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Maithreya Sitaraman

University of Chicago

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Symmetrizing elements of the kernel to characterize the solvability of the σ^+ Lights Out puzzle on various geometrical arrangements

Maithreya Sitaraman*

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Abstract. The σ^+ Lights Out puzzle is a game played on some geometrical grid wherein there is a bulb and a switch on each tile and the switches are wired so that flipping any switch changes the state of not only the bulb in its own tile but also of those in all adjacent tiles. A geometrical arrangement is said to be completely solvable if every initial configuration of 'on' and 'off' bulbs has a unique configuration of 'switch flips' to turn all the bulbs in the initial configuration off. By symmetrizing elements of the kernel of the σ^+ linear map, we present a method by which geometrical arrangements which have axes/planes of symmetry may be characterized as completely solvable or otherwise. The advantage of this method is that it can be used to reduce the complete solvability characterizations of more complicated arrangements to those of simpler arrangements, and therefore can be used to categorize these arrangements in a manner that is less cumbersome than previous polynomial approaches. We apply this method to reduce the characterizations of diamonds and surface grids on cylinders, capped-cylinders, tori, cones, capped-cones and spheres to the characterizations of rectangular grids. We also demonstrate how this method of symmetrizing elements of the kernel can be used to uncover many of the well-studied patterns regarding the characterization of $m \times n$ rectangular grids.

1. Introduction

Consider an undirected graph embedded in $\mathbb{R}^d$. At every vertex, there is a switch and a bulb. Each bulb can be in one of two states: 'on' and 'off'. Upon flipping a switch at a vertex, the bulb at that vertex and those at all adjacent vertices change their state. This is what is meant by the σ^+ game. In this paper, we will restrict our attention to finite graphs, and every explicit graph considered will be embedded in $\mathbb{R}^2$ or $\mathbb{R}^3$.

The undirected graph can often be depicted in a geometrical fashion by placing 'tiles' (finite dimensional shapes) at each vertex, and specifying an adjacency rule: a rule which allows one to determine which vertices are adjacent based on how their corresponding tiles interact. Every explicit graph considered in this paper will use tiles of dimension 2, drawn in either $\mathbb{R}^2$ or $\mathbb{R}^3$, and unless otherwise stated, two vertices will be considered to

* *Corresponding author*

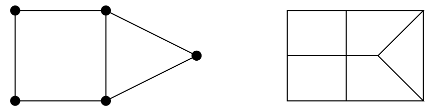


Figure 1. The diagram to the left is an undirected graph. The diagram on the right represents one way to depict the graph geometrically with the default adjacency rule. Each vertex has been represented by a tile - a two dimensional shape. If two vertices are connected by an edge, the corresponding tiles share a 1 dimensional surface.

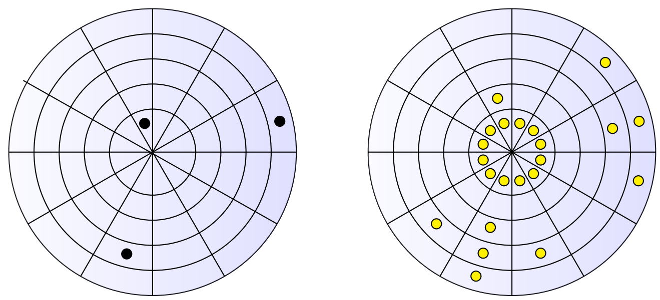


Figure 2. Both of the diagrams show the surface of a cone when one looks from the top. The black lines drawn carve the surface into tiles in a very natural way; the blue spaces between the black lines are the tiles. The diagram on the left represents which switches are flipped. The switches at the rim of the larger circle (5th rung of the cone) and the one on the fourth rung of the cone are wired according to the default adjacency rule. The switch flipped at the tip of the cone is wired according to the cone-tip rule. The diagram on the right shows which lamps will be turned on if these flips are applied on a geometrical setup in which all lamps are initially off.

be adjacent precisely when the corresponding tiles share a 1-dimensional surface. However, we will also consider cases where this is not so: for example, we will tile the surface of a cone, and will define every tile at the tip of a cone to be adjacent to every other tile at the tip. (Many of the tiles at the tip of the cone only share a single point, a 0-dimensional surface, but we will nevertheless define them to be adjacent to each other). Figure 1 demonstrates how a graph can be geometrically represented with the default adjacency rule. Figure 2 demonstrates the default adjacency rule and the cone-tip adjacency rule for a surface grid on a cone.

When a Lights Out graph is viewed as a geometrical object with tiles and an adjacency rule, we may refer to it as a 'geometrical arrangement', or a 'geometrical setup.' A geometrical arrangement is said to be completely solvable if every initial configuration of 'on' and 'off' bulbs has a unique configuration of flips that will turn it off. That is, after applying this configuration of flips to the initial configuration of 'on' and 'off' bulbs, every bulb in the geometrical arrangement will be in the off state. Such a configuration of flips can be referred to as a 'solution' to that initial configuration. The term "completely solvable" is also used in [5]. In [6], the term "reversible" is used instead.

The patterns which describe when a particular $m \times n$ rectangular grid is completely solvable have been studied in detail by Barua and Ramakrishnan [2], and also by Goldwasser, Klostermeyer, and Trapp [5]. While the rectangular σ game, which is similar except its adjacency rule requires that a switch does not affect its own tile, can be completely catogorized in a crisp statement, as shown in [2], the rectangular σ^+ game is more difficult to categorize. In both [2] and [5], the method that was used was to construct a Sylvester matrix equation $AX + XB = C$ over the field $\mathbb{Z}/2\mathbb{Z}$ (denoting 'off' by '0' and 'on' by 1), where the rectangular matrices A and B describe the default σ^+ adjacency rule, X is the rectangular matrix which denotes the configuration of flips, and C is the initial configuration which is turned off by the flips described by X. In this method, the geometrical setup will be completely solvable if and only if, for every C, there is a unique X which satisfies the equation. By the standard result on existence and uniqueness of solutions of Sylvester equations, this will happen if and only if the characteristic polynomials of A and $-B$ are relatively prime over $\mathbb{Z}/2\mathbb{Z}$, and since $-B = B$ over $\mathbb{Z}/2\mathbb{Z}$, complete solvability will hold if and only if the characteristic polynomials of A and B are relatively prime. The polynomials in question were shown in the two papers to be Fibonacci polynomials, which are elsewhere called Chebyshev-polynomials, as in [6] and [9]. By studying these, some patterns of solvabilities were established, and it was shown in [5] that the method of checking if the polynomials were relatively prime could be used to determine the solvability of a $m \times n$ grid in $O(\max(m,n) \cdot \log^2(\max(m,n)))$ time. Alternative approaches to characterize rectangular grids include studying a variation of Pascal's triangle, as was done in [7], but such approaches have not thus far proved as effective as studying Chebyshev-polynomials. Determining complete solvability is in general a difficult task and algorithmic methods need to be employed. Nevertheless, the solvability of certain problems of interest have clear and crisp answers. For instance, it has been shown linear algebraically by Sutner [8], and graph theoretically by H. Eriksson, K. Eriksson and Sjöstrand [3], that the all ones problem (equivalent to the problem of turning off lights when every light is initially on) always has a solution for every finite graph. The complete solvability of a geometric setup is substantially more difficult to categorize.

Generalized versions of the Lights Out puzzle in which each bulb has p states for a prime $p \in \mathbb{Z}$ have also been investigated on rectangular grids. Here, flipping adjacent switches causes a bulb to cycle through it's p states. In [6], Hunziker, Machiavelo and Park study Chebyshev-polynomials over $\mathbb{Z}/p\mathbb{Z}$ for a prime $p \in \mathbb{Z}$ in order to investigate whether there are infinitely many square grids that are not completely solvable in the generalized game with p states. Checking for complete solvability reduces to checking whether two polynomials over $\mathbb{Z}/p\mathbb{Z}$ are relatively prime over $\mathbb{Z}/p\mathbb{Z}$. The fact that there are infinitely many such grids for $p = 2$ is well known, and can in fact be easily deduced from the results we will present in Section 4. Hunziker, Machiavelo and Park prove that there are infinitely many square grids which are not completely solvable for $p = 3$ and formulate a conjecture which would prove the result true for all primes. In this paper, we will restrict our attention to the standard Lights Out puzzle in which there are exactly two states - i.e $p = 2$. We will be interested in a broader class of geometrical arrangements than rectangles, namely those geometrical arrangements which have axes or planes of symmetry.

The notion of symmetry is very natural to the problem of complete solvability. In [4], M. Florence and M. Frédéric show that any doubly-symmetric (about both vertical and horizontal axes) configuration on a rectangular grid can be turned off. In [3], H.Eriksson, K.Eriksson and Sjöstrand ask a very different question: "How few lamps can be lit on an infinite rectangular grid (with the usual adjacency rule) when all bulbs are initially off?". They conclude that the minimum number of lights that can be lit is 5, and furthermore show that the only way to light five bulbs is to flip switches in symmetric patterns which are called Mikado diamonds. In order to prove that the only patterns of flips that can light five switches is one of the Mikado diamonds, they first show that the pattern must be symmetric (about all vertical, horizontal and diagonal axes). To do this, they suppose that it was not and sum the asymmetric pattern with its reflection to obtain a contradiction. This method of obtaining the contradiction by reflecting an asymmetric configuration of flips about an axis of symmetry and summing it with the original configuration has the flavor of what we will exploit in this paper.

However, despite the fact that the notion of symmetry is natural to the problem, symmetrizing an element of the kernel in order to show equivalence of geometrical arrangements is, to our knowledge, a novel approach presented in this paper. This is a very powerful tool, since it can convert somewhat difficult geometrical arrangements into simpler ones. If one were to approach each problem from the polynomial approach, one would have to construct a new matrix equation each time, and investigate the gcds of the characteristic polynomials, which is substantially more cumbersome than the method presented here. Nevertheless, it should be noted that the polynomial approach, although more cumbersome, can provide additional information, since one can use it to find the number of elements of the kernel of the σ^+ linear map (details about the linear map itself are provided in Section 2). By symmetrizing elements of the kernel, we lose this information, but we also gain the information about the number of symmetric elements of the kernel. However, this is not the objective of the paper. The objective is not to find the number of elements of the kernel, but to simply identify which geometrical arrangements are completely solvable and which are not. This is a problem of great importance; given a setup of bulbs and switches wired in some way, and an initial configuration of burning bulbs, the first question we are likely to ask is whether there is a way to turn all the bulbs off, and complete solvability of the underlying setup provides a sufficient condition under which the answer will be 'yes'.

In Section 2 of the paper, we introduce the linear map M. In Section 3 of the paper, we will establish how we may exploit symmetrization of the kernel of M in some general settings. In Section 4, we prove many of the results presented in [2] and [5] regarding the patterns of characterizations of $m \times n$ rectangular grids. In Section 5, we will reduce the characterizations of diamonds and surface grids on cylinders, capped-cylinders, tori, cones, capped-cones and spheres to the characterizations of rectangular grids.

2. Introducing the linear map M

In this section, we will set the stage for the next section by introducing the σ^+ linear map M which is dependent on the particular geometrical arrangement at hand. We first label the tiles of the geometric setup as $1, 2, \ldots, n$. The order in which the tiles are labelled does not affect arguments that follow.

