0, \dots, n$ ,  $f(a_j)$  is in the partition, and  $f$  is strictly increasing or strictly decreasing on each interval of the form  $(a_j, a_{j+1})$ . In [2], Banič and Lunder generalize this definition to accommodate set-valued functions. In this more general definition,  $f(a_j)$  does not need to be a single element of the partition, but rather, it can be a closed interval whose endpoints are partition elements. This allows for vertical lines in the graph, such as we have in the graph of  $F_\lambda$ .

Markov maps are very nice to work with, because much of their dynamics can be determined by witnessing how the function moves the intervals of the form  $(a_j, a_{j+1})$ . This pattern may be encoded in what is called a transition matrix. The process for relating a Markov map to its corresponding transition matrix is described in [4, pp. 212-213]. Then, by [4, Theorem 9.2.7], if  $\rho$  is the largest eigenvalue (in modulus) of the transition matrix, then the topological entropy of the system is  $\log \rho$ .

We adapt this process so that it may apply it to set-valued functions. We illustrate this first through the following example.

**Example 8.1.** Let  $\lambda = 1/8$ . Then we partition the interval  $[0, 1]$  with

$$\begin{aligned} a_0 &= 0 \\ a_1 &= \frac{1}{8} \\ a_2 &= \frac{1}{4} \\ a_3 &= \frac{1}{2} \\ a_4 &= 1. \end{aligned}$$

This gives 4 open intervals which we denote by

$$\begin{aligned} P_1 &= \left(0, \frac{1}{8}\right) \\ P_2 &= \left(\frac{1}{8}, \frac{1}{4}\right) \\ P_3 &= \left(\frac{1}{4}, \frac{1}{2}\right) \\ P_4 &= \left(\frac{1}{2}, 1\right). \end{aligned}$$

The graph of  $F_{1/8}$  with the partition elements and intervals labeled is pictured in Figure 3.

We have 5 partition elements and 4 intervals, so our transition matrix will be  $9 \times 9$ . Each row and each column is associated with a particular partition element or interval. Each entry is either a 0 or a 1. If the item (either a partition element or interval) associated with a row is contained in the image of the item associated with a column, then we put a 1 in that entry; otherwise we put a 0.

For example,  $P_3$  is a subset of  $F_{1/8}(P_2)$ ,  $F_{1/8}(a_3)$ , and  $F_{1/8}(P_4)$ . Thus, the row associated with  $P_3$  will have 1s in the columns associated with  $P_2$ ,  $P_4$ , and  $a_3$ , and it will have 0s everywhere else. The full transition matrix for  $F_{1/8}$  is pictured in Figure 4.

The largest eigenvalue of this matrix is approximately 2.87561. We may conclude that  $h(F_{1/8}) \geq \log 2.87561$ .

As we construct the transition matrix for more values of  $\lambda = 1/2^n$ , a clear pattern emerges.

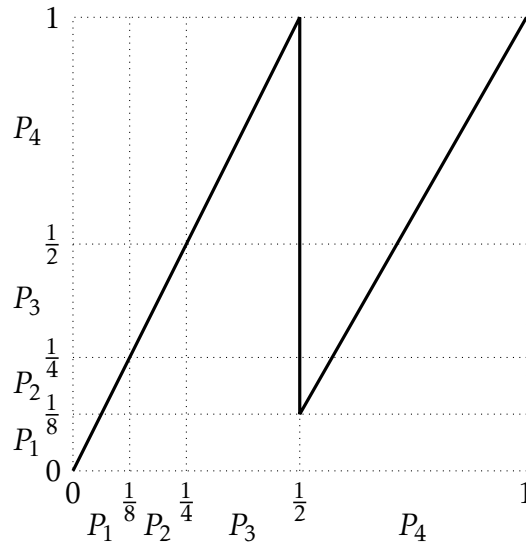


FIGURE 3. The graph of  $F_\lambda$

	$P_1$	$P_2$	$P_3$	$P_4$	$a_0$	$a_1$	$a_2$	$a_3$	$a_4$
$P_1$	1	0	0	0	0	0	0	0	0
$P_2$	1	0	0	1	0	0	0	1	0
$P_3$	0	1	0	1	0	0	0	1	0
$P_4$	0	0	1	1	0	0	0	1	0
$a_0$	0	0	0	0	1	0	0	0	0
$a_1$	1	0	0	0	0	0	0	1	0
$a_2$	0	0	0	1	0	1	0	1	0
$a_3$	0	0	0	1	0	0	1	1	0
$a_4$	0	0	0	0	0	0	0	1	1

FIGURE 4. Transition matrix for  $F_{1/8}$

**Example 8.2.** Let  $n$  be a natural number greater than or equal to 2, and let  $\lambda = 1/2^n$ . Then the partition is

$$\begin{aligned}
 a_0 &= 0 \\
 a_1 &= \frac{1}{2^n} \\
 &\vdots \\
 a_n &= \frac{1}{2} \\
 a_{n+1} &= 1,
 \end{aligned}$$

and for each  $j = 1, \dots, n+1$ , we define  $P_j = (a_{j-1}, a_j)$ . The transition matrix will have  $2n+3$  rows and  $2n+3$  columns, and we may describe the rows as follows:

- (1) The  $P_1$  row will contain a 1 in the  $P_1$  column.



- (2) For  $j = 2, \dots, n + 1$ , the  $P_j$  row will have a 1 in the  $P_{j-1}$  column, the  $P_{n+1}$  column, and the  $a_n$  column.
- (3) The  $a_0$  row will have a 1 in the  $a_0$  column.
- (4) The  $a_1$  row will have a 1 in the  $P_1$  column and the  $a_n$  column.
- (5) For  $j = 2, \dots, n$ , the  $a_j$  row will have a 1 in the  $P_{n+1}$  column, the  $a_{j-1}$  column, and the  $a_n$  column.
- (6) The  $a_{n+1}$  row will have a 1 in the  $a_n$  column and the  $a_{n+1}$  column.
- (7) All other entries are 0.

While this does not give us an explicit formula for the topological entropy of  $F_\lambda$ , for each individual value  $\lambda = 1/2^n$ , we may use this description to construct the transition matrix then calculate the largest eigenvalue to find the topological entropy.

We may also use a transition matrix to calculate a lower bound for the entropy when  $\lambda = 0$ .

**Example 8.3.** Let  $\lambda = 0$ . In this case we may use the partition  $a_0 = 0$ ,  $a_1 = 1/2$ , and  $a_2 = 1$ . Then we have  $P_1 = (0, 1/2)$ , and  $P_2 = (1/2, 1)$ . The transition matrix is

$$\begin{array}{c} P_1 \quad P_2 \quad a_0 \quad a_1 \quad a_2 \\ \begin{array}{l} P_1 \\ P_2 \\ a_0 \\ a_1 \\ a_2 \end{array} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

which has its largest eigenvalue of 3. Thus  $h(F_0) \geq \log 3$ .

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