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Shiv Priyam Raghuraman

University of Minnesota

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Abstract. We incorporate Milankovitch cycles into a recent conceptual climate model of the glacial-interglacial cycles. Investigated are the effects of orbital forcing elements such as variations in the eccentricity of the Earth's orbit and variations in the obliquity of the Earth's axis, on the expansion and retreat of stable ice sheets. The results showed that in a colder climate, variations in obliquity dominate the evolution of the ice sheet, whereas in a warmer climate, variations in eccentricity and obliquity both matter in the evolution of the ice sheet. The resulting simulations exhibited glacial cycles and also exhibited the skipped obliquity signal phenomenon.

1. Introduction

Over the last one million years, massive ice sheets across North America have periodically formed and melted during glacial periods. Milankovitch hypothesized that variations in the Earth's orbital parameters, such as obliquity of the Earth's spin axis (i.e., the tilt of the Earth's axis), eccentricity of the Earth's orbit, and the precession of the Earth as it rotates (i.e., the amount it "wobbles"), pace the glacial cycles. [11] However, orbital forcing cannot be the only reason behind the pacing of the glacial cycles, [4] since the Milankovitch cycles have been constant over the last 5 million years. Thus, there must be nonlinear feedbacks inherent to Earth's climate system. [16] One such feedback is the ice-albedo feedback, which is modeled as a dynamical system [9]. Ice-albedo feedback is a positive feedback climate process where a change in the area of snow-covered land, ice caps, glaciers or sea ice alters the albedo, i.e., the ratio of reflected radiation to the incident radiation. Budyko was interested in how ice-albedo feedback affects climate and in his 1969 paper [3], he introduces the conceptual energy balance model (EBM).

We look at Budyko's model which studies the average annual temperatures in latitudinal zones. A key feature of Budyko's model is that it assumes that the Earth has an ice cap, with the requirement that above a particular latitude $y = \eta$ there is always ice and below the latitude $y = \eta$ there's no ice. The ice line is then defined to be the edge of the ice sheet η. However, Budyko's model does not permit the ice line η to respond to changes in temperature. This drawback was solved by Widiasih in [18] where an ODE modeling the evolution of η was added. McGehee and Widiasih introduced a quadratic approximation to reduce this infinite dimensional system to a pair of ordinary differential equations. However, since η approaches either a small ice cap or the equator over time, it doesn't take into account the relative sizes of the accumulation and ablation (melting) zones in glacial advance and retreat and hence does not permit glacial cycles. Walsh et al. in [16], rectified this by adding a variable called the snow line, which was independent of the ice line.

The goal of this paper is to incorporate Milankovitch cycles into Walsh et al.'s conceptual climate model as a numerical simulation. We aim to understand which orbital forcing factor contributes more to the expansion and retreat of the ice sheet under different conditions. In the following section we introduce Budyko's EBM and Widiasih's ice sheet evolution equation. In Section 3 we draw attention to McGehee and Widiasih's work of reducing the infinite dimensional system to a pair of ordinary differential equations. In Section 4 we describe how Walsh et al. added a snow line to McGehee's and Widiasih's model. In the penultimate section, we include Milankovitch cycles into Walsh et al.'s model and discuss the simulations and note that they exhibit the "skipped obliquity" phenomenon, an idea first put forth by Huybers in [6] as a conceptual model with a physical interpretation of how the obliquity signal skips in the climate record. In the final section we summarize our findings and conclude that variations in obliquity of Earth's orbit affect the glacial cycles predominantly but variations in the eccentricity of Earth's orbit plays a role as well in a warmer climate.

2. Temperature-Ice line model

Consider Budyko's time-dependent equation [14] [5]:

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) - C\left(T-\overline{T}\right), \tag{1}$$

This equation represents the change in energy stored in the Earth's surface at $y \in [0,1]$, where y is the sine of the latitude with $y = 0$ the equator and $y = 1$ the north pole since Budyko's EBM assumes the Earth to be symmetric about the equator. The function $T = T(y,t)$(°C) is the annual average surface temperature on the circle of latitude y. The units of each side of (1) are Watts per meter squared $\left(\frac{\text{W}}{\text{m}^2}\right)$. The quantity R is the specific heat capacity of the Earth's surface, measured in units of $\frac{\text{J}}{\text{m}^2\,{}^\circ\text{C}}$. The average annual incoming solar radiation (also known as insolation), a parameter which depends on the eccentricity of Earth's orbit [9] is represented by Q. The distribution function $s(y)$ depends upon the obliquity of Earth's orbit [9], which describes the distribution of insolation across a latitude, and satisfies $\int_0^1 s(y)\mathrm{d}y = 1$. The albedo function α_y denotes the albedo at y, which as described earlier, measures the extent to which insolation is reflected back into space. Thus, the first term on the right hand side of (1) represents the energy absorbed at latitude y on the surface from the sun.

The energy reradiated into space at longer wavelengths is approximated linearly by the term $A+BT$. Before the heat escapes into space, some of it is absorbed by greenhouse gases and returned to the surface. Thus, this reradiation term is the net loss of energy from the surface to space. The energy transported from warmer latitudes to cooler latitudes is approximated by the term $C(\overline{T} - T)$ where $\overline{T}$ is the global annual average surface temperature and satisfies $\overline{T} = \int_0^1 T(y,t)\mathrm{d}y$. The positive constants $A, B,$ and C are found empirically through satellite data. [10]

The equilibrium temperature profiles are found to be: [10]

$$T^*(y) = \frac{1}{B+C}\Big(Qs(y)(1-\alpha(y)) - A + \frac{C}{B}(Q(1-\overline{\alpha}-A)\Big), \tag{2}$$

where

$$\overline{\alpha} = \int_0^1 \alpha(y)s(y)\mathrm{d}y \tag{3}$$

and

$$s(y) = s_0p_0(y) + s_2p_2(y), s_0 = 1, s_2 = -0.482. \tag{4}$$

The terms $p_0(y) = 1$ and $p_2(y) = \frac{1}{2}(3y^2 - 1)$ are the first two even Legendre polynomials. Equation (4) is within 2% of the true $s(y)$ values [12]. The albedo function is written as:

$$\alpha_\eta(y) = \begin{cases} \alpha_1, & \text{if } y < \eta \\ \alpha_2, & \text{if } y > \eta, \\ \alpha_0 = \frac{\alpha_1+\alpha_2}{2}, & \text{if } y = \eta, \end{cases} \tag{5}$$