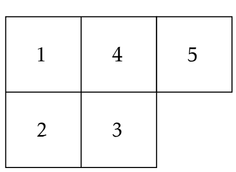
An event of flipping a particular arrangement of switches may be viewed as a vector v where $v_i = 1 \in \mathbb{Z}/2\mathbb{Z}$ if the i'th switch is flipped and $v_i = 0 \in \mathbb{Z}/2\mathbb{Z}$ if the i'th switch is not flipped, so that $\vec{v} = (v_1, ..., v_n)$ denotes the configuration of switch flips on the grid. This view imposes a natural additive structure on the set of all configurations of switch flips that did not exist previously. With this additive structure, the collection of all such vectors forms the n dimensional vector space $V = (\mathbb{Z}/2\mathbb{Z})^n$ over the field $\mathbb{Z}/2\mathbb{Z}$. Now, when all of the bulbs in a geometrical setup are off, and we flip the switches corresponding to a vector $v \in V$, some of the bulbs will be turned on. This is a configuration of 'on' and 'off' bulbs. Let Y denote the space of configurations of 'on' and 'off' bulbs. We may endow Y with an additive structure, and represent such configurations by vectors $(y_1, ...y_n) \in Y$, where $y_i = 1 \in \mathbb{Z}/2\mathbb{Z}$ if the i'th bulb is on, and $y_i = 0 \in \mathbb{Z}/2\mathbb{Z}$ if the i'th bulb is off. V and Y are then isomorphic spaces. The map $M : V \to Y$ is defined in such a way that $(Mv)_i = 1$ if the i'th bulb is 'on' after flipping the switches corresponding to the configuration described by $v \in V$ on an initially 'all off' setup and $(Mv)_i = 0$ if the i'th bulb is off after flipping switches as described by $v \in V$ on an initially 'all off' setup. Henceforth, we will use the following notation: $v_i = v(i)$ and $y_i = y(i)$. This notation will enable us to get rid of subscripts in longer expressions.

It remains to be shown that M is indeed a linear map. Since V and Y are vector spaces over $\mathbb{Z}/2\mathbb{Z}$, and since $M(0v) = M(0) = 0 = 0 \cdot M(v)$ and $M(1 \cdot v) = M(v) = 1 \cdot M(v)$, what remains to be shown is that if $v, w \in V$ are two vectors of 'flips', then $M(v+w) = Mv + Mw$. To check this, we introduce the following notation:

Definition 2.1. For any vector $u \in V$, and any tile i, define the set $\mathrm{ad}(u,i) := \{\text{tiles } j \,|\, u(j) = 1 \text{ and either } j \text{ is adjacent to } i \text{ or } j = i\}$. We will use $|\mathrm{ad}(u,i)|$ to denote the cardinality of $\mathrm{ad}(u,i)$.

Let i be any tile. Consider $(Mv)(i) + (Mw)(i)$. We observe that $(Mv)(i) + (Mw)(i) = 1$ if and only if $|\mathrm{ad}(v,i)| + |\mathrm{ad}(w,i)|$ is an odd number. Let $k = |\mathrm{ad}(v,i) \cap \mathrm{ad}(w,i)|$. We then observe that $(M(v+w))(i) = 1$ if and only if $|\mathrm{ad}(v,i)| - k + |\mathrm{ad}(w,i)| - k$ is an odd number. Since $|\mathrm{ad}(v,i)| - k + |\mathrm{ad}(w,i)| - k = |\mathrm{ad}(v,i)| + |\mathrm{ad}(w,i)| - 2k$, and $2k$ is even, these two characterizations are equivalent. Thus, $(Mv)(i) + (Mw)(i) = (M(v+w))(i)$. Since this is true for each i, we have that $Mv + Mw = M(v+w)$ as desired.

Example 2.2. What is the linear map M of the following arrangement?



All we have to note is that 1 affects 1, 2 and 4, 2 affects 1, 2 and 3, 3 affects 2, 3 and 4, 4 affects 1, 3, 4 and 5 and 5 affects 4 and 5. We use the standard basis with $e_i(j) = \delta_{ij}$. With respect to this basis, the linear map

$$M = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Note that M is a symmetric matrix, simply because i affects j if and only if j affects i.

Before we proceed, we should remark that in the above example, we used the standard basis, and with respect to this basis, the matrix we generated was simply the adjacency matrix of the associated graph plus the identity matrix. This will always be the case when we choose the standard basis, since we may then construct the matrix representing M by considering which tiles are affected by a given tile i, and the σ^+ rule dictates that the tiles that are affect by the i'th tile are simply the i'th tile itself and all adjacent tiles. In this paper however, we will never need to use matrices, and all arguments will be presented in an intrinsic manner.

Now, keeping the map M in mind, we return to the problem of characterizing a geometrical arrangement as completely solvable or otherwise. If there is some initial configuration without a solution, then $\text{rank}(M) \neq n$ and thus, by the Rank-Nullity theorem, it follows that there exists some non-trivial element of the kernel of M. Similarly, if uniqueness does not hold, then some problem has multiple solutions (multiple vectors of flips), and so the sum of these (which is nontrivial) will be in the kernel of M. Conversely, if the kernel of M is nontrivial, then there exists a nontrivial $v \in \ker(M)$ and in particular, uniqueness cannot hold since $M(v) = 0$ and $M(0) = 0$. Furthermore, by the Rank-Nullity theorem, the rank of M must not equal n and thus there must exist some vector w such that there is no v such that $Mv = w$. In particular, the initial configuration described by w cannot be turned off. Recalling that a geometrical setup is said to be completely solvable if every initial configuration of lit bulbs has a unique vector to turn it off, we then observe that complete solvability holds if and only if $\ker(M) = \{0\}$.

If v is a nontrivial element of $\ker(M)$, what does this mean in the context of the Lights Out puzzle? By definition, it means that $v \neq 0$ and $Mv = 0$. Thus, v represents a configuration of flips such that if the switches in the geometrical arrangement are flipped according to v, no bulbs will change from their initial state.

In subsequent sections, we will refer to the linear map M. If we refer to M in the context of a general theorem, it is understood that M represents an abstract linear map which describes the type of geometrical setups that the theorem holds for. If we refer to M in the context of an example or a specific application, it should be understood that M refers to the σ^+ linear map associated with the particular arrangement that we are considering. Thus, the definition of M will change according to the setup considered.

3. Symmetrizing elements of the kernel

In the introduction, we introduced a geometrical arrangement as a collection of tiles with some adjacency rule. Often, such grids can be drawn very naturally on a flat two dimensional surface, but some grids may be more naturally drawn on the surfaces of higher dimensional solids, and some grids may have tiles which are higher dimensional shapes and connections are not represented on a surface of a solid, but rather through the solid. Any of these notions is permitted by the definition of a geometrical arrangement we provided in the introduction. In this paper we primarily consider examples of geometrical arrangements which are either two dimensional and drawn on a plane, or are two dimensional surfaces of three dimensional objects. However, the results in this section are much more general in nature. We will always work in $\mathbb{R}^d$, and the tiles may have any dimension less than d. Some people might object that this generality is unnecessary since it is a fact that every finite graph can be embedded in $\mathbb{R}^3$. Nevertheless, while this is true, one should note that if the original geometrical arrangement is drawn more naturally in $\mathbb{R}^d$ for $d > 3$, we might prefer to determine the complete solvability directly instead of first embedding it in $\mathbb{R}^3$. For this reason, we choose to work in $\mathbb{R}^d$.

We saw at the end of the previous section that a geometrical arrangement is completely solvable if and only if there is no nontrivial element of the kernel of the linear map M. Bearing this in mind, we first present the main theorem, which is applied in the rest of the paper. Let us say we are in $\mathbb{R}^d$. In the statement of the theorem as well as in the rest of the paper, we use the word 'plane' to denote a translated linear subspace of dimension k for any $k < d$. We use the word 'axis' to refer to a 1-dimensional plane.

Theorem 3.1. *Consider a geometrical arrangement of tiles with associated linear map $M : V \to Y$, where the tiles of the geometrical arrangement are represented in $\mathbb{R}^d$. Let $a_1, ..., a_n$ be perpendicular planes about which the geometrical arrangement is symmetric, each plane having a dimension less than d. Then, there exists $v \in \ker(M)$ nontrivial if and only if there exists $\tilde{v} \in \ker(M)$ nontrivial which is symmetric about all planes $a_1, ..., a_n$.*

That is, if $v \in \ker(M)$ is nontrivial, then v can be symmetrized about all the planes $a_1, ...a_n$.

Proof. Before we begin the proof, let us introduce some notation. Let us denote the operation of reflection about plane a by R_a. Also, for a vector $u \in V$ and a tile i, $\mathrm{ad}(u, i)$ will be defined as in Definition 2.1.

We will prove the theorem by induction on n, but we will first show that if a is a plane about which the geometrical setup is symmetric, then $v \in \ker(M) \Leftrightarrow R_a v \in \ker(M)$. To this end, let $v \in \ker(M)$. Then

$$(M(R_a v))(i) = |\mathrm{ad}(R_a v, i)| \pmod 2 = |\mathrm{ad}(v, R_a(i))| \pmod 2 = (Mv)(R_a(i)) = 0$$

for each i. Thus, $R_a v \in \ker(M)$. Conversely, if $R_a v \in \ker(M)$, then we have just shown that $R_a^2 v \in \ker(M)$, and since $v = R_a^2 v$, $v \in \ker(M)$. Let us now begin the inductive process. For $n = 1$, take $v \in \ker(M)$ nontrivial. Then, set:

$$\tilde{v} = \begin{cases} v & \text{if } v \text{ is symmetric about } a_1 \\ v + R_{a_1} v & \text{otherwise} \end{cases}$$

The claim is that $\tilde{v}$ is a nontrivial element of the kernel which is symmetric about a_1. If v is already symmetric about a_1, there is nothing to show. If v is not symmetric about a_1, then since $v \in \ker(M) \Leftrightarrow R_{a_1} v \in \ker(M)$, we have that $M\tilde{v} = Mv + M(R_{a_1} v) = 0 + 0 = 0$, so $\tilde{v} \in \ker(M)$. Also, if $\tilde{v} = 0$, then $v(i) = (R_{a_1} v)(i)$ for every tile i which means that v would have to be symmetric about a_1. Therefore, since v is not symmetric about a_1, then $\tilde{v} \neq 0$ and thus is nontrivial. To show that $\tilde{v}$ is symmetric about a_1 it is sufficient to show that the tiles i where $\tilde{v}(i) = 0$ are symmetrically arranged about a_1 because then since the whole geometrical setup is symmetric about a_1, it will follow that the tiles i where $\tilde{v}(i) = 1$ are also symmetrically arranged about a_1. Thus, we observe that $\tilde{v}(i) = 0 \Leftrightarrow v(i) = (R_{a_1} v)(i) \Leftrightarrow (R_{a_1} v)(R_{a_1}(i)) = v(R_{a_1}(i)) \Leftrightarrow \tilde{v}(R_{a_1}(i)) = 0$, and thus $\tilde{v}$ is symmetric about a_1. This proves the case for $n = 1$.

Now, assume that there exists a nontrivial vector $v \in \ker(M)$ which is symmetric about $a_1, ..., a_{n-1}$. The claim is that there is a nontrivial vector $\tilde{v} \in \ker(M)$ which is symmetric about $a_1, ..., a_n$. As before, set

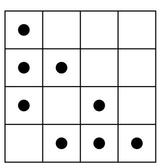
$$\tilde{v} = \begin{cases} v & \text{if } v \text{ is symmetric about } a_n \\ v + R_{a_n} v & \text{otherwise} \end{cases}$$

By our base case, we know that $\tilde{v} \in \ker(M)$, $\tilde{v}$ is nontrivial, and $\tilde{v}$ is symmetric about a_n. It remains to be shown that $\tilde{v}$ is symmetric about $a_1, ..., a_{n-1}$. If v is already symmetric about a_n, then we are done. Otherwise, take some axis a_k, and by perpendicularity of the planes, we have that since v is symmetric about a_k, $R_{a_n} v$ is symmetric about a_k. Thus,

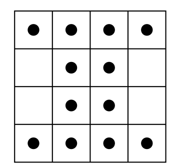
$$\tilde{v}(i) = v(i) + (R_{a_n} v)(i) = v(R_{a_k}(i)) + (R_{a_n} v)(R_{a_k}(i)) = \tilde{v}(R_{a_k}(i)),$$

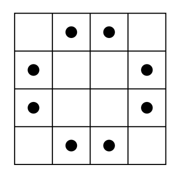
and thus $\tilde{v}$ is symmetric about a_k. This concludes the proof. □

Example 3.2. Symmetrizing elements of the kernel on a 4×4 grid.



