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Ellen Borgeld[1], Elizabeth Meena[2], and Shelly Smith[3]

[1]Mathnasium of Littleton
[2]Rock Valley College
[3]Grand Valley State University

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Abstract. We investigate a variation of Sudoku that, in addition to using a limited number of numerical clues, also uses arrows indicating the sequence of entries within each block. We begin with an overview and strategies of the game, including smaller versions. For three sizes of Arrow Path puzzles, we determine the number of blocks that may be used to build the puzzles, find upper and lower bounds on the minimum number of numerical clues required to create puzzles, and find the number of 4×4 puzzles that use a minimum number of clues.

1. Introduction

Although not originally developed in the United States, the game of Sudoku has recently become extremely popular in the U.S., with Sudoku puzzles becoming a standard item in magazines, newspapers, and puzzle books. Given the unique qualities that they have, Sudoku puzzles are a fascinating topic for mathematicians to study. In a standard Sudoku puzzle, the player is given a 9×9 grid partitioned into nine 3×3 **blocks**. The puzzle initially contains numerical clues in at least 17 of the 81 **cells** [2], and the player's task is to complete the puzzle by placing the digits 1 through 9 in the remaining cells so that each digit appears exactly once in every row, column, and block. There are numerous variations of the standard puzzle, and in this paper, we will focus on a variation called **Arrow Path Sudoku** [4]. Arrow Path Sudoku is not widely known and there is currently only one known example on the internet. From the research done by the authors it appears that mathematicians have not previously studied this particular variation. In an Arrow Path Sudoku puzzle, there is one arrow in each cell that points in the direction of a row, column, or diagonal within that block. Each arrow points to the subsequent digit in that block, with the arrow in the cell containing 9 pointing to the cell containing 1. Note that there are only eight possible directions for the arrows in an Arrow Path Sudoku puzzle, with examples shown in Figure 1. For ease of notation, we identify the cells within an individual block c_0 to c_8, beginning in the upper left-hand corner and ending in the lower right-hand corner.

* *Corresponding author*

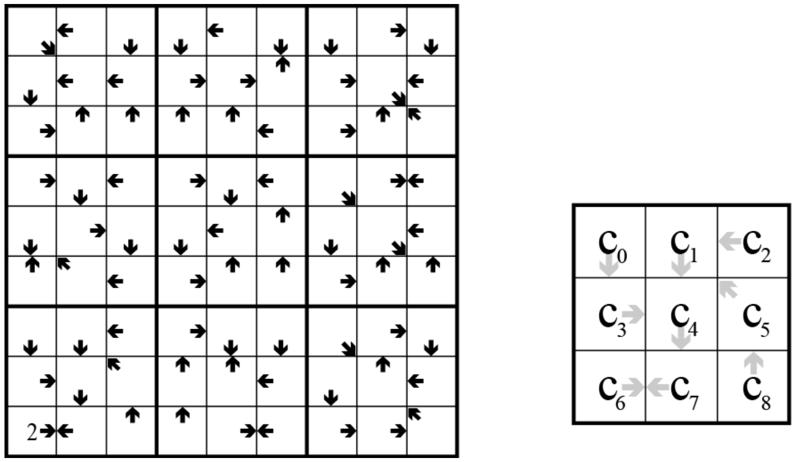


Figure 1. Arrow Path Sudoku puzzle [4] and its lower left block with cells labeled.

Compared to standard Sudoku, an Arrow Path Sudoku puzzle contains only a small number of numerical clues in addition to the arrows. An Arrow Path Sudoku puzzle is solved by the player filling in all cells while simultaneously satisfying the arrows as well as the usual conditions placed on rows, columns, and blocks. For the puzzle in Figure 1, the logical starting point for solving the puzzle is the lower left block, because it is the only block containing a numerical clue. Since the arrow in the cell containing 2 points to the right, 3 must be contained in either cell c_7 or c_8. It is not possible for 3 to be placed in cell c_7, since the left arrow indicates that 4 would then have to be placed in cell c_6, which is already filled. Thus we place 3 in cell c_8. Next, we have two options, cell c_2 and cell c_5, in which to place 4. Note that cell c_2 has only one arrow pointing to it, from the cell containing 3, so we must place 4 there. We have two choices of cells in which to place the 5, but cell c_0 also has only one arrow pointing to it, from the cell containing 4, so we see that 5 should be in that cell. We continue to test the placement of numbers in remaining cells, following the arrows to make sure the placement does not result in a contradiction, until the block is complete.

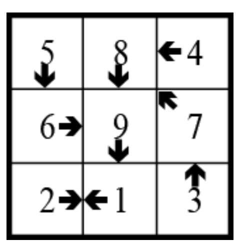


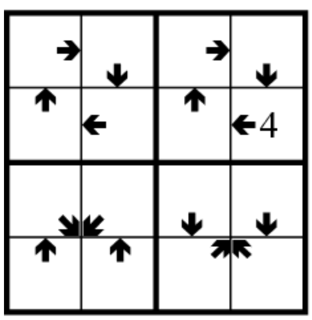
Figure 2. Completed initial block.