where $\alpha_1 < \alpha_2$ and α_1 denotes the albedo of the surface having no ice and α_2 denotes the albedo of the surface having ice. Using (4) and (5), the equilibrium temperature profiles (2) are even, piecewise quadratic functions having a discontinuity at η. Note that η parametrizes (3) and hence (2), so $T^*(y)$ is written as $T^*_\eta(y)$. Therefore, for each value of η there are infinitely many equilibrium temperature functions. McGehee and Widiasih define $T^*(\eta)$ as $T^*(\eta) = \frac{\lim_{y\to\eta^-} T^*(y) + \lim_{y\to\eta^+} T^*(y)}{2}$ and the equilibrium temperature at the ice line as

$$T^*_\eta(\eta) = \frac{1}{B+C}\Big(Qs(\eta)(1-\alpha_0) - A + \frac{C}{B}(Q(1-\overline{\alpha}-A)\Big), \tag{6}$$

where $\alpha_0 = \frac{\alpha_1+\alpha_2}{2}$. Budyko's EBM however doesn't allow the ice line η to respond to temperature changes. This disadvantage was resolved by Widiasih in [18] where an ordinary differential equation modeling the evolution of η was added, giving the following system:

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C\left(T - \overline{T}\right) \tag{7a}$$

$$\frac{d\eta}{dt} = \rho(T(\eta,t) - T_c), \tag{7b}$$

where ρ is a parameter which controls the relaxation time of the ice sheet and T_c is a critical temperature above which ice melts and below which ice forms. The above system describes how the temperature distribution $T(y,t)$ evolves according to Budyko's equation (7a) and the evolution of η in (7b). If $T(\eta,t) > T_c$, the ice sheets melt and retreat toward the pole. If $T(\eta,t) < T_c$, the ice sheets expand and move toward the equator. [16]

3. Approximation to finite dimensional system

As mentioned earlier, the equilibrium solutions of Budyko's equation (7a) are even and piecewise quadratic, with a discontinuity at η when using (4) and (5). This prompted McGehee and Widiasih to introduce a quadratic approximation to the infinite dimensional system (7) and we note the final result below while the reader can find all details in [10]. Thus, the infinite dimensional system (7) is approximated by the system of ODEs

$$\dot{w} = -\tau(w - F(\eta)) \tag{8a}$$

$$\dot{\eta} = \rho(w - G(\eta)), \tag{8b}$$

where $F(\eta), G(\eta)$ are given below in (9) and (10) respectively, $\tau = \frac{B}{R}$ and ρ are time constants. One can prove that for fixed η the variable w is a translate of the global average temperature. In [10], it is proven that there exists a stable equilibrium point with a small ice cap, and a saddle equilibrium point with a large ice cap, for all $\rho > 0$, for standard parameter values (See Table 1). [16] Note that $F(\eta)$ is a cubic polynomial because $P_2(\eta) = \frac{\eta^3-\eta}{2}$,

$$F(\eta) = \frac{Q}{B}\left((1-\alpha_0)A + \frac{C}{B+C}(\alpha_2 - \alpha_1)\left(\eta - \frac{1}{2} + s_2 P_2(\eta)\right)\right), \tag{9}$$

and $G(\eta)$ is a quadratic polynomial because $p_2(\eta) = \frac{3\eta^2-1}{2}$,

$$G(\eta) = -\frac{Q}{B+C}s_2(1-\alpha_0)p_2(\eta) + T_c. \tag{10}$$

The constants $A, B, C, Q, \alpha_0, \alpha_1, \alpha_2, s_2$, and T_c are given in Table 1. But, as η approaches either the equator or a small ice cap over time, it does not take into account the relative sizes of the ablation (melting) and accumulation zones when ice sheets advance and retreat and hence doesn't permit glacial cycles. In [16], this was resolved by adding a variable called the snow line independent of the ice line (which will now become the albedo line), which is presented below.

4. Snow line addition

The accumulation and ablation of ice play a fundamental role in the theory of glacial cycles, serving to control the terminus advance and retreat, the ice volume, and the geometry of the surface of the ice sheet [2]. Abe-Ouchi et al in [1] found the fast retreat of the ice sheet was due to significantly enhanced ablation, i.e., the ablation rate for a large, advancing ice sheet was necessarily much smaller than the ablation rate for a retreating ice sheet, in order to faithfully reproduce the last four glacial cycles. [16] This simple idea was incorporated into Walsh et al.'s model below (Systems (11) and (12)). The model (Systems (11) and (12)) vector field has a line of discontinuity that produces a switch to the alternate regime. Walsh et al.'s model below will share some similarities with [17], in the way that it "flip-flops."

Considering system (8) again, independent snow and ice lines are introduced to incorporate accumulation and ablation zones. Walsh et al. begin by recasting the role played by η, interpreting η henceforth as the snow line. We denote by ξ the (more slowly moving) ice line, i.e., the edge of the ice sheet (see Figure 1 below). The ablation zone has extent $\eta - \xi$ (when $\eta > \xi$), while the accumulation zone has size $1 - \eta$. [16] The temperature-ice

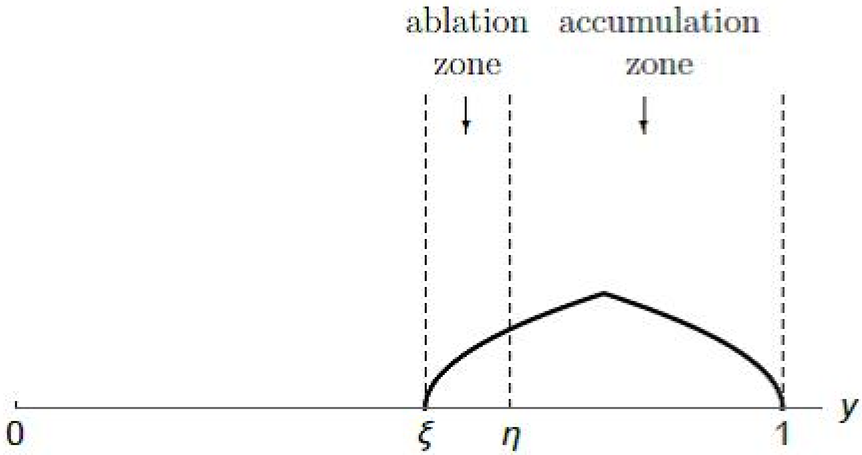


Figure 1. Walsh et al. model set-up. η is the snow line and ξ is the ice line. For the sake of illustration, the shape presented represents a glacier. Taken from Walsh et al [16], reproduced with permission from the authors.