Consider a 4×4 grid. There are many elements of the kernel which are asymmetric about both vertical and horizontal axes. An example is illustrated in the diagram to the left. Theorem 3.1 tells us that there is also an element of the kernel which is symmetric about both the vertical and horizontal axes. If we apply the procedure in Theorem 3.1 with a_1 being the horizontal axis, and a_2 being the vertical axis, we obtain an element of the kernel symmetric about the two axes which has been represented in the diagram to the top right. If we flip the switches of every tile with a black dot, notice that no bulbs will change state. Theorem 3.1 also tells us that there is an element of the kernel symmetric about both the diagonal axes, since they are perpendicular. Labelling the diagonal axes as a_1 and a_2 in any order, we can apply the procedure described in the theorem to obtain the element of the kernel depicted in the bottom right, which is symmetric about the two diagonal axes.





The next two theorems and related corollaries use the above result and apply it to some general situations. They are practically useful and will be applied throughout the paper. Lemma 3.4, which will follow soon, is not a lemma for the theorems that follow but it helps us apply the theorems. The result presented in the lemma is very well known, and can be easily worked out for oneself, but we leave the proof in for the sake of clarity. The notion of 'length' mentioned in the lemma as well as in the example below is just the number of tiles of a standard line. In this paper, geometrical dimensions always refer to the number of tiles.

Example 3.3. What will we mean by a standard line of tiles? What is a linear map associated with a standard line of tiles of length 4?

The diagram on the left shows a standard line of tiles of length 4. In the case of a line of length 4, label the tiles from left to right by $1,2,3,4$ in that order. A standard line of length n can be similarly labelled, and the adjacency rule will then be: tile 1 is adjacent to itself and tile 2, tile n is adjacent to itself and tile $n-1$, and, for all $k \in \{2,...,n-1\}$, tile k is adjacent to tile $k-1$, itself and tile $k+1$. For $i \in \{1,2,3,4\}$, let $e_i = \delta_{ij}$ be the i'th standard basis vector. Then, with respect to this basis,

$$M = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Lemma 3.4. *Consider a standard line of tiles, of length n. Then, the line is completely solvable iff* $n \not\equiv 2 \pmod 3$*. Consider a ring (a line connected onto itself) of length n. Then the ring is completely solvable iff* $n \not\equiv 0 \pmod 3$*.*

Proof. We can label the positions of the line as $1,2,3..,n$. The line is not completely solvable if and only if there is a nontrivial element v of the kernel. We search for such an element. If $v(1) = 0$, then to ensure $(Mv)(1) = 0$, we must require that $v(2) = 0$. Assume $v(1),...,v(j) = 0$. Then to ensure $(Mv)(j) = 0$, we must have that $v(j+1) = 0$. Therefore, $v(i) = 0$ for all i. This is the trivial solution. Now, assume that $v(1) = 1$. Then, to ensure $(Mv)(1) = 0$, we must have that $v(2) = 1$, and continuing the argument, we see that the pattern of v must be $1,1,0,1,1,0,1,1,0,1,1,0...$ and so on. But we also require this to be satisfied from the other end, so in particular, we require $v(n) = 1$, $v(n-1) = 1$, $v(n-2) = 0$. This can happen if and only if $n = 2 \pmod 3$. Thus, the line is completely solvable if and only if $n \not\equiv 2 \pmod 3$.

The argument for a ring is very similar. Suppose that there exists a nontrivial $v \in \ker(M)$. Then, we must flip some switch. Let us label the tile in which this switch is flipped as the first tile. There must be exactly one adjacent tile whose switch is also flipped. Label this tile as the second tile, and flip its switch. In this manner, label the tile adjacent to tile 2 that is not tile 1 as tile 3, and continuing in this manner, we may label all the tiles. Note that tile 1 is adjacent to tile 2 and tile n. We cannot flip the switch of tile n in order to ensure that $(Mv)(1) = 0$. Arguing as before, the pattern must be $1,1,0,1,1,0,...$ and since $v(n) = 0$, we may find an element of the kernel if and only if $n \equiv 0 \pmod 3$. □

The theorems that are soon to follow allow us to reduce the characterization of a larger geometrical arrangement to that of a smaller arrangement, utilizing the symmetrization process we observed in Theorem 3.1. The following definition provides us with a notion that is essential to make this reduction.

Definition 3.5. Consider a geometrical setup S. A *generalized line* of tiles (denoted L) of S is a collection of tiles that has two properties:

(1) S is symmetric about a plane which bisects every tile of the L.

(2) For any given tile $i \in L$, the tiles of $S - L$ which are adjacent to i exist in symmetric pairs about the plane.

Example 3.6. What are some examples of generalized lines?

(1) We can place three rings of tiles, each of length n, one on top of each other to form a cylinder of height 3. Let us choose for simplicity to perfectly align the rings when we stack them so that any tile in the middle ring is adjacent to exactly two tiles of the setup minus the ring: the tile above and the tile below. Note that the cylinder is symmetric about a plane of symmetry which bisects every tile of the middle-ring. Note also that given any tile on the ring, the two tiles which we noted were in the setup minus the ring and were adjacent to the tile form a symmetric pair about the plane. Thus, the middle-ring of the cylinder is a generalized line of tiles. Let us take a moment to look at the adjacency rule of the ring. The rule is similar to that of a line, but the difference is that the ends affect each other.

(2) Consider a square grid (like the one in Example 3.2), and consider the collection of tiles along one of the diagonals. Note that a diagonal axis of symmetry bisects every tile in the diagonal. Moreover, the tiles of the square minus the diagonal that are adjacent to any tile on the diagonal exist in symmetric pairs about the diagonal axis. Thus, the collection of tiles on the diagonal is a generalized line of the square. The adjacency rule here is that no two tiles on the diagonal are adjacent to each other. If we use the term 'diagonal' in the context of another geometrical setup, it will mean a generalized line in which no two tiles on the line are adjacent to each other.

We saw two examples of generalized lines with two different adjacency rules. There is no restriction on the adjacency rule of a generalized line other than the fact that the rule must be symmetric with respect to the plane of symmetry that bisects the line.

Theorem 3.7. *Consider an arbitrary geometrical setup drawn in $\mathbb{R}^d$ which has a generalized line. If the generalized line is not completely solvable, then the setup is not completely solvable.*

Proof. Denote the generalized line by L, the setup by S, the setup minus the line by $S-L$, and the restriction of a vector v to a collection of tiles X by $v\restriction_X$. Denote by M_X the linear map associated with a collection of tiles X, when considered in isolation (Note that for $X \neq S$, M_X is a smaller matrix that M). Denote by $M_{XS} : X \to S$ the following linear map: For $x \in X$, take $y \in S$ such that $y\restriction_{S-X} = 0$ and $y\restriction_X = x$; then, $M_{XS}x = My$.

Suppose first that $S-L$ is completely solvable. Then, since L is not completely solvable, there exists some nontrivial $w_0 \in \ker(M_L)$. Set $v\restriction_L = w_0$. The flipping of the switches of v on L induces some of the lights of $S-L$ to be switched on. Because a plane of symmetry bisects every tile of L, the induced pattern will be symmetric about this plane. Since $S-L$ is completely solvable, there exists a unique solution w which will turn this induced initial configuration off. By uniqueness, w must be symmetric about the plane of symmetry (since the original configuration of 'on' and 'off' lights was symmetric about the plane, and if w is not symmetric about the plane, we can reflect it about the plane and the reflection will also be a solution, thereby contradicting uniqueness). Take any tile $i \in L$. The tiles of $S-L$ adjacent to i exist as symmetric pairs about the plane, and w, defined only on $S-L$, is symmetric about the plane bisecting L. Therefore, we have that the tiles j adjacent to i such that $w(j) = 1$ must exist as symmetric pairs about i. Therefore, there is no net effect on the state of the bulb in tile i when w is applied. Noting this, take $v\restriction_L = w_0$, and

$v\restriction_{S-L} = w$. For $i \in L$, we have:

$$(Mv)(i) = (M_{LS}v\restriction_L)(i) + (M_{(S-L)S}v\restriction_{S-L})(i) = (M_{LS}w_0)(i) + (M_{(S-L)S}w)(i) = 0 + 0 = 0$$

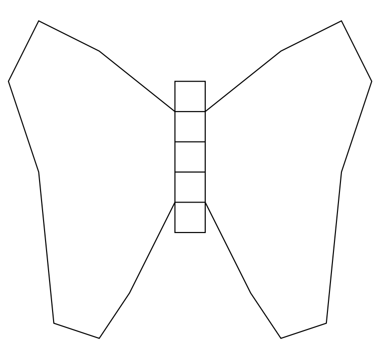
For $i \in S-L$, we have that $(Mv)(i) = (M_{LS}w_0)(i) + (M_{(S-L)S}w)(i) = 0$ by the definition of w. Thus, we see that $v \in \ker(M)$ so the setup is not completely solvable.

Now suppose that $S-L$ is not completely solvable. Then, by Theorem 3.1, there exists a nontrivial symmetric (about the plane bisecting L) element $w \in \ker(M_{S-L})$. So, set $v\restriction_{S-L} = w$ and $v\restriction_L = 0$. Then, since $v\restriction_L = 0$, the generalized line does not contribute to the state of any bulb, so for a tile $i \in S-L$, $(Mv)(i) = (M_{(S-L)S}w)(i) = 0$, since $w \in \ker(M_{S-L})$. For a tile $i \in L$, note that the tiles $j \in S-L$ adjacent to i such that $w(j) = 1$ exist in symmetric pairs about i, and therefore, their contributions cancel. Thus $(Mv)(i) = 0$. Hence, $v \in \ker(M)$, so the setup is not completely solvable. □

We wish to emphasize that the theorem above makes no restriction that the portions of the setup on either side of the line cannot affect each other (We will see several instances where the symmetric portions affect each other when we consider the shapes such as the odd-cylinder and the odd-cone). One might also wonder why we required in the definition of a generalized line that for any given tile i on the generalized line, the tiles of the setup minus the generalized line which are adjacent to i exist in symmetric pairs about the plane. Does this not follow from the fact that a plane of symmetry bisects every tile on the generalized line? The answer is no. We will explain the reason for this in Example 3.9.

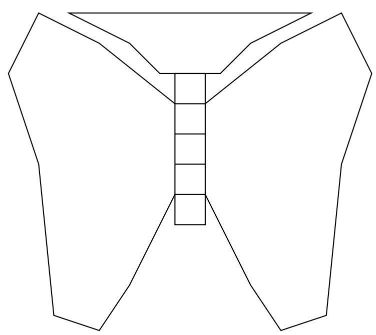
When we are working with specific geometrical arrangements, it will often be the case that the generalized line in question is either a standard line or a ring. It is therefore worthwhile giving some special attention to these cases. We see as a direct consequence of Lemma 3.4 and Theorem 3.7 that if a geometrical setup has a standard line of length $l \equiv 2 \pmod 3$ for a generalized line, then the setup is not completely solvable. Likewise, if a geometrical setup has a ring of tiles of length $l \equiv 0 \pmod 3$ as a generalized line, then the setup is not completely solvable.

Example 3.8. Lights Out on a butterfly with central body length 5. (A butterfly is a being with a standard line as a central body and two identical wings on either side. The wings are tiled in some manner unknown to us.)



The wonderful thing about Theorem 3.7 is that even though we know nothing about the tiling of the wings, it guarantees that the butterfly is not completely solvable, since it is symmetric about a plane bisecting a line of length $5 \equiv 2 \pmod 3$ and the tiles of the setup minus the line that are adjacent to any tile i of the line exist in symmetric pairs about the plane. Needless to say, since we have no information about the shape and tiling of the wings, we cannot possibly explicitly find an element of the kernel. Nevertheless, Theorem 3.7 guarantees that one exists.