We then move on to the block immediately to the right of our initial block, because we can use our partial solution to help us fill in this block, keeping in mind that standard Sudoku rules must be followed as well as the Arrow Path Sudoku rules. We see that both 1 and 2 must be in one of the top two rows; we can then experiment with placing subsequent numbers in the cells unless we reach a contradiction either in the arrow path or in standard Sudoku rules. Using this method, we can complete the rest of the puzzle, aided by the previously completed blocks. The solution to this and other sample puzzles are at the end of this article.

2. Number Blocks and Arrow Blocks

When analyzing Arrow Path Sudoku puzzles, the 9×9 puzzles are quite complex, thus we will start our analysis of Arrow Path puzzles by considering two smaller sizes. We will further simplify our analysis by beginning with individual blocks, in particular, determining how many different blocks can be used in Arrow Path puzzles of various sizes. Once we have a thorough understanding of the blocks, then we can investigate putting the blocks together to form puzzles, and we can determine the minimum number of numerical clues that must be given to guarantee a unique solution.

Definition 2.1. *Shidoku* is a 4×4 version of Sudoku comprised of four 2×2 blocks. The numbers 1 through 4 are contained in each row, column, and block. *Rokudoku* is a 6×6 version of Sudoku consisting of six 2×3 blocks. The numbers 1 through 6 are contained in each row, column, and block. [3]



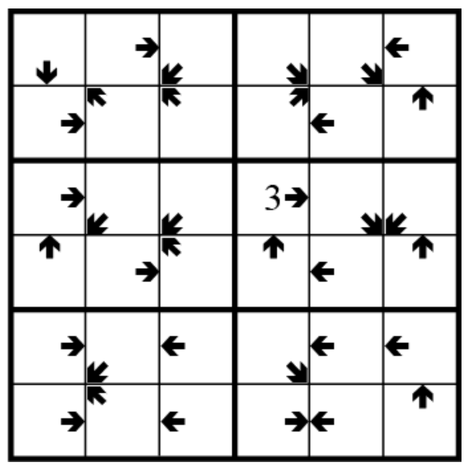


Figure 3. Arrow Path Shidoku and Arrow Path Rokudoku puzzles.

In our analysis of individual Arrow Path blocks, we consider the blocks from two different perspectives. First, we begin with a block that already contains numbers, and our goal is to add arrows to the cells so that the numbers follow the resulting arrow path. Second, we place arrows in an empty block, and our goal is to place the digits so that they satisfy the arrows. In these investigations, we let m and n be natural numbers with $m, n \geq 2$, and we define the following two types of $m \times n$ blocks.

Definition 2.2. A *number block* is an $m \times n$ block containing the digits 1 through mn.

Definition 2.3. When referring to the arrows inside a block, we will restrict ourselves to using only *reasonable* arrows. That is, the arrows must be pointing to other cells within the block and not outside of the block. A block that is filled with only reasonable arrows is an *arrow block*. If we place the digits 1 through mn in an arrow block such that the direction of the arrows is satisfied, then the sequence of arrows form an *arrow path*.

2	1
3	4

Figure 4. A Shidoku number block.

The block on the left in Figure 5 shows an example of a Shidoku arrow block containing only reasonable arrows, while the Shidoku block on the right contains unreasonable arrows and thus is not an arrow block.

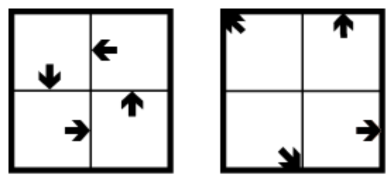


Figure 5. Blocks containing reasonable (left) and unreasonable (right) arrows.

3. Counting Number Blocks

Filling in the cells of an $m \times n$ number block simply requires a permutation of the numbers 1 through mn. Thus, for an $m \times n$ block, there are $(mn)!$ ways to place the numbers 1 through mn in the cells of the block. However, we are particularly interested in blocks that may be used to create Arrow Path puzzles, so we would like to determine how many such usable number blocks are possible.

Definition 3.1. A *valid number block* is a number block for which an arrow path exists. A number block for which an arrow path does not exist is an *invalid number block*.

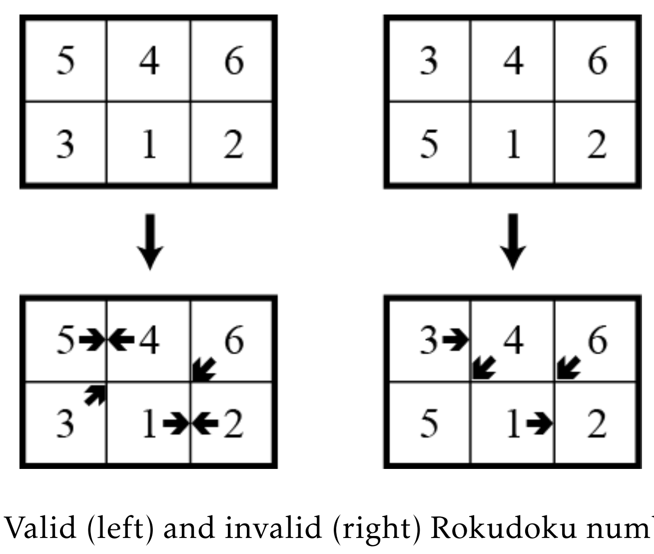


Figure 6. Valid (left) and invalid (right) Rokudoku number blocks.