line-snow line model (Systems (11) and (12)) is a non smooth system with state space [16]

$$\mathcal{B} = \{(w, \eta, \xi) : w \in \mathbb{R}, \eta \in [0,1], \xi \in [0,1]\}$$

defined as follows. Pick parameters $b_0 < b < b_1$ representing ablation rates, and a parameter a denoting the accumulation rate. When $b(\eta - \xi) - a(1 - \eta) < 0$, so that accumulation exceeds ablation and the ice sheet advances [16], *set*

$$\dot{w} = -\tau(w - F(\eta)) \tag{11a}$$

$$\dot{\eta} = \rho(w - G_{-}(\eta)) \tag{11b}$$

$$\dot{\xi} = \epsilon(b_0(\eta - \xi) - a(1 - \eta)). \tag{11c}$$

The function $F(\eta)$ in (16a) is given by (9), $G_{-}(\eta)$ in (11b) is given by (10) but with $T_c = T_c^{-} = -5°\text{C}$, and $\epsilon > 0$ is a time constant for the movement of the ice line.

When ablation exceeds accumulation $(b(\eta - \xi) - a(1 - \eta) > 0$ and the ice sheet retreats, *set*

$$\dot{w} = -\tau(w - F(\eta)) \tag{12a}$$

$$\dot{\eta} = \rho(w - G_{+}(\eta)) \tag{12b}$$

$$\dot{\xi} = \epsilon(b_1(\eta - \xi) - a(1 - \eta)). \tag{12c}$$

where $F(\eta)$ in (12a) is given by (9), $G_{+}(\eta)$ in (12b) is given by (10) but with $T_c = T_c^{+} = -10°\text{C}$.

The relative sizes of ablation rates b_0 and b_1 were motivated by [1]. The choice of different T_c-values is motivated by [15], in which a linear interpolation between $T_c = -13°\text{C}$ and $T_c = -3°\text{C}$ is introduced to model changes in deep ocean temperature. The idea behind it is that a large advancing sheet implies a colder world overall, so that less energy is required to form ice (and vice versa for a retreating ice sheet).

Walsh et al. thus arrived at a 3-dimensional system having a plane of discontinuity

$$\Sigma = \{(w,\eta,\xi) : b(\eta-\xi) - a(1-\eta) = 0\} = \left\{(w,\eta,\xi) : \xi = \left(1+\frac{a}{b}\right)\eta - \frac{a}{b}\right\}. \tag{13}$$

A trajectory in (w,η,ξ)-space passing through Σ switches from advancing mode to glacial retreat, or vice versa [16], similar in spirit to the flip-flop model in [17].

5. Milankovitch forcing

5.1. **Equilibrium Solutions.** The temperature-ice line model (System (8)) is now forced with Milankovitch cycles, i.e., we make important parameters such as Q and s_2 depend on eccentricity of the Earth's orbit and obliquity of Earth's axis respectively. We would like to emphasize here that both these orbital forcing elements are time dependent [8]. The work of Laskar [8], was instrumental in incorporating these elements in our numerical simulation below. McGehee and Lehman in [9] showed that insolation, Q is a function of e, the eccentricity of the Earth's orbit, given by

$$Q = Q(e) = \frac{Q_0}{\sqrt{1-e^2}}, \tag{14}$$

where Q_0 is the insolation assuming the eccentricity of Earth's orbit is 0 (See Table 1 for value). McGehee and Widiasih in [10] proved that the function s_2 in (4) actually depends on β, the obliquity of Earth's axis, given by

$$s_2 = s_2(\beta) = \frac{5}{16}(-2+3\sin^2\beta). \tag{15}$$

Recall that $F(\eta)$ and $G(\eta)$ are functions of Q and s_2, which makes $F(\eta)$ and $G(\eta)$ functions of e and β:

$$F(\eta) = \frac{Q_0}{B(\sqrt{1-e^2})}\left((1-\alpha_0)A + \frac{C}{B+C}(\alpha_2-\alpha_1)\left(\eta - \frac{1}{2} + \frac{5}{16}(-2+3\sin^2\beta)\cdot\frac{\eta^3-\eta}{2}\right)\right), \tag{16}$$

$$G(\eta) = -\frac{Q_0}{(B+C)\sqrt{1-e^2}}\frac{5}{16}(-2+3\sin^2\beta)(1-\alpha_0)\cdot\frac{3\eta^2-1}{2} + T_c. \tag{17}$$

Note that the accumulation and ablation parameters a, b, b_0, b_1 are dimensionless con-

Parameter	Value	Units
Q_0	343	W m^{-2}
A	202	W m^{-2}
B	1.9	W m^{-2}(°C)$^{-1}$
C	3.04	W m^{-2}(°C)$^{-1}$
α_1	0.32	dimensionless
α_2	0.62	dimensionless
α_0	0.47	dimensionless
T_c^+	−10	°C
T_c^-	−5.5	°C
s_2	−0.482	dimensionless

Table 1. Parameter values (taken from [16])

stants, τ, ϵ have units (seconds)$^{-1}$, and ρ has units (seconds)$^{-1}$(°C)$^{-1}$. Consider the system of ODEs in (8) again:

$$\dot{w} = -\tau(w - F(\eta))$$
$$\dot{\eta} = \rho(w - G(\eta)).$$

In order to obtain the equilibrium solutions, we set the derivatives equal to 0. That is,

$$w = F(\eta) = G(\eta) \Rightarrow F(\eta) - G(\eta) = 0.$$

Recall that $F(\eta)$ is a cubic polynomial in η due to $P_2(\eta) = \frac{\eta^3-\eta}{2}$ and $G(\eta)$ is a quadratic polynomial in η due to $p_2(\eta) = \frac{3\eta^2-1}{2}$. We wrote a MATLAB code to solve the cubic equation $F(\eta) - G(\eta) = 0$ using a numeric solver and out of the three roots found for every kYr, one is discarded since it does not belong to the range $[0,1]$. These roots are the snow line values that are dependent on both e and β since $F(\eta)$ and $G(\eta)$ are dependent on these orbital elements. We aim to find out which factor η_{sink} is more dependent on because it signifies the extent of the stable ice sheet.

Using MATLAB we plot the stable and unstable snow lines along side the eccentricity and obliquity curves (which are due to Laskar [8]) over the last million years. Using the eyeball metric, we see in Figure 2 that for $T_c = -10$°C, which has a small stable (sink) ice cap near the pole, η, the snow line, is varying in sync with the obliquity curve. The snow line (sink) curve and the obliquity curve have the same amplitude and to compute the period, we look at the range -700 kYr to -600 kYr. The η_{sink} curve has 2.5 cycles in the period, implying that one cycle has a 40,000 year period, which coincides with the obliquity cycle period as seen in Figure 2. Variations in the obliquity signal driving the ice sheet formation and melting was also predicted by McGehee and Lehman [9] and so by forcing a system (8) with Earth's orbital elements, this figure supplements their findings.