Example 3.9. Why do we need the seemingly extra assumption that for "any given tile i on the generalized line, the tiles of the setup minus the generalized line which are adjacent to i exist in symmetric pairs about the plane"? Does this not follow from the fact that the setup is symmetric about a plane that bisects every tile of the generalized line?



In the diagram to the left, the central column satisfies the following two properties: (i) It is a standard line (ii) If we drew an axis which bisected it, the setup would be symmetric about the axis. However, we cannot for certain say anything about the solvability of the setup for the following reason: The topmost of the five tiles is adjacent to some collection of tiles directly above it which is not in line. It is very possible that one of the tiles in the collection is its own reflection, in which case the tile cannot possibly have a symmetric pair that about the axis. We have no way to assert, therefore that the standard line is a generalized line. Thus, Theorem 3.7 cannot be applied to this arrangement.

Theorem 3.10. *Consider an arbitrary geometrical setup drawn in $\mathbb{R}^d$ which has a generalized line. If the generalized line is completely solvable, then the solvability of the setup is equivalent to the solvability of the setup minus the generalized line.*

Proof. As before, denote the generalized line by L, the setup by S, the setup minus the line by $S-L$, and the restriction of a vector v to a collection of tiles X by $v\restriction_X$. Denote M_X by the linear map associated with a collection of tiles X, when considered in isolation.

Assume that the setup is not completely solvable. Then, by Theorem 3.1, there exists an nontrivial element $v \in \ker(M)$ which is symmetric about the plane that bisects L. There is no net effect on the bulbs in L by $v\restriction_{S-L}$ since the tiles in $S-L$ that are adjacent to any tile $i \in L$ exist in symmetric pairs about the plane, and v is symmetric about the plane. Therefore, $v\restriction_L \in \ker(M_L)$. Since L is completely solvable, it follows that $v\restriction_L = 0$. Therefore since there is no contribution from $v\restriction_L$ to the bulbs in $S-L$, it follows that $v\restriction_{S-L} \in \ker(M_{S-L})$. Thus, $S-L$ is not completely solvable.

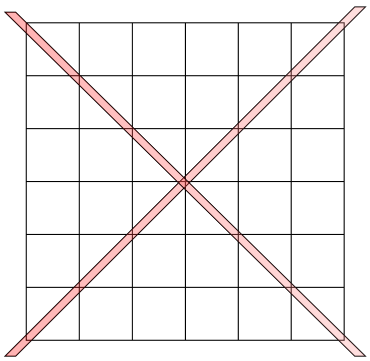
Assume that $S-L$ is not completely solvable. By a proof identical to the second portion of Theorem 3.7, we see that the setup is not completely solvable. Hence, the solvabilities of L and $S-L$ are equivalent. ☐

In a similar vein to the remarks after Theorem 3.7, we will give some special attention to the case when the generalized line in question is either a standard line or a ring. We see as a direct consequence of Lemma 3.4 and Theorem 3.7 that if a geometrical setup has a standard line of length $l \not\equiv 2 \pmod 3$ for a generalized line, then the solvability of the setup is equivalent to the solvability of the setup minus the standard line. Likewise, if a geometrical setup has a ring of tiles of length $l \not\equiv 0 \pmod 3$ as a generalized line, then the solvability of the setup is equivalent to the solvability of the setup minus the ring.

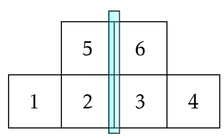
Corollary 3.11. *Consider a geometrical setup drawn in $\mathbb{R}^d$ which has a diagonal for a generalized line. Then, the solvability of the setup is equivalent to the solvability of the setup minus the diagonal.*

Proof. As remarked earlier, a diagonal is a generalized line in which no two tiles of the line are adjacent to each other. Such a line L is always completely solvable since there can not exist any nontrivial element of $\ker(M_L)$ (as each tile stands alone). Hence, applying Theorem 3.10, we see that the solvability of the setup is equivalent to the solvability of the setup minus the diagonal. □

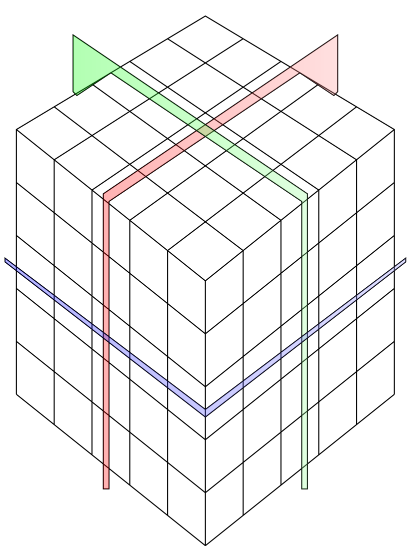
Example 3.12. Is the 6×6 square grid completely solvable?



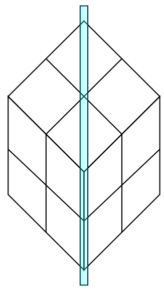
Applying Corollary 3.11 twice in succession, we see that the solvability of the 6×6 square is equivalent to the solvability of the (4)-pyramid, the structure depicted below. By Theorem 3.1, the pyramid is not completely solvable iff there is a nontrivial element of the kernel which is symmetric about the cyan axis. We will try to find an element v of the kernel of the pyramid which is symmetric about the cyan axis. First suppose $v(1) = v(4) = 0$. Then, $(Mv)(1) = 0 \implies v(2) = 0$ and by symmetry, $v(3) = 0$. Further, $(Mv)(2) = 0 \implies v(5) = 0$ and by symmetry $v(6) = 0$. This is the trivial element of the kernel. If $v(1) = v(4) = 1$, then $(Mv)(1) = 0 \implies v(2) = 1$ and by symmetry, $v(3) = 1$. $(Mv)(2) = 0 \implies v(5) = 1$ and by symmetry, $v(6) = 1$. But then, $(Mv)(5) = 1$ which shows that $v \notin \ker(M)$. Thus, there is no nontrivial element of the kernel symmetric about the cyan axis, and by Theorem 3.1, the 6×6 square is completely solvable.



Example 3.13. Is the surface grid of a $5 \times 5 \times 5$ cube completely solvable?

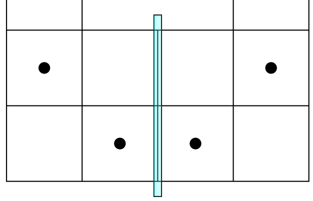


Three planes of symmetry are depicted in the diagram. Each plane bisects a ring which is of length $5 \cdot 4 = 20$ and $20 \not\equiv 0 \pmod 3$. Therefore, first apply Theorem 3.10 using the red plane and the solvability of the cube is equivalent to the solvability of the cube-minus-red-ring. Since the sections on either side of the red ring do not affect each other, the solvability of the cube minus the red ring is equivalent to the solvability of one of the sections. Now, such a section is symmetric about a line that the green plane cuts. This line has length $5 + 2 \cdot 2 = 9 \not\equiv 2 \pmod 3$. Therefore, by Theorem 3.10, the solvability of the side is equivalent to the solvability of the section minus the green line, which is comprised of two portions on either side of the green plane that do not affect each other. Such a portion is symmetric about a line that the blue plane bisects. The length of this line is $4 \not\equiv 2 \pmod 3$. Therefore, applying Theorem 3.10 the solvability of the section-minus-the-green-line is equivalent to the solvability of the section-minus-the-green-line minus the blue line. Thus, the solvability of the cube is equivalent to the solvability of (a) below.

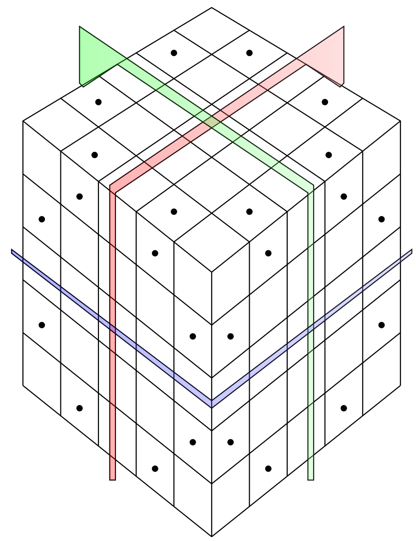


(a)

Now, (a) has a diagonal axis of symmetry (cyan). By Corollary 3.11, characterizing the geometrical setup is equivalent to characterizing the setup minus the diagonal, which is depicted in (b), on which a nontrivial element of the kernel has been found. Therefore, the $5 \times 5 \times 5$ cube is not completely solvable. The element of the kernel has been depicted in (c) below.



(b)



(c)

4. The solvability patterns for $m \times n$ rectangular grids

Obviously, the solvability characterization of a $m \times n$ grid is the same as that of a $n \times m$ grid. Bearing this in mind, we have the following propositions wherein it is understood that the same result holds for an $n \times m$ grid.

Proposition 4.1. *If $m \equiv 2 \pmod 3$ and n is odd, then the $m \times n$ grid is not completely solvable.*

Proof. Note that if n is odd, there is a central column about which the grid is symmetric. Moreover, since $m \equiv 2 \pmod 3$, the length of the column $l \equiv 2 \pmod 3$. Since, the tiles of the grid minus the central column that surround any tile i of the central column exist in symmetric pairs about i, the proposition is a direct corollary of Theorem 3.7. □

Proof. We can also prove this in a constructive way. On any odd-numbered column, make the components of v follow the pattern $1,1,0,1,1,0,1,1,0,...,0,1,1$ as we go down successive rows, and let the restriction of v to even columns be 0. Then, $v \in \ker(M)$. □

In the first proof of the above proposition, notice that it was very important for n to be odd, so that the rectangle could be symmetric about a column (vertical line of tiles). In a rectangle with an even number of columns, the vertical axis of symmetry does not bisect a column, but instead traces a line that separates two columns (which we will henceforth refer to as a separating-line). Thus, it is not possible for us to apply Theorem 3.7, and Proposition 4.1 cannot be extended to include even n. For a simple counter-example, the reader may check that the 2×2 square is completely solvable although $2 \equiv 2 \pmod 3$ (One quick way to verify this would be to apply Corollary 3.11).

Note that the above proposition is equivalent to the first statement in the third part of Proposition 4.5 from [2]. Another interesting observation is that the computer game made by *Tiger Electronics* which was studied by M. Anderson and T. Feil in [1] featured a 5×5 grid. This proposition shows that such a grid is not completely solvable. This not only means that some initial configurations will not have solutions but also means that solutions to symmetric initial configurations may be asymmetric, which is likely to have made the solutions to puzzles in the computer game less obvious. Note that if complete solvability holds and an initial configuration is symmetric about some plane a_1, then its solution must also be symmetric about a_1, since if it is not, we may reflect the solution about a_1 and the reflection will also be a solution to the initial configuration, thereby contradicting uniqueness. For geometrical arrangements which are not completely solvable, there is no such restriction.

We should also remark here that the above result implies that there are infinitely many square grids that are not completely solvable. The analogous statement when each bulb has three states was shown to be true in [6]. For a general prime p, the question is to our knowledge still open, and [6] includes a conjecture which would imply that the statement is true for every prime p.

Proposition 4.2. *If $m \not\equiv 2 \pmod 3$, then the $m \times n$ grid is completely solvable if and only if the $m \times (2n+1)$ grid is completely solvable.*

Proof. Since $2n+1$ is always odd, there will be a column about which the $m \times (2n+1)$ grid is symmetric. Also, given any tile i in the column, the tiles of the grid minus the column adjacent i exist in symmetric pairs about the column. Thus, by Theorem 3.10, since the length of the column $l = m \not\equiv 2 \pmod 3$, we have that the solvability of the $m \times (2n+1)$ grid is equivalent to the solvability of the $m \times (2n+1)$ grid without the central column, and since the two $m \times n$ grids on either side of the column do not affect each other, we have that the solvability of the $m \times (2n+1)$ is equivalent to that of an $m \times n$ grid, which proves the result. □

Note that the above two propositions mean that we can categorize all grids with one or two odd sides either explicitly, or in terms of smaller grids with two even sides. This is because if n is odd, then either $m \equiv 2 \pmod 3$, in which case we apply Proposition 4.1, or else $m \not\equiv 2 \pmod 3$ in which case we apply Proposition 4.2 in order to reduce it to an $m \times \frac{n-1}{2}$ grid. We can obviously do the same thing with the roles of m and n reversed.