For example, we can add arrows to the number block on the top left in Figure 6 that each point to the subsequent digit in the block, whereas there is no arrow that can point from 2 to 3 or from 5 to 6 in the number block on the right. Invalid number blocks exist because the arrows in a cell can only point to the cells in the same row, column, or diagonal as the cell itself. If we think of the block as a chess board, the arrows can only play queen's moves. Thus, a number block is valid if and only if the consecutive digits in the block are a queen's move away from each other.

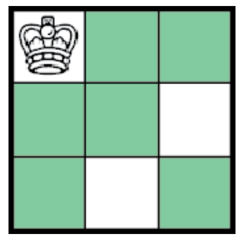


Figure 7. Only queen's moves are allowed.

Proposition 3.2. *There are 24 valid Shidoku number blocks.*

Proof. In a 2×2 block, every cell in the block is a queen's move away from all the other cells, therefore, every permutation of the four entries results in a valid Shidoku number block. □

Proposition 3.3. *There are 288 valid Rokudoku number blocks.*

Proof. We use inclusion-exclusion to count the invalid Rokudoku number blocks and then subtract that number from the total of 720 possible number blocks. There are two ways that consecutive numbers cannot be placed in a block, in cells c_0 and c_5, or in cells c_2 and c_3, because these pairs of cells are not queen's moves apart so no arrow path is possible. When creating representatives of sets of invalid number blocks, without loss of generality we will place 1 in cell c_0. There are six sets of invalid number blocks, with representatives shown in Figure 8, labeled S_1 through S_6.

Sets S_1 and S_2 each have cardinality 24, because the four remaining entries may be placed in any order. For sets S_3 through S_6, we could switch the order of the two consecutive numbers in cells c_2 and c_3, as well as place the three remaining entries in any order, thus each set contains 12 blocks. We see that $|S_1 \cap S_i| = 4$ for $i = 4, 5, 6$, and similarly, $|S_2 \cap S_j| = 4$ for $j = 3, 4, 5$. All other pairwise intersections of sets, or intersections of three or more sets,

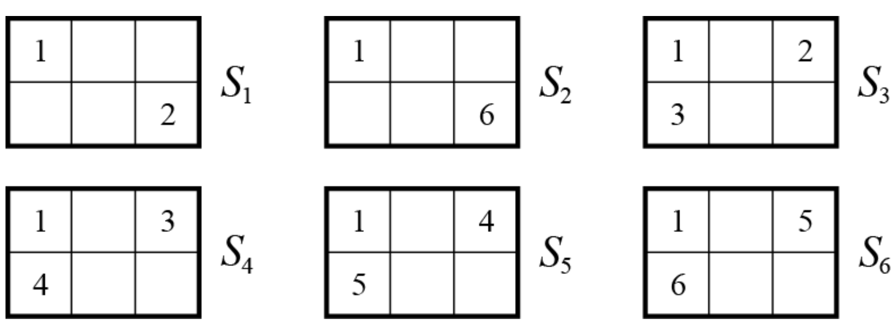


Figure 8. Representatives of sets of invalid number blocks.

are empty. Using inclusion-exclusion, the cardinality of the union of all six sets is 72. So far we have counted all invalid Rokudoku number blocks with 1 in cell c_0, but we could place any number 1 through 6 in c_0 and shift the other entries modulo 6, resulting in 432 invalid blocks. Subtracting this result from the 720 possible Rokudoku blocks gives us the desired result. □

Theorem 3.4. *There are 35,280 valid Sudoku number blocks.*

Proof. When counting Sudoku number blocks, the 46 different sets of unsolvable 3×3 number blocks we would need to consider make using inclusion-exclusion less practicable. Instead, we calculated this result using a C++ computer program that checks the validity of each permutation with 1 in cell c_0. We used a one-dimensional array to represent a 3×3 number block and programmed the computer to count each array in which every entry in the corresponding number block is only a queen's move away from the previous entry. This resulted in a count of 3,920 distinct valid number blocks. To count the set of all possible blocks with any number in c_0, we choose the digit in c_0 in one of 9 ways, then shift the remaining entries modulo 9, resulting in a total of 35,280 valid Sudoku number blocks. □

4. Counting Arrow Blocks

Now that we have counted number blocks that permit an arrow path, we return to our second approach to consider $m \times n$ blocks that initially contain arrows, and determine which can be filled with numbers 1 through mn that satisfy those arrows. First we count the number of ways to choose arrows for an $m \times n$ block; the 3×4 block shown in Figure 9 shows all of the reasonable arrows that we may choose from.