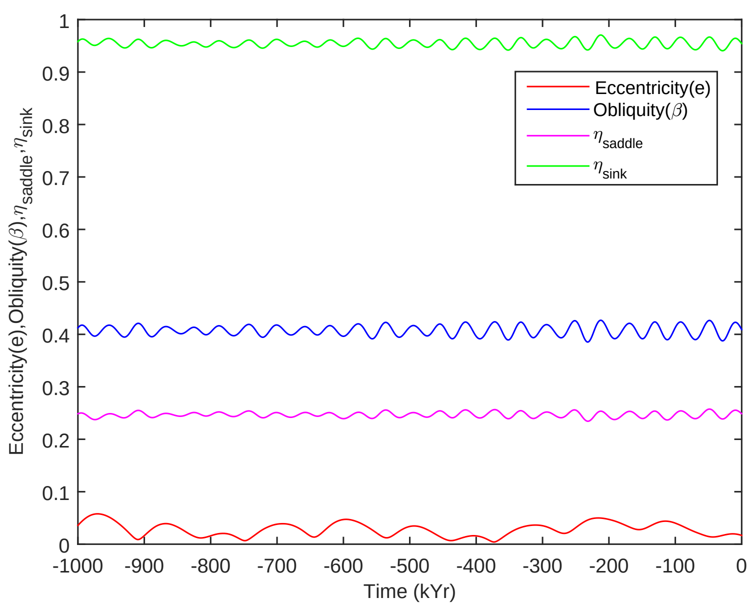


Figure 2. Ice lines affected by Milankovitch cycles when $T_c = -10$°C. Units of the vertical axis are dimensionless.

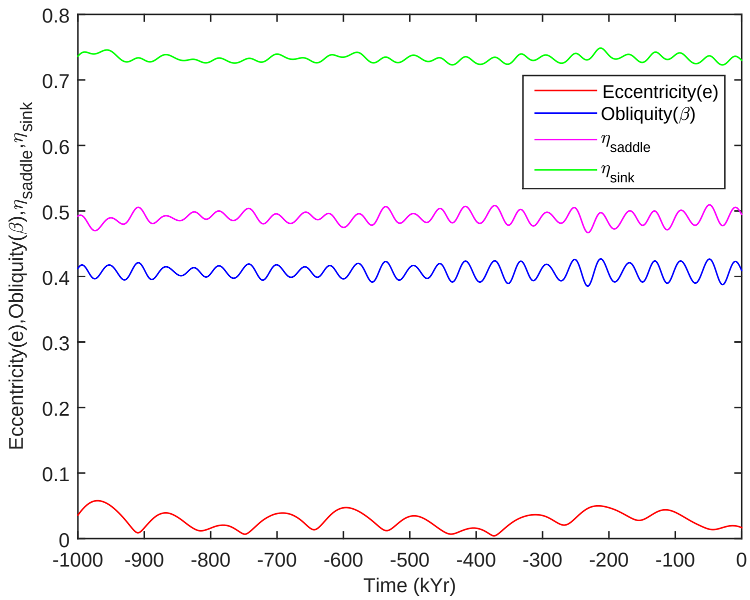


Figure 3. Ice lines affected by Milankovitch cycles when $T_c = -5.5°C$. Units of the vertical axis are dimensionless.

Using the eyeball metric again in Figure 3, we observe that the stable (sink) ice cap, does not vary only with the β signal. Instead, it depends on both e and β. Although the amplitude of the η_{sink} signal is similar to the obliquity signal, the dependence on e can be seen by comparing the periods of the eccentricity and η_{sink} signals. In the η_{sink} signal, one cycle can be measured from the trough at 650 kYr to the trough at 550 kYr. This period of 100,000 years matches the period of the eccentricity signal as seen in the same figure. The unstable (saddle) large ice cap below varies closely with β. Given that these observations were done by eyeballing the figures, we did a power spectrum analysis to confirm the results.

We computed the power spectrum using the Fast Fourier Transform algorithm in MATLAB in order to claim that the stable snow line is forced by eccentricity or obliquity. The power spectrum aims to tell us how much of the signal is at a particular frequency. In Figure 4 we notice that the η_{sink} and obliquity signals follow the same shape. We note the spikes at frequencies around 0.019 and 0.025 of the stable snow line, which also correspond to the obliquity signal spikes. The ice cap movement follows the obliquity cycle with a 40,000 year period ($\frac{1\times1000}{0.025} = 40,000$). This further confirms that the stable ice line follows obliquity cycle as also seen in Figure 1. In Figure 5, we note that the power signal for η_{sink} is mimicking the eccentricity signal and not just the obliquity signal. The spikes of the stable snow line demonstrate the effect of the eccentricity. We see that the snow line follows the eccentricity cycles of period 100,000 years. In Figure 3 we could only hypothesize that the ice cap movement is also determined by the eccentricity signal, but we have confirmed from Figure 5 that this is the case. Now that we have explored in depth which orbital forcing is more dominant in different temperature settings in the static case, i.e., without producing glacial cycles, we now focus on the dynamics of Walsh et al.'s model when we include Milankovitch forcing.

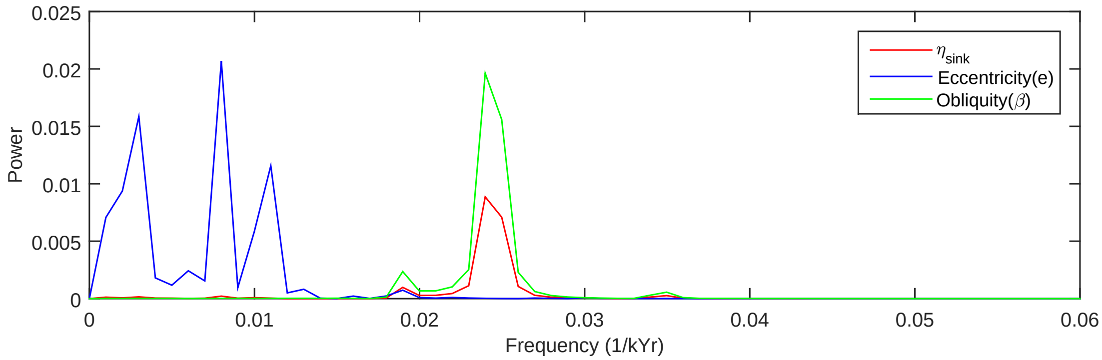


Figure 4. Power spectrum of stable ice line, eccentricity, and obliquity over the last one million years when $T_c = -10°$C.