Corollary 4.3. *Suppose that the $m \times n$ grid is completely solvable. Then: (i) If $m \not\equiv 2 \pmod 3$, then the $m \times (2^k(n+1)-1)$ grid is completely solvable for every $k \in \mathbb{N} \cup \{0\}$. (ii) If $m, n \not\equiv 2 \pmod 3$, then the $(2^k(m+1)-1) \times (2^l(n+1)-1)$ grid is completely solvable for every $k, l \in \mathbb{N} \cup \{0\}$.*

Proof. Let us prove part (i) first. For $k = 0$, the result is trivial since $2^0(n+1)-1 = n$. Assume that (i) is true for $k \in \mathbb{N}$: that is, the $m \times (2^k(n+1)-1)$ grid is completely solvable. Then, for $k+1$, we see that since $m \not\equiv 2 \pmod 3$, we may apply Proposition 4.2

and conclude that the $m \times 2(2^k(n+1)-1)+1$ grid is completely solvable. That is: the $m \times 2^{k+1}(n+1)-1$ grid is completely solvable, which proves (i).

Now let us prove (ii). Note that it suffices to show that $2^k(m+1)-1 \not\equiv 2 \pmod 3$ for all k, because we can then apply (i) to conclude (ii). Before we begin an inductive argument, we will first show that if $x \not\equiv 2 \pmod 3$, then $2x+1 \not\equiv 2 \pmod 3$. To see this, note that if $x \in \mathbb{N}$ and $2x+1 = 2 \pmod 3$, then since $2x+1$ is always odd, $2x+1 = 5+6j$ for some $j \in \mathbb{N} \cup \{0\}$. Thus, $x = \frac{4+6j}{2} = 2+3j$, so $x \equiv 2 \pmod 3$. Thus, if $x \not\equiv 2 \pmod 3$, then $2x+1 \not\equiv 2 \pmod 3$.

Now we may proceed with an inductive argument to prove (ii). For $k = 0$, we easily see that $2^0(m+1)-1 = m \not\equiv 2 \pmod 3$. Assume for k that $2^k(m+1)-1 \not\equiv 2 \pmod 3$. Take $x = 2^k(m+1)-1$. Then, $2^{k+1}(m+1)-1 = 2(2^k(m+1)-1)+1 = 2x+1 \not\equiv 2 \pmod 3$, which completes the inductive argument and proves (ii). □

A special case of this corollary is with $m = n = 1$, and the 1×1 square is trivially completely solvable, and $1 \not\equiv 2 \pmod 3$. We then apply the corollary to see that the $(2^k-1) \times (2^l-1)$ grid is completely solvable for every $k, l \in \mathbb{N}$. This is the result quoted in the first part of Proposition 4.5 of [2].

One can also compare the above result with part ii) of Proposition 8 in [5]. The results are very similar; moreover, the reverse direction that is quoted there is a simple consequence of the next proposition, since $(2n+1) = n+(n+1)$, and $2(n+j(n+1))+1 = n+(2j+1)(n+1)$.

Proposition 4.4. *Assume that a $m \times n$ rectangle is not completely solvable. Then, the $m' \times n'$ rectangle is not completely solvable whenever $m' \equiv m \pmod{m+1}$ and $n' \equiv n \pmod{n+1}$.*

Proof. Assuming that a $m \times n$ rectangle is not completely solvable, then by Theorem 3.1, we may find an element of the kernel w which is symmetric about both the horizontal and vertical axes. Now consider a $(m+k(m+1)) \times (n+k'(n+1))$ rectangle. Mark the columns $n+1, 2(n+1), \ldots k'(n+1)$, and the rows $m+1, 2(m+1), \ldots k(m+1)$ and set $v(i) = 0$ whenever i is a tile on these marked rows or columns. What we have then done is we have divided the big rectangle into little rectangles of size $m \times n$, separated by the marked rows and columns. On each little rectangle R_j, set $v\restriction_{R_j} = w$. We claim that v is in the kernel of the $(m+k(m+1)) \times (n+k'(n+1))$ rectangle.

First, take a tile i on one of the marked rows and columns. If i is on the intersection of a marked row and a marked column, then the none of the adjacent tiles are flipped when v is applied and thus $(Mv)(i) = 0$. Otherwise, i is adjacent to three tiles on its marked row/column, and two other tiles i_1 and i_2 which belong to two different little rectangles. The relative positions of i_1 and i_2 with respect to their rectangles are such that if the two rectangles were superimposed, i_1 and i_2 would be reflections of each other about a horizontal/vertical axis of symmetry. Thus, since w is identically placed on both rectangles and w is symmetric about the horizontal and vertical axes of symmetry, $w(i_1) = w(i_2)$. Thus, $Mv(i) = 0 + w(i_1) + w(i_2) = 0$.

Now, take a tile i in one of the little $m \times n$ rectangles; we see that the marked rows and columns isolate the rectangle from all other flipped switches of v and therefore $(Mv)(i) = (M_{m \times n}w)(i) = 0$. ($M_{m \times n}$ is the linear map of the $m \times n$ grid which contains i). Hence v is in the kernel of the $(m+k(m+1))\times(n+k'(n+1))$ rectangle, and since k,k' were arbitrary, the proposition follows. □

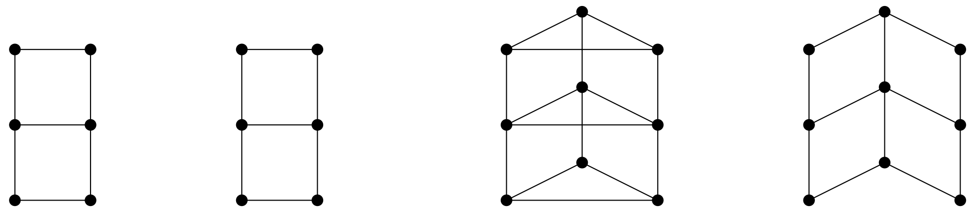
5. Using symmetry to predict Lights Out solvabilities of various arrangements of tiles

Previous investigation into the solvability of rectangular grids has primarily exploited polynomial algebra. Polynomial approaches can become difficult when we move to non-rectangular arrangements of tiles. This is because, in order to determine which polynomials must be relatively prime for the solvability to hold, we must construct a Sylvester rectangular-matrix equation (of the form $AX+XB=C$) as explicitly demonstrated for example in [2]. As the shapes change, it may be difficult to construct the rectangular matrix equation and determine whether the characteristic polynomials are relatively prime. In the case of a cylinder or torus, there is an obvious choice for the matrices and hence the matrix equation, as shown in [2], where a polynomial based solution is given. The reason why the matrix equation could be very naturally constructed in those cases is because the cylinder and torus are inherently linked to rectangles: Lights out on the surface of a cylinder or torus is equivalent to that on a rectangle with a different adjacency rule that allows tiles on opposite sides to affect each other. Depending on the problem however, it may not be so easy, and the matrices (and hence characteristic polynomials) can become ugly. Fortunately, for many of these shapes, an approach by symmetrizing the kernel is much cleaner, and in this section, we will reduce the characterizations of surface grids on cylinders, capped-cylinders, tori, cones, capped-cones and spheres as well as the characterizations of diamond grids (in that order) to the characterizations of rectangular grids.

Lights Out on the surface of a cylinder. When we refer to a cylinder, we refer to a cylindrical shell with a hole through the middle, without caps at the ends. Tile a cylinder by drawing circles at regular intervals of height and drawing lines that run vertically from the bottom to the top. We will call the columns of tiles that exist between two lines as columns, and the lines that separate the columns will be called separating-lines. We may call a cylinder a (n,h) cylinder if it has a height h and circumference n, where n and h correspond to the number of tiles along the circumference and the number of tiles in each column respectively. The $n=1$ case is equivalent to a line of tiles with length h, which we categorized in Lemma 3.4. The $n=2$ case is equivalent to a $h\times 2$ rectangle, which can be seen by representing the $(2,h)$ cylinder by its associated graph (with each tile being a vertex and the connecting edges representing the adjacency rule). The $h\times 2$ rectangle will be completely solvable if and only if h is even (Hint: Look for elements of the kernel which are symmetric about the vertical axis (the axis that is parallel to the long side of the rectangle)). However, for $n>2$, it is not true that the graph of a (n,h) cylinder is equivalent to the graph of a $h\times n$ rectangle. This is because in such a rectangle, the first column and n'th column do not affect each other. The example below illustrates this difference. Henceforth, we may therefore consider $n>2$.

Example 5.1. The graphs associated with a $(2,3)$ and a $(3,3)$ cylinder.

Graph (a) is the associated graph of a $(2,3)$ cylinder. Graph (b) is the associated graph of the 3×2 rectangle. Note that they are the same. Graph (c) is the associated graph of a $(3,3)$ cylinder. Graph (d) is the associated graph of a 3×3 square. Note that they are not the same.



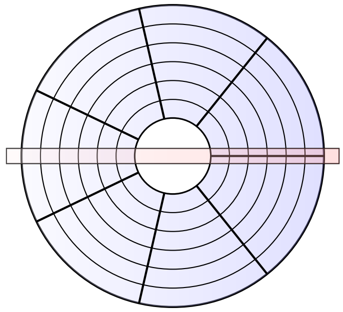
(a) (b) (c) (d)

Proposition 5.2. *Let $n > 2$. If $h \equiv 2 \pmod 3$, then the cylinder is not completely solvable. If $h \not\equiv 2 \pmod 3$, then the cylinder is solvable if and only if the $(n-1)\times h$ rectangle is solvable.*

Proof. Consider a plane that bisects any vertical column. The cylinder is symmetric about this plane. Moreover, taking any tile i in the column which the plane bisects, the adjacent tiles to i that are in the cylinder minus the column exist in pairs about the plane. If $h \equiv 2 \pmod 3$ then the cylinder is not completely solvable by Theorem 3.7. If $h \not\equiv 2 \pmod 3$, then by Theorem 3.10, the solvability of the cylinder is equivalent to the solvability of the cylinder minus the column, which is a $(n-1)\times h$ rectangle. □

Note that for the case where $h \equiv 2 \pmod 3$, we could have alternatively explicitly constructed an element of the kernel: label the circumference position of a tile by $i = 1,...,n$ and the height position of a tile by $j = 1,...,h$, so that the coordinate of any tile can be represented by (i,j). If $h \equiv 2 \pmod 3$, then for each i, define v by letting $v(i,1), v(i,2), ..., v(i,h)$ follow the pattern $1,1,0,1,1,0,...,0,1,1$ (Note Lemma 3.4). We claim that $v \in \ker(M)$. To see this, let us look at any position (i,j). $(Mv)(i,j)$ is not affected by adjacent columns since the contribution from the columns to either side cancels out. Thus, the only contribution to the state of the bulb in a tile is from the same column. Since each (i,j) is adjacent to exactly two tiles in its column that are associated with nonzero components of v, it follows that $(Mv)(i,j) = 0$. Thus, $v \in \ker(M)$ and the cylinder is not completely solvable.

Example 5.3. Consider a cylinder of height 6 and circumference 7. Is this cylinder completely solvable?



This is a top view of a cylinder of height 6 and radius 7. The red rectangle represents a plane of symmetry which cuts through the cylinder as described in the proof of Proposition 5.2. We first notice that $6 \not\equiv 2 \pmod 3$. Therefore, by Proposition 5.2, the solvability of this cylinder is equivalent to the solvability of a 6×6 square, which is completely solvable by Example 3.12. Therefore, this cylinder is completely solvable.