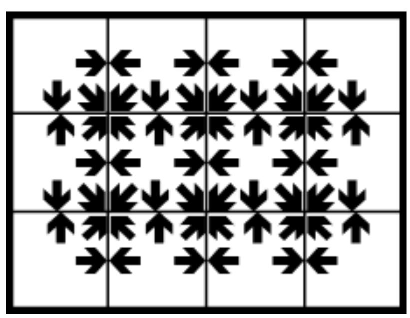


Figure 9. Reasonable arrow choices for a 3×4 block.

We assume that $m, n \geq 2$, so every block will contain exactly four corner cells. Each corner cell contains one of three possible arrows, therefore there are 3^4 ways to choose arrows for the corners. The exterior cells along the edge of the block (that are not corner cells) each contain one of five reasonable arrows. There are $2(m-2)$ cells with five possible choices of arrows in exterior columns, and $2(n-2)$ cells with five possible arrows in the exterior rows, resulting in $5^{(2n+2m-8)}$ possible reasonable arrows for the exterior cells. We may place any of the 8 possible arrows in each cell of the of the $(m-2) \times (n-2)$ rectangle of interior cells, giving us $8^{(m-2)(n-2)}$ possible arrow choices for the set of interior cells.

The total number of ways that arrows can be placed in the block is the product of these results, leading us to the following theorem.

Proposition 4.1. *Let m and n be natural numbers with $m, n \geq 2$. There are*

$$3^4 5^{(2n+2m-8)} 8^{(m-2)(n-2)}$$

$m \times n$ arrow blocks.

A Shidoku block consists solely of four corner cells. Since each corner cell has three options (an arrow pointing to each of the other three cells in the block), there are 81 possible Shidoku arrow blocks. A 2×3 block will have 4 corner cells and 2 exterior cells. Thus, the number of Rokudoku arrow blocks is $3^4 5^2 8^0 = 2,025$. Similarly, we can compute the number of Sudoku arrow blocks: $3^4 5^4 8^1 = 405,000$.

Definition 4.2. A **solvable arrow block** has an arrow path that corresponds to one or more number blocks.

We have already counted the number of possible arrow blocks for any given $m \times n$ block. However, not all of these arrow blocks are solvable. That is, it is not possible to fill an $m \times n$ block with the digits 1 through mn such that they satisfy the arrow path. For example, consider the Shidoku arrow block in Figure 10. Note that if 1 is placed in cell c_0, then 2 would have to be placed in cell c_1. But since the arrow in cell c_1 points back to cell c_0, there would have to be 3 in cell c_0, which is already filled by 1. Clearly, a contradiction is reached, and the arrow block is said to be unsolvable. The arrow blocks counted in Theorem 4.1 include both solvable and unsolvable arrow blocks.

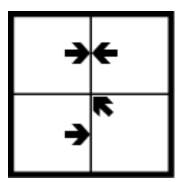


Figure 10. An unsolvable arrow block.

A solvable arrow block can be represented by a **digraph**.

Definition 4.3. A *digraph* is a graph in which each edge has a direction assigned to it, and the directed edges are called *arcs*. A digraph is *strongly connected* if and only if there is a path from any vertex to any other vertex that respects the direction of each arc. A *Hamiltonian cycle* is a cycle that contains every vertex of the graph exactly once.

Given an arrow block, we construct the associated digraph by creating a vertex to represent each cell in the block. We then draw an arc from one vertex to another if the arrow in the first cell (vertex) points toward the second cell (vertex). This is illustrated in Figure 11; for example, the arrow in cell c points toward cells a and b, and in the digraph we see arcs from c to a and from c to b. Furthermore, an arrow path in an arrow block corresponds to a Hamiltonian cycle in the digraph. Hence, an arrow block is solvable if and only if it corresponds to a digraph that contains a Hamiltonian cycle.

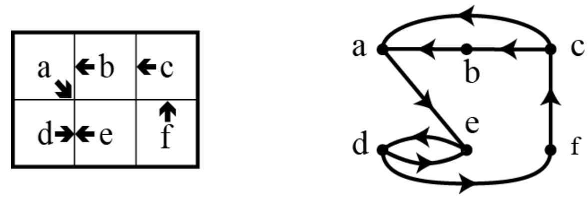


FIGURE 11. Rokudoku arrow block and corresponding digraph.

Proposition 4.4. *There are 6 solvable Shidoku arrow blocks and 48 solvable Rokudoku arrow blocks.*

Proof. In a Shidoku arrow block, as long as all four arrows point to a distinct cell, the block is solvable. The first cell has 3 options for arrows, the second cell has 2 options, and the last two are uniquely determined.