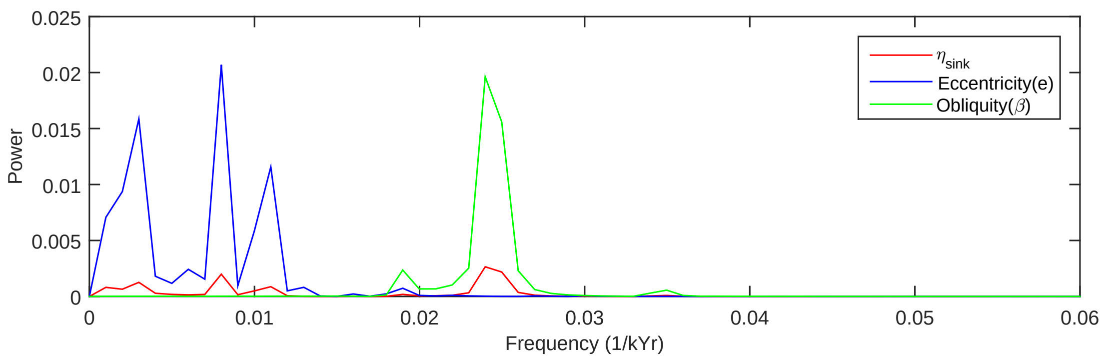


Figure 5. Power spectrum of stable ice line, eccentricity, and obliquity over the last one million years when $T_c = -5.5°$C.

5.2. **Ice line-Snow line dynamics.** We first show that we do get the similarities to the "flip-flop" model in [17] with Walsh et al.'s model (System (8)). Since we calculated η in Section 5.1 (Figures 2 and 3), we can also calculate ξ using (13) and finally find the discontinuity equation given by the first equality in (13). We begin with $T_c = -10°$C, where we have a small ice cap, i.e., η_{sink} is close to the pole (Figure 2). Using MATLAB, assuming $a = 1.05, b = 1.75, b_1 = 5, b_0 = 1.5, D = b(\eta-\xi)-a(1-\eta)$ and sink values of η which are found in Figure 2 and ξ given by $\xi = \left(1+\frac{a}{b}\right)\eta - \frac{a}{b}$, we get $D < 0$ which means that there is more accumulation than ablation. Recall that when $D < 0$, we set $T_c = -5.5°$C and we switch to the ODEs in system (11) and we have Figure 7. We note that for $T_c = -5.5°$C we have a larger ice cap, i.e., η_{sink} is closer to the equator as compared to η_{sink} in the $-10°$C case. For the same values of a, b, b_1, b_0 as above, we get $D > 0$, which means that there is more ablation than accumulation. However, recall that when $D > 0$, we set $T_c = -10°$C and switch to the ODEs in system (12), which corresponds exactly to Figure 6 above. But after this, $D < 0$ and we switch back to Figure 7. Thus, for these parameter values, glacial cycles do occur, where glaciation ($D < 0$) as seen in Figure 6 and deglaciation ($D > 0$) as seen in Figure 7 take place one after the other. In particular, although the model (Systems (11) and (12)) is constantly in the "flip-flop" state, D never crosses 0. If it were