Lights Out on the surface of capped cylinders. We add caps to either one end, or two ends of the cylinder. Such cylinders are called one-capped and two-capped cylinders respectively. The cap is a single tile that affects and is affected by all the tiles in the rung which is adjacent to it. Let h denote the height of the cylinder without the caps, and n denote the number of tiles making up the circumference. If $n = 1$, the capped cylinder is simply a line of tiles of length $n+1$ or $n+2$ (depending on the number of caps). Thus, we may consider $n > 1$.

Proposition 5.4. *Let $n > 1$.*

(i) If n is even, the solvability conditions of the capped and regular cylinder are the same (for both the one-capped cylinder and the two-capped cylinder).

(ii) If n is odd, then the one-capped cylinder is completely solvable if and only if $h \not\equiv 1 \pmod 3$ and the $(n-1) \times h$ rectangle is completely solvable.

(iii) If n is odd, then the two-capped cylinder is completely solvable if and only if $h \not\equiv 0 \pmod 3$ and the $(n-1) \times h$ rectangle is completely solvable.

Proof. Denote the whole setup (the capped-cylinder) by S. Denote the collection of tiles corresponding to the caps by C. Denote the capped-cylinder minus the caps by $S - C$ (a regular cylinder). Notation such as M_{S-C} and $M_{(S-C)S}$ will be carried over from Theorem 3.7.

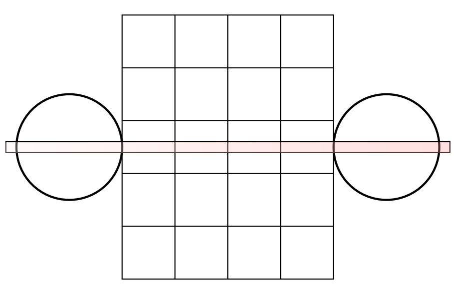
Proof of i): If n is even, then consider a plane of symmetry that cuts through two opposite vertical separating-lines (not columns, but the lines that separate the tiles). If the capped cylinder is not completely solvable, then there is a nontrivial element $v \in \ker(M)$ which is symmetric about this plane. Due to symmetry about the plane, $v(j) = 1$ on an even number of tiles on the top rung and on an even number of tiles on the bottom rung. Since a tile in $S - C$ is adjacent to a cap if and only if it is in the nearest rung (either bottom or top rung), it follows that if i is a cap tile, $(M_{(S-C)S} v \upharpoonright_{S-C})(i) = 0$. Thus, since $v \in \ker(M)$, $0 = (Mv)(i) = (M_{(S-C)S} v \upharpoonright_{S-C})(i) + (M_{CS} v \upharpoonright_C)(i)$, and thus, $(M_{CS} v \upharpoonright_C)(i) = 0$ which is only possible if $v(i) = 0$ (since the caps are disjoint from one another and flipping a switch on each cap affects that cap). Hence, $v \upharpoonright_C = 0$. Thus, when we flip switches in a configuration described by v, the caps do not contribute to the state of the bulbs in $S - C$. Therefore, $v \upharpoonright_{S-C} \in \ker(M_{S-C})$, and the (n,h) uncapped cylinder $S - C$ is not completely solvable.

Conversely, if the regular (uncapped) cylinder $S-C$ is not completely solvable, then there is a nontrivial element $w \in \ker(M_{S-C})$ which is symmetric about the plane which cuts through two opposite vertical separating-lines. Set $v \upharpoonright_{S-C} = w$ and $v \upharpoonright_C = 0$. Then, v is in the kernel of the capped cylinder, since the caps neither affect other tiles (since $v \upharpoonright_C = 0$) nor are affected by tiles (since the number of tiles j that are adjacent to the cap and for which $w(j) = 1$ is an even number, because w is symmetric about the plane), which means that the capped cylinder is not completely solvable. Thus, the solvabilities of the capped cylinder and the regular cylinder are equivalent, which proves i).

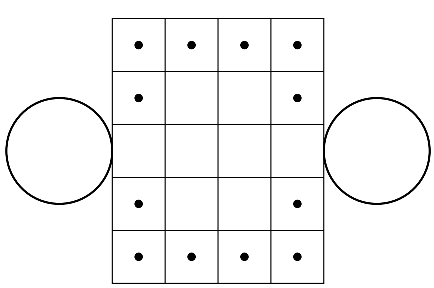
Proof of ii) and iii): If n is odd, then consider a plane of symmetry that bisects a column, and cuts through a separating-line on the opposite side. Let us call the column that the plane bisects plus the cap(s) the 'extended column'. The extended column is therefore of length $h + 1$ for a one capped cylinder and $h + 2$ for the two capped cylinder. The cylinder is symmetric about the plane that bisects the extended column, and moreover the tiles of the setup minus the extended column that are adjacent to any tile i of the extended column exist in symmetric pairs about the plane. Therefore, by Theorem 3.7, if $h+1 \equiv 2 \pmod 3$, i.e $h \equiv 1 \pmod 3$, then the one-capped cylinder is not completely solvable and if $h+2 \equiv 2 \pmod 3$, i.e $h \equiv 0 \pmod 3$, then the two-capped cylinder is not completely solvable. If $h+1 \not\equiv 2 \pmod 3$, then by Theorem 3.10, the solvability of the one capped cylinder is equivalent to the solvability the one-capped cylinder minus the extended column, which is a $(n-1) \times h$ rectangle. If $h+2 \not\equiv 2 \pmod 3$, then by Theorem 3.10, the solvability of the two-capped cylinder is equivalent to the solvability of the two-capped cylinder minus the extended column, which, is a $(n-1) \times h$ rectangle. This proves the proposition. □

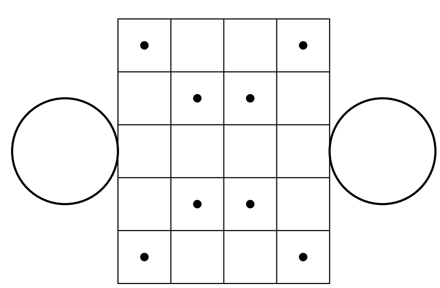
In the proofs of ii) and iii), we introduced the notion of an 'extended column'. One might wonder why this was necessary, since the plane bisects the column and the setup would thus be symmetric about the column, even if we do not add the caps. The reason is that if we did not add caps to the column, it would not be true that given any tile i in the column L, the tiles adjacent to i that are in $S-L$ would exist in symmetric pairs around that tile. For example, if the cylinder has a cap at the top, then the topmost tile of the column is adjacent to a cap tile which is in $S-M$. However, there is no tile adjacent to the column which is a symmetric pair of this cap, since the reflection of the cap about the plane of symmetry is itself.

Example 5.5. Consider a two-capped cylinder of height 4 and circumference 5. Is this cylinder completely solvable?



The diagram on the left is a representation of a capped cylinder of height 4 and circumference 5. A capped cylinder can be formed from the representation by rolling the top and bottom of the rectangle inwards, and afterwards folding the circular caps inwards. The red rectangle represents the plane of symmetry as described in the proof of Proposition 5.4 when n is odd. The extended line of symmetry comprises of every tile that the plane of symmetry bisects, including the caps. Since $4 \not\equiv 0 \pmod 3$, it follows that the solvability of this capped cylinder is equivalent to the solvability of a 4×4 square. The 4×4 square is not completely solvable by Example 3.2, and looking at Example 3.2 we can construct two elements of the kernel, depicted below.





Lights Out on the surface of a torus. A torus is just a cylinder whose ends are connected. In this spirit, we can label the height of the cylinder which is self connected as h, the circumference of this cylinder as n, and call the resulting torus a (n,h) torus. If $n = 1$ or $h = 1$, then the torus just becomes a cylinder of height 1. If $n = 2$ or $h = 2$, then the graph of the torus is the same as the graph of cylinder of height 2. To see this, just construct the associated graphs of the two shapes as we did in Example 5.1. We may now consider $n > 2$, $h > 2$.

Proposition 5.6. *Let $n,h > 2$. If either $n \equiv 0 \pmod 3$ or $h \equiv 0 \pmod 3$, then the torus is not completely solvable. Otherwise, the torus is solvable if and only if a $(n-1)\times(h-1)$ rectangle is solvable.*

Proof. Label the tiles of the torus by (i,j) where $i \in \{1,...,n\}$ gives the circumference position and $j \in \{1,...,h\}$ gives the height position of the cylinder which was bent to make the torus. If $n \equiv 0 \pmod 3$, then, for each j, take

$$(v(1,j), v(2,j), ..., v(n,j)) = (1,1,0,1,1,0,...,1,1,0).$$

Then it is easy to verify that $v \in \ker(M)$. If $h \equiv 0 \pmod 3$, then, for each i, take

$$(v(i,1), v(i,2), ..., v(i,h)) = (1,1,0,1,1,0,....,1,1,0).$$

Once again, it is easy to verify that $v \in \ker(M)$ (Recall Lemma 3.4).

If $n,h \not\equiv 0 \pmod 3$, then consider a plane of symmetry which bisects one of the rungs of the cylinder used to create the torus. The torus is symmetric about this plane. We look for nontrivial elements $\tilde{v} \in \ker(M)$ which are symmetric about this plane, and observe that if we take any tile (i,j) on this rung, all tiles of the setup minus the ring that are adjacent to (i,j) exist in symmetric pairs about the plane (This is the statement that requires $h > 2$). The torus minus the ring is nothing but a cylinder with height $h-1$ and circumference n. Since the length of the rung $n \not\equiv 0 \pmod 3$, the solvability of the torus is equivalent to the solvability of the $(n,h-1)$ cylinder by Theorem 3.10. Since $h \not\equiv 0 \pmod 3$, we have that $h-1 \not\equiv 2 \pmod 3$. Because, $h-1 \not\equiv 2 \pmod 3$ and $n > 2$, we can apply Proposition 5.2 to conclude that the solvability of the torus is equivalent to the solvability of a $(n-1)\times(h-1)$ rectangle. □

Lights Out on the surface of a cone. For the purposes of the Lights Out puzzle, the difference between a cone and a cylinder is the tip. We play Lights Out on the cone with the rule that every tile at the tip of the cone affects every other tile at the tip. Just like a cylinder we can talk of a cone of height h and circumference n. Since the $n = 1$ case is equivalent to a line with length h, and the $n = 2$ case is equivalent to a $2 \times h$ rectangle, we may consider $n > 2$.

Proposition 5.7. *Let $n > 2$. If $h = 1$, the cone is not completely solvable. If $h = 2$, and n is even, the cone is completely solvable. If $h > 2$ and n is even, the cone is completely solvable if and only if $h \not\equiv 1 \pmod 3$ and the $(n-1)\times(h-2)$ rectangle is solvable. If $h \geq 2$ and n is odd, then the cone is completely solvable if and only if $h \not\equiv 2 \pmod 3$ and the $(n-1)\times(h-2)$ rectangle is solvable.*

Denote the setup (the cone) by S. Denote the collection of tiles present at the tip of the cone by T. Denote the rung below the tip by R. The bottom part of the cone that comprises of a $(n,h-2)$ cylinder is thus $S-T-R$. We carry over notation such as M_X from Theorem 3.7. For the sake of description, we orient the cone so that the tip is at the top of the cone and the base is at the bottom. We may therefore say that a tile i is 'directly above' tile j (or j is 'directly below' i) if i and j are adjacent tiles on the same column and i is above j with respect to this orientation.

If $h = 1$, then every tile of the cone affects every other tile in the cone, and since $n > 2$, the cone has more than two tiles. Thus, choose two tiles i_1 and i_2 and define the vector v by $v(i_1) = v(i_2) = 1$, $v(j) = 0$ for all $j \notin \{i_1, i_2\}$. Then, $v \in \ker(M)$. Thus, the cone is not completely solvable. Henceforth, consider $h > 1$.