We previously proved that there are 288 valid Rokudoku number blocks. Each valid Rokudoku number block corresponds to a solvable Rokudoku arrow block, but multiple number blocks correspond to the same arrow block because an arrow block can be labeled in 6 ways by shifting the solution modulo 6. In fact, every solvable arrow block has a unique solution with 1 in cell c_0, and so corresponds to exactly 6 valid number blocks, resulting in 48 solvable Shidoku arrow blocks. □

It is worth noting a relationship between valid Shidoku and Rokudoku number blocks and solvable arrow blocks. There are 6 solvable Shidoku arrow blocks, and each of these arrow blocks can be labeled in 4 ways by starting the arrow path in cell c_0 with either 1, 2, 3, or 4. These four solutions of the six solvable arrow block correspond to the 24 valid number blocks. Similarly, each solvable Rokudoku arrow block can be labeled 6 ways, corresponding to the 288 valid number blocks. In order to understand this better, we take another look at the correspondence between arrow blocks and digraphs.

Given a solvable Shidoku arrow block, each vertex in the corresponding digraph has only one arc directed away from it. Therefore, the digraph contains only one Hamiltonian cycle. An exhaustive check of the 48 Rokudoku arrow blocks verifies that each Rokudoku arrow block has only one Hamiltonian cycle in the corresponding digraph. For each solvable Shidoku or Rokudoku arrow block, there exists a unique Hamiltonian cycle in the corresponding digraph. However, we no longer have this same relationship when analyzing Sudoku blocks.

For 3×3 Sudoku arrow blocks, we found that there are 32 solvable arrow blocks that have two distinct Hamiltonian cycles and thus correspond to more than one number block containing 1 in cell c_0. An example is shown in Figure 12. Each of these 32 blocks has 18, rather than 9, solutions.

Theorem 4.5. *There are 3,888 solvable Sudoku arrow blocks. Of these, 32 have two distinct Hamiltonian cycles in the corresponding digraph.*

Proof. This was simply too big to verify by hand, so we wrote a computer program that checked all Sudoku arrow blocks for Hamiltonian cycles. The program identified 3,888

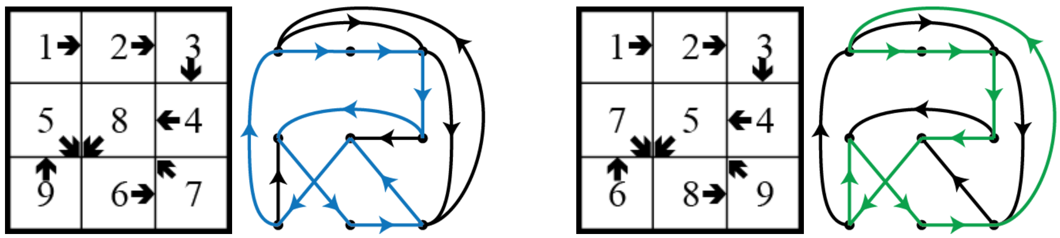


Figure 12. One arrow block with two Hamiltonian cycles highlighted in digraph.

solvable Sudoku arrow blocks, 32 of which contained two Hamiltonian cycles. Each of the 3,856 arrow blocks with a unique Hamiltonian cycle may be labeled in 9 ways, by choosing any entry for cell c_0 and following the arrow path, and the arrow blocks with two Hamiltonian cycles each have 18 possible solutions. The combined number of solutions for all 3,888 solvable Sudoku arrow blocks is 35,280, which is consistent with the number of valid Sudoku number blocks identified in Theorem 3.4. □

5. Arrow Path Squares and Puzzles

Now that we have developed a better understanding of individual blocks, we can begin looking at combining these blocks into **squares** with the goal of forming Arrow Path puzzles, and then determine the minimum number of numerical clues that are needed to create a puzzle. The following definition is specific to Arrow Path Sudoku, but the definitions for 4×4 Shidoku and 6×6 Rokudoku are analogous.

Definition 5.1. An *Arrow Path Sudoku square* is a 9×9 grid comprised of nine solvable Arrow Path Sudoku blocks such that the numbers 1 through 9 may be placed in the grid with each number appearing exactly once in each row, column, and block. An Arrow Path Sudoku square is partitioned into *bands* and *pillars*. A band is a horizontal grouping of 3 blocks, and a pillar is a vertical grouping of 3 blocks.

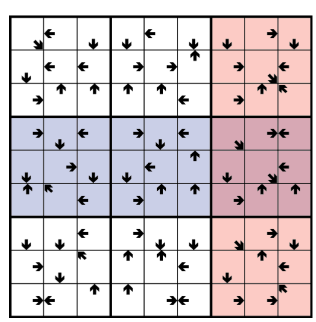


Figure 13. Arrow Path Sudoku square with a band (blue) and pillar (red) highlighted.

An Arrow Path square is essentially a potential Arrow Path puzzle, and may be transformed into a puzzle by placing one or more numerical clues that will lead to a unique solution. In this section we will count the number of Arrow Path Shidoku squares, as well as the number of Arrow Path Shidoku puzzles that may be created from these squares using the minimum number of numerical clues. We will also determine upper and lower bounds for the minimum number of clues required to transform Arrow Path Rokudoku and Arrow Path Sudoku squares into puzzles.