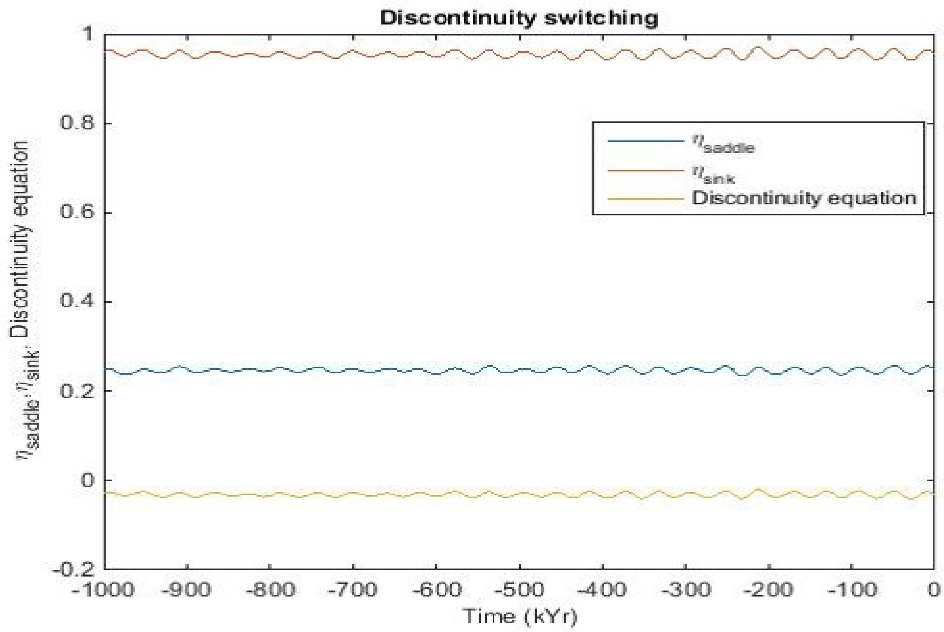


Figure 6. Discontinuity equation and switching mechanism when $T_c = -10°C$

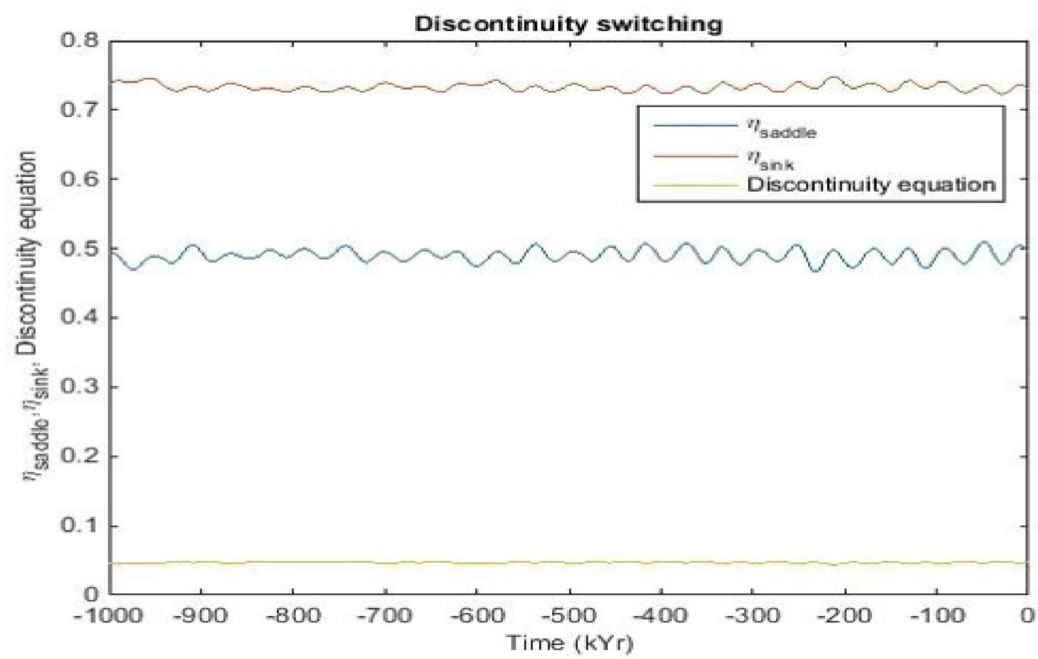
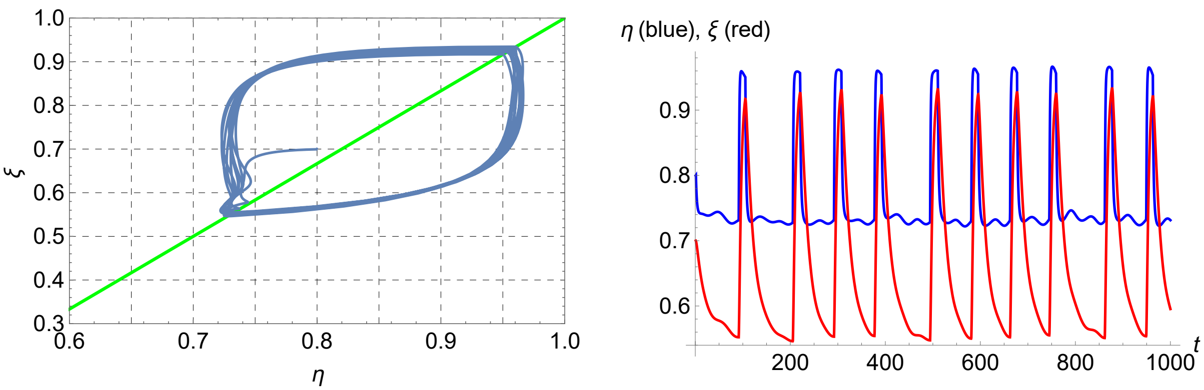


Figure 7. Discontinuity equation and switching mechanism when $T_c = -5.5°C$

0, accumulation equals ablation and hence there would be no trigger to go to the next state.

We now make $b = b_0$ to reduce our parameter space. First, we force the model (Systems (11) and (12)) with two Milankovitch parameters, obliquity and eccentricity. Using



(**a**) Discontinuity plane cycles with full Milankovitch forcing

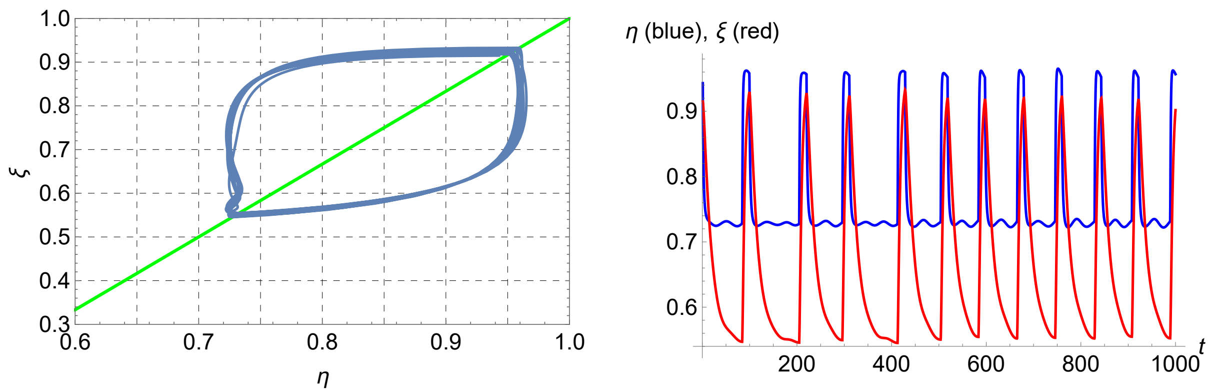
(**b**) Evolution of ice and snow lines over time with full Milankovitch forcing

Figure 8. $b = b_0 = 1.5, b_1 = 5, a = 1, \rho = \epsilon = 4 \times 10^{-2}$

Mathematica, in Figure 8(a) we note that we get cycles, that is, we constantly move from a glaciation state (blue curve below the green straight line) to a deglaciation state (blue curve above the green straight line) and vice versa. When the trajectory crosses the discontinuity plane, ablation exceeds accumulation and retreat begins (deglaciation) or accumulation exceeds ablation and ice sheets expand toward the equator (glaciation) (larger values of ξ). Now that we have established that glacial cycles are possible in Walsh et al.'s model, we aim to see how the snow and ice lines vary with time and how they behave with orbital forcing one at a time.

Figure 8(b) describes the evolution of the snow line and ice line over time, where $t = 0$ represents one million years ago and $t = 1000$ represents present day. Note that the ice line (red) increases slowly, denoting a slow descent into a long glacial period and an expanding ice sheet and then a short interglacial period follows where the ice sheet relatively quickly retreats as seen in the steep increase of ξ. This is consistent with paleoclimate data [13]. Huybers in [6] made a case that obliquity must be the trigger for ice sheet retreat since otherwise ice sheets were too massive to melt rapidly without any external factor. Most importantly, what we notice in Figure 8(b) is that deglacial events were not occurring every obliquity cycle and instead skipped two obliquity cycles at multiple times, such as at $t = 200, 400, 800$. Measuring from peak to peak, we see that these skips were between 80 and 120 kyr which is exactly what Huybers predicted in [6]. However, when we force Walsh et al.'s model ((Systems (11) and (12))) only with obliquity (Figure 9(b)), that is, we remove variations due to eccentricity by assuming $Q(e) = Q_0$, we notice that near $t = 800$ there is no obliquity cycle which is skipped. Also note that the skipping of cycles near $t = 400$ in Figure 8(b) is pushed back to $t = 300$ in Figure 9(b). This leads us to consider the possibility that eccentricity still does play a role in these cycles. In Figure 10(b) we see exactly this, as there are skipping of cycles consistently when obliquity is removed, i.e., $s_2 = s_2(\beta)$. This is in agreement with Huybers' revised outlook in [7] where he notes that eccentricity is also responsible for deglaciation. Table 2 summarizes the explorations and their results.