Proof of Proposition 5.7 for even n: If n is even, then choose a plane of symmetry which cuts through two opposite vertical separating-lines. If there is a nontrivial element $v \in \ker(M)$, then there exists a nontrivial element $\tilde{v} \in \ker(M)$ which is symmetric about this plane. By symmetry about the plane, $\tilde{v}\restriction_T(i) = 1$ for an even number of tiles $i \in T$. Thus, for any tile $i_0 \in T$, $(M_{TS}\tilde{v}\restriction_T)(i_0) = 0$. Taking any $i \in T$, we can now note that since the only adjacent tiles to i are in T or R,

$$(M\tilde{v})(i) = (M_{TS}\tilde{v}\restriction_T)(i) + (M_{RS}\tilde{v}\restriction_R)(i) = 0 + (M_{RS}\tilde{v}\restriction_R)(i) = 0 + \tilde{v}(j),$$

where j is the tile in R located directly below i. Since $\tilde{v} \in \ker(M)$, $(M\tilde{v})(i) = 0$ and so $\tilde{v}(j) = 0$. Since every $j \in R$ is directly below some $i \in T$, it follows that $\tilde{v}\restriction_R = 0$. If $h = 2$, then this means that we must require $\tilde{v}\restriction_T = 0$ also, or else bulbs in R will be lit. Thus, $\tilde{v} = 0$ is the only symmetric element in $\ker(M)$ and the cone with height $h = 2$ is completely solvable. Now let $h > 2$. Because $\tilde{v}\restriction_R = 0$, R effectively isolates the nonzero components of $\tilde{v}$ in T and the nonzero components of $\tilde{v}$ in $S-T-R$. It follows that $\tilde{v}\restriction_{S-T-R} \in \ker(M_{S-T-R})$, since $v \in \ker(M)$ and thus, for any $i \in S-T-R$,

$$\begin{aligned} 0 &= (M\tilde{v})(i) \\ &= (M_{(S-T-R)S}\tilde{v}\restriction_{S-T-R})(i) + (M_{RS}\tilde{v}\restriction_R)(i) \\ &= (M_{(S-T-R)S}\tilde{v}\restriction_{S-T-R})(i) + 0 \\ &= (M_{(S-T-R)S}\tilde{v}\restriction_{S-T-R})(i). \end{aligned}$$

Thus, the $(n, h-2)$ cylinder $S-T-R$ is not completely solvable.

Conversely, if the $(n, h-2)$ cylinder $S-T-R$ is not solvable, then there exists an element $\tilde{w} \in \ker(M_{S-T-R})$ which is symmetric about a plane which passes through two opposite separating-lines. Construct $\tilde{v}$ by requiring $\tilde{v}\restriction_{S-T-R} = \tilde{w}$, $\tilde{v}\restriction_R = 0$, and requiring, for $i \in T$, $\tilde{v}(i) = \tilde{w}(j)$ where $j \in S-T-R$ is the tile in the same column as i and two rungs below i (so if we call the tip the first rung, j is on the third rung). Note that since $\tilde{w}$ is symmetric about the plane which passes through opposite separating-lines, the number of nonzero components of $\tilde{w}$ in the third rung of the cone is an even number and thus, the number of nonzero components of $\tilde{v}$ at the tip is an even number. Thus, for a tile $i \in T$, $(M\tilde{v})(i) = (M_{TS}\tilde{v}\restriction_T)(i) + (M_{RS}\tilde{v}\restriction_R)(i) = 0 + 0 = 0$. For a tile $j \in R$, denote by $i_1 \in T$ the tile directly above j and by $i_2 \in S-T-R$ the tile directly below j. Then, because $\tilde{v}(i_1) = \tilde{v}(i_2)$ and $\tilde{v}\restriction_R = 0$, we have that

$$(M\tilde{v})(j) = (M_{RS}\tilde{v}\restriction_R)(j) + \tilde{v}(i_1) + \tilde{v}(i_2) = 0 + 2\tilde{v}(i_1) = 0.$$

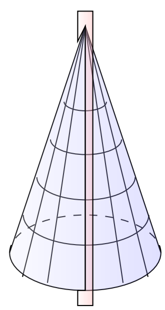
For $j \in S-T-R$,

$$(M\tilde{v})(j) = (M_{RS}\tilde{v}\restriction_R)(j) + (M_{(S-T-R)S}\tilde{v}\restriction_{S-T-R})(j) = 0 + (M_{(S-T-R)S}\tilde{w})(j) = 0$$

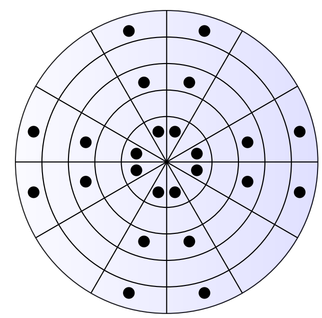
since $w \in \ker(M_{S-T-R})$. Thus, $\tilde{v} \in \ker(M)$ and thus, the (n,h) cone is not completely solvable.

Therefore, for n even, and $h > 2$ the (n,h) cone is completely solvable if and only if the $(n,h-2)$ cylinder is completely solvable. By Proposition 3.1, this will happen if and only if $(h-2) \not\equiv 2 \pmod 3$ and the $(n-1)\times(h-2)$ rectangle is solvable. Since $(h-2) \not\equiv 2 \pmod 3 \Leftrightarrow h \not\equiv 1 \pmod 3$, the result follows for even n. □

Example 5.8. Example for n even: What is the solvability characterization of a cone with circumference 12 and height 5?



This is a cone of height 5 and circumference 12. In particular, the circumference is even. The plane of symmetry (red) cuts through a vertical separating-line and the separating-line opposite it. By Proposition 5.7, since $5 \not\equiv 1 \pmod 3$, the solvability of the cone reduces to the solvability of a 11×3 rectangle. Since $11 \equiv 2 \pmod 3$, and 3 is odd, it follows by Proposition 4.1 that this cone is not completely solvable. The diagram on the right is a top view of the cone, and the arrangement shown is an element of the kernel. That is, if we flip the switches of every tile with a black dot, no bulbs will change state.

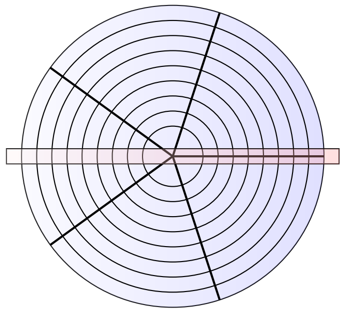


Proof of Proposition 5.7 for odd n: If n is odd, then choose a plane of symmetry that bisects a column L and goes through the separating-line on the opposite side. Given any tile $i \in L$, the tiles in $S-L$ that adjacent to i (if i is the topmost tile of L, this refers to every tile of $S-L$ at the tip) exist in symmetric pairs about the plane. Hence, by Theorem 3.7, if $h \equiv 2 \pmod 3$, the cone is not completely solvable. If $h \not\equiv 2 \pmod 3$, then by Theorem 3.10, it follows that the solvability of the setup is equivalent to the solvability of $S-L$. $S-L$ is symmetric about the very same plane introduced earlier, and moreover, the tiles at the tip of $S-L$ exist in symmetric pairs about the plane. Denote by T_2 the tip of $S-L$, and by R_2 the rung adjacent to T_2 that is below T_2.

Suppose that $S-L$ is not completely solvable. Then, there exists a nontrivial $v \in \ker(M_{S-L})$ which is symmetric about the plane. Since the tiles of T_2 exist in symmetric pairs about the plane, it follows that the number of tiles $i \in T_2$ such that $v(i) = 1$ is an even number. Thus, $(M_{T_2 S} v\restriction_{T_2})(i) = 0$ for all $i \in T_2$. As we argued in the proof for even n, this implies that $v\restriction_{R_2} = 0$, and $v\restriction_{(S-L)-T_2-R_2} \in \ker(M_{(S-L)-T_2-R_2})$. Since $(S-L)-T_2-R_2$ is a $(h-2)\times(n-1)$ rectangle, this implies that the $(h-2)\times(n-1)$ rectangle is not completely solvable.

Conversely, suppose the $(h-2)\times(n-1)$ rectangle $(S-L)-T_2-R_2$ is not completely solvable. Then, there exists a nontrivial element $w \in \ker(M_{(S-L)-T_2-R_2})$ that is symmetric about the plane cutting through $S-L$. Define v by the following: $v\restriction_{(S-L)-T_2-R_2} = w$, $v\restriction_{R_2} = 0$, and for a tile $i \in T_2$, $v(i) = w(j)$ where j is the tile on the same column as i and two rungs lower. By similar reasoning as in the proof for even n, $v \in \ker(M_{(S-L)})$. Thus, $S-L$ is not completely solvable. Thus, we have shown that if $h \not\equiv 2 \pmod 3$, the solvability of the cone S is equivalent to the solvability of $S-L$ which is equivalent to the solvability of a $(h-2)\times(n-1)$ rectangle. This completes the proof. □

Example 5.9. Example for n odd: What is the solvability characterization of a cone with circumference 5 and height 9?



This is top-view of a cone of height 9 and circumference 5. In particular, the circumference is odd. The plane of symmetry (red) cuts through a vertical line and the column of tiles opposite it. By Proposition 5.7, since $9 \not\equiv 2 \pmod 3$, the solvability of the cone reduces to the solvability of a 4×7 rectangle. Since $4 \not\equiv 2 \pmod 3$, it follows by Proposition 4.2 that the solvability is equivalent to a 4×3 rectangle which is (again applying Proposition 4.2) equivalent to the solvability of a 4×1 rectangle. Since $4 \not\equiv 2 \pmod 3$, we can then conclude by Lemma 3.4 that the 4×1 rectangle is completely solvable and thus the cone in question is completely solvable.

Lights Out on the surface of a capped cone. The cap here has similar properties to the cap of a cylinder. That is, if we orient the cone such that the tip points upwards, then the cap is a single tile placed over the base of the cone, which is adjacent to every tile at the bottom-most rung of the cone. Let h be the height of the cone without the cap and n the number of tiles making up the circumference of the cone.

Proposition 5.10. *If n is even, the solvability conditions of the capped and regular cone are the same. If n is odd, then the capped cone is completely solvable if and only if $h \not\equiv 1 \pmod 3$ and the $(n-1) \times (h-2)$ rectangle is completely solvable.*

Proof. Similar arguments to those given to categorize the capped cylinder and those given to categorize the regular cone can be adapted to prove this. Details are left to the reader. □

Lights Out on the surface of a sphere. We divide the sphere by introducing latitudes and longitudes, and play Lights Out on the resulting grid. At the poles, the rule is the same as that we used for a cone - every switch at the pole affects every light at the pole. We denote the number of tiles along the latitude by n and the number of tiles from pole to pole by h. Once again, if $n = 1$, then the sphere is equivalent to a line of length h and if $n = 2$, then the sphere is equivalent to a $2 \times h$ rectangle. Thus, we can take $n > 2$.