We will again start small by considering 4×4 Arrow Path Shidoku squares. We saw that every Shidoku number block is valid, so any Shidoku square can be made into an Arrow Path Shidoku square simply by adding the appropriate arrows and removing the numbers. Rosenhouse and Taalman have shown that there are 288 standard Shidoku squares [3]. When we counted 2×2 arrow blocks, we saw that each arrow block corresponds to four number squares because we can place any number 1 through 4 in cell c_0 then follow the arrow path to complete the block. An Arrow Path Shidoku square can be labeled similarly, so we know that every Arrow Path Shidoku square must correspond to at least 4 different number squares. Thus, we can have at most 72 Arrow Path Shidoku squares. In fact, there are fewer than 72 because some Arrow Path squares correspond to 8 different Shidoku squares. This occurs when there is either vertical symmetry in both pillars or horizontal symmetry in both bands. Figure 14 shows two distinct Shidoku number squares that correspond to the same Arrow Path square, due to the horizontal symmetry within each band.

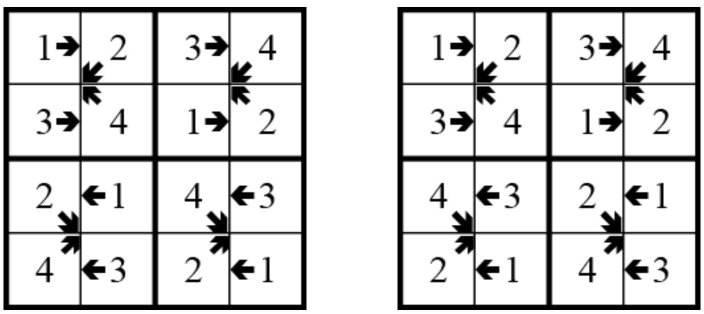


Figure 14. Two distinct Shidoku number squares corresponding to the same Arrow Path square.

Proposition 5.2. *There are 56 Arrow Path Shidoku squares.*

Proof. Each of the 288 Shidoku squares corresponds to an Arrow Path Shidoku square. We found that 40 Arrow Path Shidoku squares each correspond to a unique number square and the remaining 16 correspond to 2 distinct number squares. Since each of the distinct number squares corresponds to 4 labelings, the 40 Shidoku squares correspond to 160 labelings, and the other 16 Shidoku squares correspond to 128 labelings. Thus, we have accounted for all 288 possible labelings of Shidoku squares identified by Rosenhouse and Taalman. □

Increasing the size to Rokudoku and Sudoku squares results in a huge jump in number and complexity. While there are only 6 solvable 2×2 arrow blocks to form into Arrow Path Shidoku squares, there are 48 solvable 2×3 arrow blocks that we may use to create Arrow Path Rokudoku squares. The number of Sudoku squares is known to be $6,670,903,752,021,072,936,960 \approx 6.671 \times 10^{21}$ [1], but the number of Arrow Path Rokudoku and Arrow Path Sudoku squares is currently an open question. However, even without that information, we can gain some understanding of how we can turn Arrow Path squares into puzzles, so we will continue by considering Arrow Path Shidoku puzzles.

Arrow Path puzzles are Arrow Path squares with some numerical clues filled in such that there is a unique way to complete the remaining numerical entries so that the entries satisfy the arrow paths and each number appears exactly once in each row, column, and block. When looking at puzzles, we will start small and begin with Arrow Path Shidoku puzzles, where we are particularly interested in counting puzzles that use only a minimum number of numerical clues. We have already found all possible Arrow Path Shidoku squares, so a little investigation yields the following results:

There are 40 Arrow Path Shidoku squares that require only 1 numerical clue in order to create an Arrow Path Shidoku puzzle. We have already shown that 40 of the 56 possible Arrow Path Shidoku squares each correspond to 4 number squares that differ by shifting the entries modulo 4. A single numerical clue in any cell is sufficient to create a puzzle with a unique solution.

The remaining 16 Arrow Path Shidoku squares each need 2 numerical clues in order to create an Arrow Path Shidoku puzzle. Recall that these 16 Arrow Path Shidoku squares each correspond to 8 Shidoku squares. Due to the symmetry within the bands or pillars of these squares, there are two ways to complete the square when placing 1 in cell c_0. In other words, when one of the bands or pillars has a single solution for the numerical clue, the other band or pillar can be completed in 2 ways. Thus, we must have a clue in the second band or pillar as well, depending on whether the symmetry is horizontal symmetry within the bands or vertical symmetry within the pillars, respectively.

Theorem 5.3. *There are 10,752 Arrow Path Shidoku puzzles that contain a minimum number of numerical clues.*

Proof. We will first count the number of puzzles that correspond to Arrow Path Shidoku squares with only 4 possible labelings. With these squares, we need only one numerical clue to determine the solution to the puzzle. We have 16 cells to choose where to place the numerical clue and four different choices for the entry in that cell. Our choice of the numerical clue will determine which of the four possible labelings is the puzzle's solution. Therefore, we have 64 puzzles that correspond to each Arrow Path Shidoku square of this type. The set of 40 such squares can be used to create 2,560 possible puzzles.