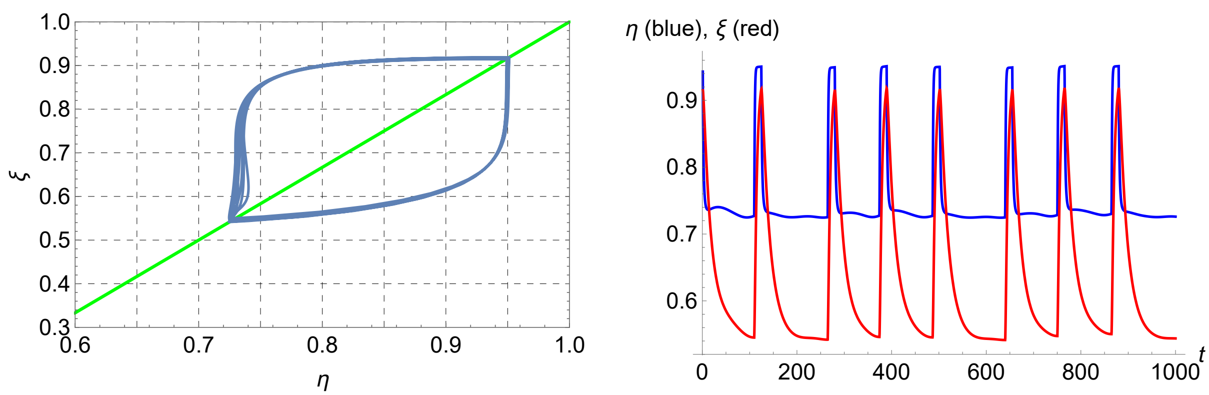
In Figures 11, 12, and 13 we make the time constant for the snow line 100 times faster than the time constant for the ice line. Interestingly, we note that there is not much



(a) Discontinuity plane cycles with obliquity only forcing

(b) Evolution of ice and snow lines over time with obliquity only forcing

Figure 9. $b = b_0 = 1.5, b_1 = 5, a = 1, \rho = \epsilon = 4 \times 10^{-2}$

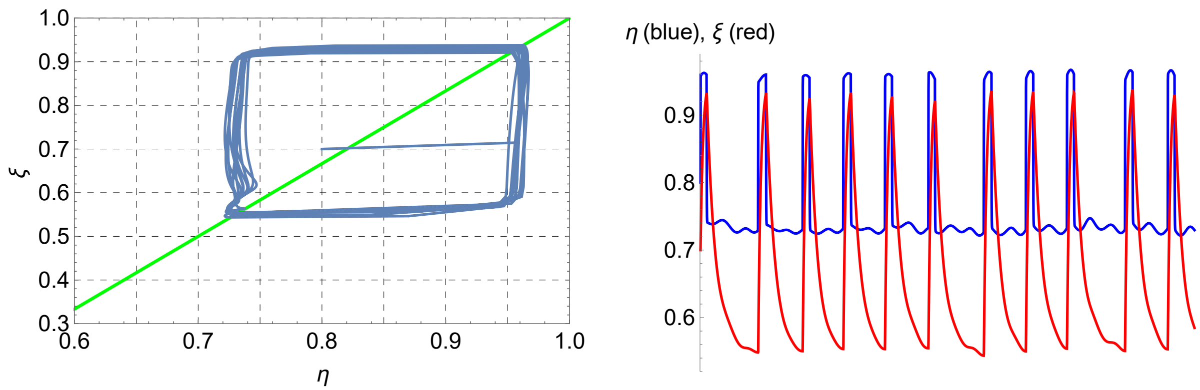


(a) Discontinuity plane cycles with eccentricity only forcing

(b) Evolution of ice and snow lines over time with eccentricity only forcing

Figure 10. $b = b_0 = 1.5, b_1 = 5, a = 1, \rho = \epsilon = 4 \times 10^{-2}$

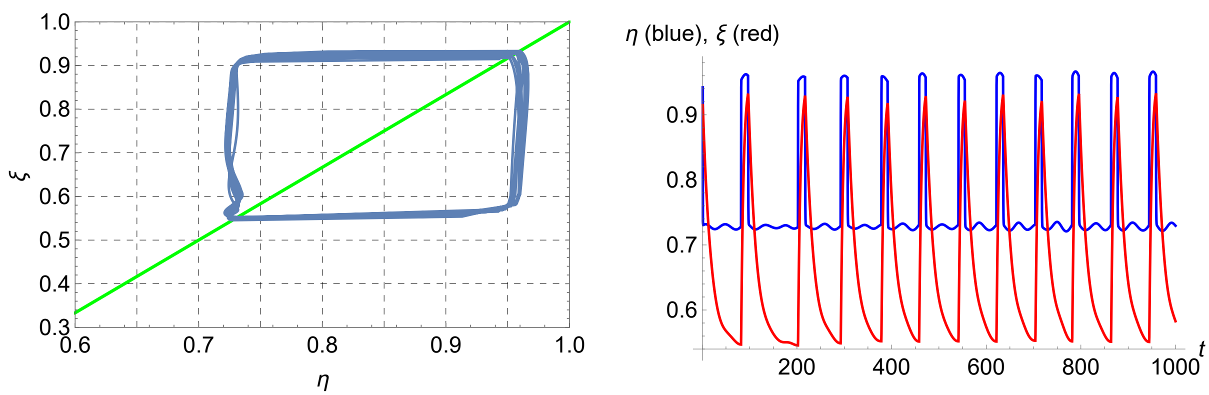
of a change in the evolution of the ice lines and snow lines over time even though the trajectory across the discontinuity plane has changed. We now increase the ablation parameter b_1 to its limiting case $b_1 = 45$, beyond this value we do not get the expected glacial-interglacial cycles. When we force the system fully with Milankovitch cycles we observe in Figure 14(a) that the trajectory does not spend much time ablating and quickly crosses the discontinuity plane to start accumulating ice slowly. We expect this since $b_1 = 45$ is a large ablation value and pushes the ice sheets to retreat faster. In Figure 14(b) we see the rapid interglacial period and slow descent into the glacial period as well and still have the skipping of some obliquity cycles.