Proposition 5.11. *Let $n > 2$. If $h = 2$, the sphere is completely solvable if and only if n is even. If $h = 3$, the sphere is not completely solvable. If $h = 4$, the sphere is always completely solvable. If $h > 4$, and n is even, the sphere is completely solvable if and only if $h \not\equiv 0 \pmod 3$ and the $(n-1)\times(h-4)$ rectangle is solvable. If $h > 4$ and n is odd, then the sphere is solvable if and only if $h \not\equiv 2 \pmod 3$ and the $(n-1) \times (h-4)$ rectangle is solvable.*

Proof. If $h = 2$, and n is odd, then, set $v = 1$ (flip all the switches). Then, one can verify that $v \in \ker(M)$, so the geometrical setup is not completely solvable. If $h = 3$, then, since $n > 2$, choose any two tiles at the north pole and flip those two switches. Flip also the switches of the identically placed tiles in the south pole. Do not flip any tiles in the middle. Then, if we denote this configuration of flips by v, then one can verify that $v \in \ker(M)$ so the geometrical setup is not completely solvable. For a general sphere, denote the sphere by

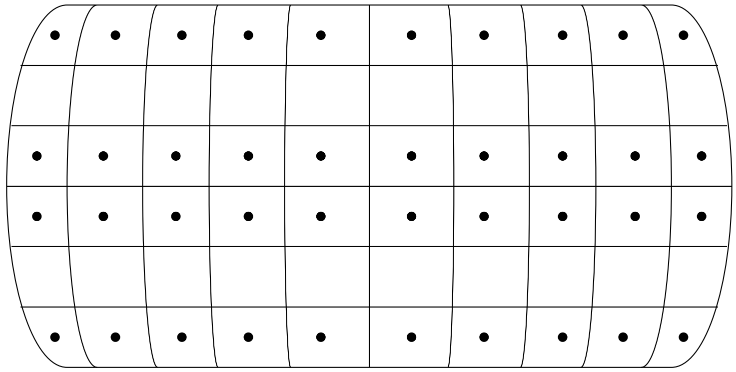
S, the poles by P_1 and P_2. An arbitrary pole may be labelled P. Denote by R_1 and R_2 the rungs of the sphere which are adjacent to P_1 and P_2 respectively and denote an arbitrary rung by R.

Let n be even. Then, arguing as we did for the case of a cone, we take a plane of symmetry which goes through two opposite separating-lines, and look for nontrivial elements $v \in \ker(M)$ which are symmetric about this plane. Because v is symmetric about the plane, at each pole, the number of tiles $i \in P$ such that $v(i) = 1$ must be even. If $h = 2$, this means that we cannot flip any switches of either pole, so the sphere is completely solvable. For $h > 2$, by an argument identical to that presented in the case of a cone, the even number of flips at the poles implies that we require $v\restriction_{R_1} = v\restriction_{R_2} = 0$. If $h = 4$, this means that we cannot flip switches on the second and third rungs, and to ensure the bulbs on these rungs remain off, we cannot flip any switches on the poles either. Thus, the sphere is completely solvable for $h = 4$. Now assume $h > 4$. Then, arguing as we did in the case of a cone, the solvability of a sphere is equivalent to the solvability of the remaining portion, which is a $(n, h-4)$ cylinder, and therefore, we have that the sphere is completely solvable if and only if $h - 4 \not\equiv 2 \pmod 3$ and the $(n-1) \times (h-4)$ rectangle is solvable. That is, for $h > 4$, the sphere is completely solvable if and only if $h \not\equiv 0 \pmod 3$ and the $(n-1)\times(h-4)$ rectangle is completely solvable.

Let n be odd. Take a plane of symmetry that bisects a column L and cuts the separating-line on the opposite side. The sphere is symmetric about this plane and moreover, if we take any tile $i \in L$, the tiles of $S - L$ that are adjacent to i exist in symmetric pairs about the plane. Thus, if $h \equiv 2 \pmod 3$, the sphere is not completely solvable by Theorem 3.7. If $h \not\equiv 2 \pmod 3$, by Theorem 3.10, the solvability of the sphere S is equivalent to the solvability of $S - L$. $S - L$ is symmetric about the plane we described earlier. For $h = 4$, we search for an element $v \in \ker(M)$ which is symmetric about this plane. By arguments presented in Proposition 5.7, the number of tiles i at a given pole such that $v(i) = 1$ is an even number, and thus $v\restriction_{R_1+R_2} = 0$. Thus, to ensure that $(Mv)(i) = 0$ for every $i \in R_k$ $(k \in \{1, 2\})$, we require that $v\restriction_{P_1+P_2} = 0$. Thus, $v = 0$ is the only element of the kernel that symmetric about the plane, and the sphere of height 4 is therefore completely solvable. Now let $h > 4$. Using similar arguments as we did to prove Proposition 5.7 for odd n, we can find that the solvability of $S - L$ is equivalent to the solvability of a $(n-1) \times (h-4)$ rectangle, and the proposition follows for odd n. □

Example 5.12. Lights Out on the surface of the earth. By convention, there are five primary latitude lines, and let us choose to draw 10 longitudinal lines. If we draw these lines on the surface of the earth, is the resulting grid completely solvable?

The latitudes divide earth into six portions, so the height $h = 6$, and the longitudes divide the earth into 10 portions, so the circumference is 10. Since 10 is even, and $h \equiv 0 \pmod 3$, we have that the sphere is not completely solvable. An element of the kernel can be seen below. In the schematic drawing presented, it is understood that every tile at the top (resp. bottom) row affects every other tile in the top (bottom) row, since these rows represent the circles of tiles around the poles. It is also understood naturally that the tiles on opposite sides of the same row affect each other.



Lights Out on the surface of a diamond. A (n) pyramid is a pyramid formed by considering a line of length n, a second line of length $(n-2)$ symmetrically placed on top of this line, a third line of length $(n-4)$ symmetrically placed on top of the previous two and so on. A diamond is a geometrical arrangement that is constructed from two pyramids as follows: If n is even, then the diamond consists of two (n) pyramids placed back to back, and if n is odd, then the diamond consists of two (n) pyramids facing opposite directions, with their bases glued together. A (10) diamond and a (7) diamond are illustrated in Figure 3.

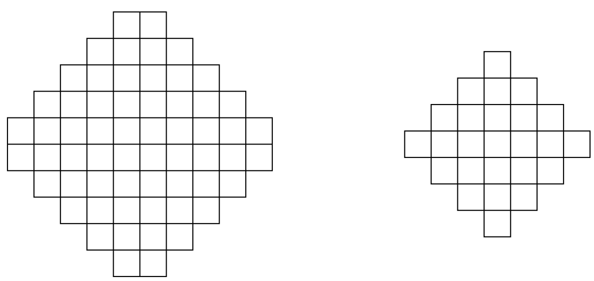


FIGURE 3. Illustration of a (10) diamond (left) and a (7) diamond (right)

Further, it is important to note that an (n) pyramid is completely solvable if and only if a $(n+2)\times(n+2)$ square is completely solvable. This can be see by considering the $(n+2)\times(n+2)$ square and searching for an element v of the kernel which is symmetric about planes passing through the diagonals, as was done in Example 3.12.

Proposition 5.13.

(i) *A* (2) *diamond is completely solvable; a* (4) *diamond is not completely solvable.*

(ii) *If n is even, $n>4$, and $\frac{n}{2}$ is odd, then the solvability conditions of an* (n) *diamond and an* $(\frac{n-2}{2})$ *diamond are the same.*

(iii) *If n is even, $n>4$, and $\frac{n}{2}$ is even, then the* (n) *diamond is completely solvable if and only if* $n \not\equiv 1 \pmod 3$ *and the $\frac{n}{2}\times\frac{n}{2}$ square is completely solvable.*

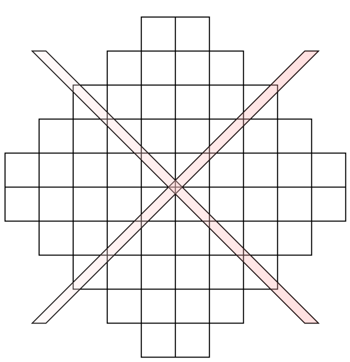
(iv) *If n is odd, then the* (n) *diamond is completely solvable if and only if* $n \not\equiv 2 \pmod 3$ *and the $n\times n$ square is completely solvable.*

Proof. Denote the setup (diamond) by S. A (2) diamond is just a 2×2 square, and is thus completely solvable. On the (4) diamond, define $v(i)=0$ if i is a tile on one of the diagonals, and define $v(i) = 1$ if i is not on a diagonal. Then, $v \in \ker(M)$ so the (4) diamond is not completely solvable, thus proving (i). If n is even, and $\frac{n}{2}$ is odd, then we notice that the diagonals separate the diamond into four smaller diamonds of size $\frac{n-2}{2}$. Since the diamond is symmetric about an axis that passes through a diagonal L, and the tiles of $S - L$ that are adjacent to any tile $i \in L$ exist in symmetric pairs about the axis, we can then apply Corollary 3.11 twice in succession and find that the solvability of the (n) diamond is equivalent to the solvability of the $\frac{n-2}{2}$ diamond, thus proving (ii). If $\frac{n}{2}$ is even, then the diagonals separate the diamond into four smaller arrangements, which consist of two $(\frac{n}{2} - 2)$ pyramids separated by a column of length $\frac{n}{2}$. By Corollary 3.11, the solvability of the (n) diamond is equivalent to the solvability of the smaller arrangements. If $\frac{n}{2} \equiv 2 \pmod 3$ (i.e $n \equiv 1 \pmod 3$), then each arrangement is not completely solvable by Theorem 3.7, and so the n diamond is not completely solvable. If $n \not\equiv 2 \pmod 3$, then, by Theorem 3.10 the solvability of the smaller arrangements is equivalent to the solvability of an $\frac{n}{2} - 2$ pyramid, which is equivalent to the solvability of an $\frac{n}{2} \times \frac{n}{2}$ square, which proves (iii).

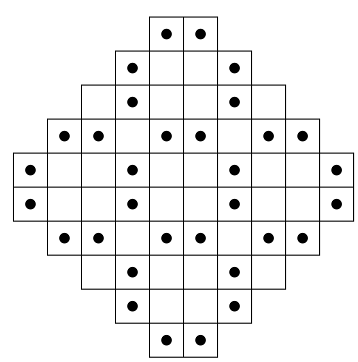
If n is odd, then we consider the horizontal axis of symmetry which bisects a column L of length n. The tiles in $S - L$ that are adjacent to any tile $i \in L$ exist in symmetric pairs about the axis. Therefore, If $n \equiv 2 \pmod 3$, the (n) diamond is not completely solvable by Theorem 3.7. If $n \not\equiv 2 \pmod 3$, then by Theorem 3.10, the solvability of diamond is equivalent to the solvability of a $(n - 2)$ pyramid, which is equivalent to the solvability of a $n \times n$ square, which proves (iv). □

Since the proposition is relatively straightforward for odd numbers, we will illustrate a couple of examples highlighting solutions for even number diamonds.

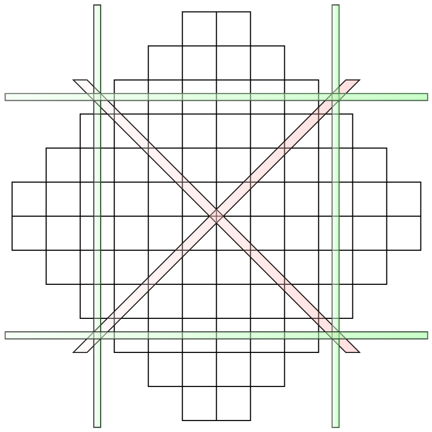
Example 5.14. Is the (10) diamond completely solvable?



The red axes of symmetry are perpendicular diagonal axes. Since 10 is even, and $\frac{10}{2}$ is odd, we apply Proposition 5.13 to determine that the solvability of a (10) diamond is equivalent to the solvability of a (4) diamond, and thus the (10) diamond is not completely solvable. An element of the kernel is illustrated to the right.



Example 5.15. Is the (12) diamond completely solvable?



12 is even and $\frac{12}{2} = 6$ is even. Applying Proposition 5.13, since $12 \not\equiv 1 \pmod 3$, we have that the solvability of a (12) diamond is equivalent to the solvability of a 6×6 square. By Example 3.12, the 6×6 square is completely solvable. Thus, the (12) diamond is completely solvable. In the proof of Proposition 5.13 iii) which addresses diamonds with similar properties to the (12) diamond, the red axes of symmetry are applied first, and then the green axes of symmetry are applied to the four separated smaller arrangements.

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Student biographies

Maithreya Sitaraman: (*Corresponding author:* maithreya@uchicago.edu) Maithreya Sitaraman is currently a third year undergraduate mathematics student at the University of Chicago. He plans to pursue a doctorate in mathematics, perhaps giving special attention to algebraic and topological questions. He is interested in pursuing a career as an academic.