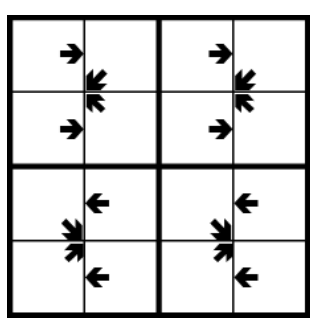


Figure 15. This Arrow Path square requires two clues, one in each band.

The Arrow Path Shidoku squares with 8 different possible labelings require two clues in order to create a puzzle with a unique solution. Consider the Arrow Path Shidoku square in Figure 15, which we saw previously in Figure 14. Without loss of generality, placing a numerical clue in the top band (in any of the 8 cells) will result in a unique solution for that band. This clue can be any of the numbers 1 through 4. The bottom band will then have two possible labelings, so we must place a second clue in one of the 8 cells of that band, and we have two choices for the second clue. Thus, this square yields 512 different possible puzzles. The reasoning is similar for Arrow Path Shidoku squares with vertical symmetry within both pillars. Since we have 16 squares like this with symmetry in both bands or both pillars, we have 8,192 possible puzzles of this type, which yields a total of 10,752 different Shidoku Arrow Path puzzles using the minimum number of numerical clues. □

6. Minimum Number of Clues for Larger Arrow Path Puzzles

As with Arrow Path squares, moving from Arrow Path Shidoku puzzles to larger Arrow Path puzzles creates a significant increase in the difficulty of our tasks. While we may not currently know the number of different Arrow Path Rokudoku puzzles using the minimum number of clues, we can determine upper and lower bounds for the minimum number of clues needed to turn an Arrow Path Rokudoku square into a puzzle. The Arrow Path Rokudoku puzzle in Figure 3 is an example of a puzzle that requires only one numerical clue to have a unique solution, but some puzzles require more clues.

As with Arrow Path Shidoku puzzles, we can create Arrow Path Rokudoku puzzles containing bands with horizontal symmetry. We cannot, however, create an Arrow Path Rokudoku puzzle that contains pillars with vertical symmetry because none of the 48 solvable Rokudoku arrow blocks have vertical symmetry. It is possible then to create a Rokudoku puzzle with up to three bands that each contain horizontal symmetry, as shown in Figure 16. This symmetry increases the number of clues needed because each separate band will have at least two solutions, regardless of the solutions selected in the other bands, so we must place at least one numerical clue in each band.

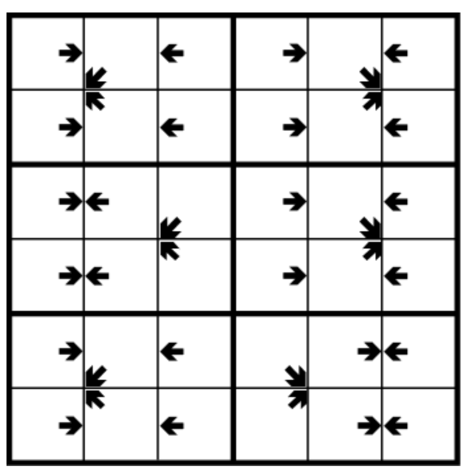


Figure 16. Arrow Path Rokudoku square containing three bands with horizontal symmetry.

Proposition 6.1. *An Arrow Path Rokudoku puzzle requires at most four clues.*

Proof. Every Rokudoku arrow block has a unique Hamiltonian cycle, so we will need at most one clue in each of the six blocks in an Arrow Path Rokudoku square. Furthermore, as long as we have one clue in two blocks of each pillar and at least one clue in each band in the square, the resulting puzzle will have a unique solution. With these four given numerical clues, we can solve the four corresponding blocks using the arrow paths. The solutions to the remaining two blocks are uniquely determined by the rules of standard Sudoku because both blocks will have the rest of band and pillar already solved. We can show by construction that there exists a Rokudoku arrow path puzzle that requires four clues; it is shown in Figure 17. Thus, four clues is a tight upper bound on the minimum number of clues needed for an Arrow Path Rokudoku puzzle. □

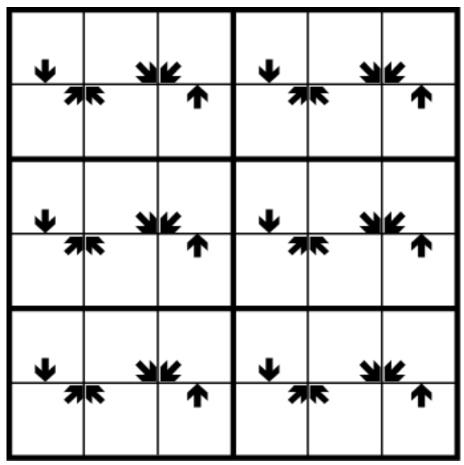


Figure 17. Arrow Path Rokudoku square requiring 4 clues to create a puzzle.