(a) Discontinuity plane cycles with full Milankovitch forcing

(b) Evolution of ice and snow lines over time with full Milankovitch forcing

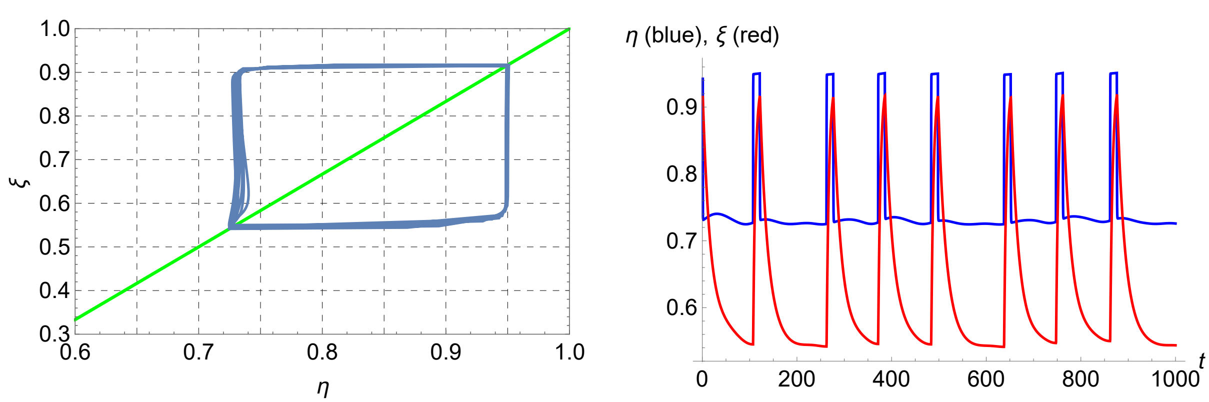
Figure 11. $b = b_0 = 1.5, b_1 = 5, a = 1, \rho = 100\epsilon = 4$



(a) Discontinuity plane cycles with obliquity only forcing

(b) Evolution of ice and snow lines over time with full Milankovitch forcing

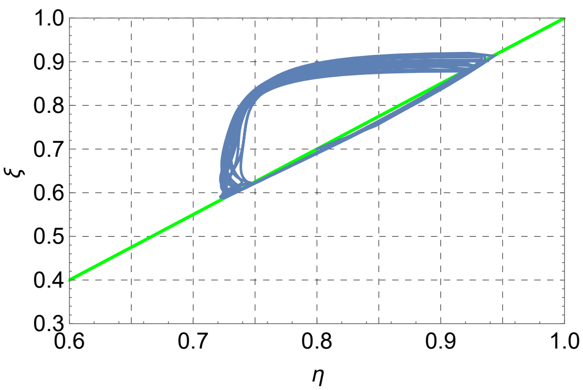
Figure 12. $b = b_0 = 1.5, b_1 = 5, a = 1, \rho = 100\epsilon = 4$



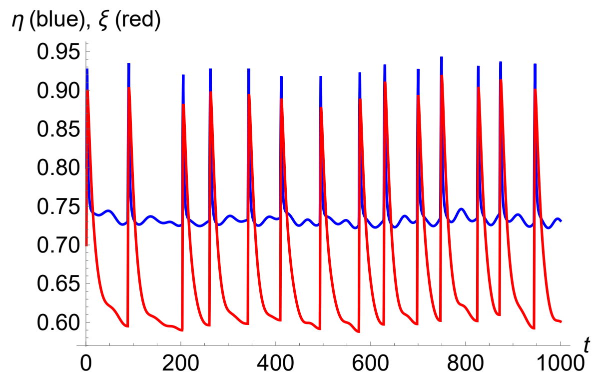
(a) Discontinuity plane cycles with eccentricity only forcing

(b) Evolution of ice and snow lines over time with eccentricity only forcing

Figure 13. $b = b_0 = 1.5, b_1 = 5, a = 1, \rho = 100\epsilon = 4$



(a) Discontinuity plane cycles with full Milankovitch forcing



(b) Evolution of ice and snow lines over time with full Milankovitch forcing

Figure 14. $b = b_0 = 2, b_1 = 45, a = 1, \rho = \epsilon = 4 \times 10^{-2}$

6. Discussion

Exploration	Results
If $T_c = -10°C$, which orbital forcing element drives the stable ice sheet expansion and retreat?	Obliquity (See Figures 2 and 4)
If $T_c = -5.5°C$, which orbital forcing element drives the stable ice sheet expansion and retreat?	Eccentricity and Obliquity (See Figures 3 and 5)
Do glacial cycles occur with full Milankovitch forcing incorporated into Walsh et al.'s model (Systems (11) and (12))?	Yes (See Figures 6,7,8(a),11(a),14(a)) and notice skipped obliquity cycles in Figure 8(b)
What happens when we force Walsh et al.'s model (System (11) and (12)) with variations in obliquity only?	No obliquity cycles skipped at certain times (See Figure 9(b))
What happens when we force Walsh et al.'s model (System (11) and (12)) with variations in eccentricity only?	Obliquity cycles skipped consistently (See Figure 10(b)), signaling that eccentricity is also responsible for deglaciation (which is in agreement with Figures 3 and 5)

Table 2. Summary of explorations and their results

Walsh et al.'s unique model (Systems (11) and (12)) was based on a finite approximation of an infinite dimensional model (System (7)) comprised of Budyko's energy balance model, an ODE describing the behavior of the edge of the ice sheet, and Walsh et al.'s snow line addition to account for glacial accumulation and ablation zones. Parameters such as insolation Q and s_2 in the temperature-ice line model were made to depend on Milankovitch cycles, that is, eccentricity of the Earth's orbit and the obliquity of the Earth's axis respectively and we observed that deglaciation and glaciation do occur mostly due to obliquity in the colder climate and in the warmer climate also due to eccentricity. Skipping of obliquity cycles was also seen which is in agreement with Huybers' model [6]. Thus, with

Walsh et al.'s new simple conceptual model we were able to produce glacial cycles and force them with Milankovitch cycles that shed more light on the behavior of ice and snow lines.

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Student biography

Shiv Priyam Raghuraman (raghu043@umn.edu) graduated in 2016 from the University of Minnesota, Twin Cities in where he received a B.S. in mathematics and a minor in physics. The work in this paper spurred his interest in climate science, and he is currently pursuing a Ph.D. in Atmospheric and Oceanic Sciences at Princeton University.