As we have already done with Arrow Path Shidoku and Arrow Path Rokudoku puzzles, we can determine the upper bound on the minimum number of clues needed for an Arrow Path Sudoku puzzle. However, this time we will need more constraints, and we do not know if the upper bound for the number of clues needed is a tight upper bound.

Theorem 6.2. *Given an Arrow Path Sudoku square that has at most one block in each band or pillar that contains two Hamiltonian cycles, at most six clues are required to create an Arrow Path Sudoku puzzle.*

Proof. By assumption, our Arrow Path Sudoku square will include at least six blocks that contain a unique Hamiltonian cycle. Placing one numerical clue in each of these blocks will be sufficient to determine a unique solution for these six blocks using the arrow path rules. The three remaining blocks must be in different bands and in different pillars, so they can be solved using the rules of standard Sudoku. □

The number of numerical clues required is not necessarily an indication of the difficulty of solving a puzzle. For example, Figure 18 shows an example of an Arrow Path Sudoku puzzle containing a block with two Hamiltonian cycles which is more challenging to solve than the puzzle in Figure 1. Nevertheless, the puzzle requires only one initial clue to yield a unique solution.

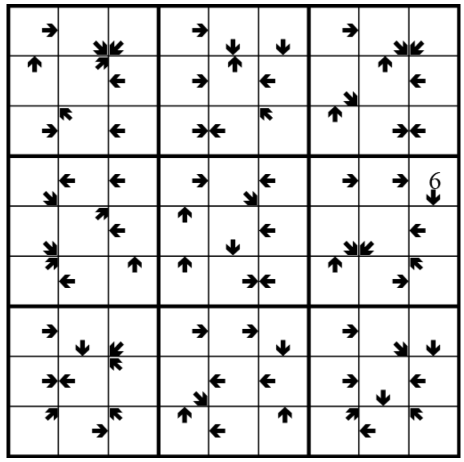


Figure 18. One block in this puzzle contains two Hamiltonian cycles.

7. Looking Forward

When creating Arrow Path squares larger than 4×4, the number of valid Arrow Path blocks to choose from increases rapidly, as does the number of ways to combine the blocks. We used results for standard Shidoku [3] to determine the number of Arrow Path Shidoku squares and puzzles, but the complexity of the larger Rokudoku and Sudoku versions increases so dramatically that the techniques used for Arrow Path Shidoku squares are not easily adapted to the larger versions.

As in Theorem 6.2, some partial results are potentially within reach if restrictions are placed on the assumptions about the properties of the individual blocks being used. In particular, limiting our choices to a small subset of the solvable Arrow Path blocks available might facilitate counting the number of Arrow Path squares we could create using that subset of blocks. Furthermore, restricting the symmetry in bands or pillars, or the number of blocks with two Hamiltonian cycles, could make it easier to find upper and lower bounds on the minimum number of clues needed to form a puzzle, and to count the number of Arrow Path puzzles that may be created.

8. Acknowledgement

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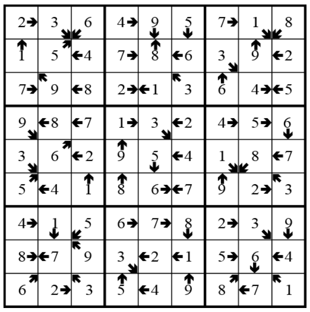
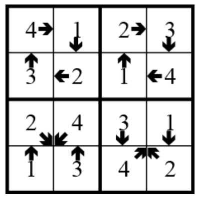
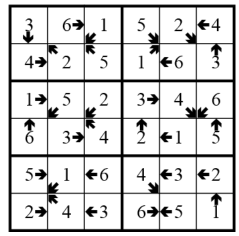
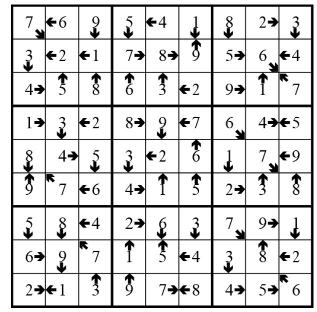
Puzzle Solutions









Student biographies

Ellen Borgeld: Ellen is an instructor at the Mathnasium of Littleton outside of Denver, Colorado where she teaches math to students in grades 2 through 12. She focuses on making math more accessible and understandable for all her students and enjoys seeing their confidence grow.

Elizabeth Meena: (*Corresponding author:* E.Meena@RockValleyCollege.edu) Elizabeth Meena is a recent graduate of the M.S. in Mathematics program at Northern Illinois University. She is in her first year as a full-time faculty member in the mathematics department at Rock Valley College, a community college in Rockford, IL. In her classroom, she loves to incorporate puzzles, games, and anything that can help make mathematics more enjoyable for her students.