The Minnesota Journal of Undergraduate Mathematics

Sums of Powers of Fibonacci and Lucas Numbers

Crista Arangala, Burke Hrovat , and Jacob Kelner,

Department of Mathematics and Statistics
Elon University, Elon, North Carolina

The Minnesota Journal of Undergraduate Mathematics

Volume 2 Number 1 (2016)

Sponsored by
School of Mathematics
University of Minnesota
Minneapolis, MN 55455

The Minnesota Journal of Undergraduate Mathematics

Volume 2 Number 1 (2016)

Sums of Powers of Fibonacci and Lucas Numbers

Crista Arangala, Burke Hrovat* , and Jacob Kelner,

Department of Mathematics and Statistics
Elon University, Elon, North Carolina

1. Introduction

The Fibonacci numbers 1,1,2,3,5,8,... have fascinated mathematicians for hundreds of years. This familiar list is a particular member of a more general family of sequences, defined recursively as $G_n = G_{n-1} + G_{n-2}$, where $n \geq 2$. In the Fibonacci sequence above, $G_0 = 1$ and $G_1 = 1$. Another famous sequence in this family, generated by $G_0 = 2$ and $G_1 = 1$, is known as the Lucas sequence:

$$2, 1, 3, 4, 7, 11, \ldots$$

Properties of Fibonacci and Lucas sequences, and the relationships between them, have been the focus of a considerable amount of research. The well known Binet formulas, which provide closed form rules to calculate the Fibonacci and Lucas numbers, both incorporate the so-called golden ratio $(1+\sqrt{5})/2$, hinting at a deep connection between the sequences:

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}}\alpha^n - \frac{1}{\sqrt{5}}\beta^n, \\ L_n &= \alpha^n + \beta^n, \end{aligned}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

In addition to these two famous sequences, many have studied the more general family of sequences that satisfy similar recursive behaviors. A number of properties related to Fibonacci and Lucas numbers can be seen in [1], [5] and [10]. In addition to these properties, the properties that sums, differences, and powers of the Fibonacci numbers possess

* *Corresponding author*

have been of considerable interest. For example, sums and differences of Fibonacci numbers are also Fibonacci numbers [7]. In [6], the authors discuss the relationship between powers of Fibonacci numbers and Pascal's triangle.

Those who have focused on the powers of Fibonacci numbers have discovered sum identities such as

$$F_{m+1}F_m = \sum_{k=1}^{m} F_k^2.$$

The work by Clary and Hemenway [4] that explores sums of cubed Fibonacci numbers has inspired this work to seek alternative closed forms for the sum of powers of Fibonacci numbers. In [8], Melham discusses a closed form for the sum of fourth powers of Fibonacci and Lucas numbers. The authors of [13] focus on powers of Fibonacci and Lucas polynomials; however these results could lead to interesting results for Fibonacci and Lucas sequences as well. In this paper, we find alternative closed forms for sums of non-consecutive second and fourth powers of Fibonacci and Lucas numbers. In addition, we continue by presenting a closed form for sums of non-consecutive sixth and eighth powers of Lucas numbers. In particular, when finding the sum of powers of non-consecutive numbers, we seek to find unique closed forms that integrates numbers lying in between the addends.

2. Fibonacci and Lucas Numbers and their Identities

Before continuing, we formally define the Fibonacci numbers for all integers $n \geq 2$ as

$$F_0 = 0, \ F_1 = 1,$$
$$F_n = F_{n-1} + F_{n-2}.$$

The Lucas numbers are defined similarly, as

$$L_0 = 2, \ L_1 = 1,$$
$$L_n = L_{n-1} + L_{n-2}.$$

We note the following identities which will be useful in the ensuing sections.

Theorem 1. [11] *Fundamental Identity*

$$L_m^2 - 5F_m^2 = 4(-1)^m.$$

From [12] we have the

Theorem 2. *Cassini's Identities*

$$F_{m-1}F_{m+1} - F_m^2 = (-1)^m.$$

$$L_{m+2}L_m - L_{m+1}^2 = 5 \cdot (-1)^m.$$

3. Sum of Squares with Indices Differing By Two

We begin our exploration by looking at sums of squared Fibonacci and Lucas numbers that are a fixed distance from each other. In Lemma 1, we look at when the indices differ by 2 and in Theorem 3 generalize to when the indices differ by an even integer k. Using Cassini's Identities and the Fundamental Identity one can show Lemma 1.

Lemma 1. *Index Difference of Two for Fibonacci Numbers Squared*

$$F_m^2 + F_{m-2}^2 = 3F_{m-1}^2 + 2(-1)^{m-1}$$
$$L_m^2 + L_{m-2}^2 = 3L_{m-1}^2 - 10(-1)^{m-1}$$

for all integers $m > 2$.

However we wish to generalize this property. A closed form for the sum of two squared Fibonacci numbers, or Lucas numbers, of distance k apart where k is an even integer is presented in Theorem 3. This result as well as many that follow will look to express sums of powers of Fibonacci, or Lucas numbers, based on terms between the original two addends, similar to that in Lemma 1.

Theorem 3.

$$F_m^2 + F_{m-k}^2 = L_k F_{m-\frac{k}{2}}^2 + 2F_{\frac{k}{2}}^2(-1)^{m-\frac{k}{2}},$$
$$L_m^2 + L_{m-k}^2 = L_k L_{m-\frac{k}{2}}^2 - 10F_{\frac{k}{2}}^2(-1)^{m-\frac{k}{2}},$$

and in general,

$$W_m^2 + W_{m-k}^2 = L_k W_{m-\frac{k}{2}}^2 - 10abF_{\frac{k}{2}}^2(-1)^{m-\frac{k}{2}}$$

for integer m and even integer k, where $W_0 = a + b, W_1 = a\alpha + b\beta, W_m = a\alpha^m + b\beta^m, \alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Note that in the Fibonacci sequence, $a = \frac{1}{\sqrt{5}}$ and $b = -\frac{1}{\sqrt{5}}$, while in the Lucas sequence $a = b = 1$.

Proof. We will show that $L_k W_{m-\frac{k}{2}}^2 = W_m^2 + W_{m-k}^2 + 10abF_{\frac{k}{2}}^2(-1)^{m-\frac{k}{2}}$.

$$\begin{aligned}
L_k W_{m-\frac{k}{2}}^2 &= (\alpha^k + \beta^k)(a\alpha^{m-k/2} + b\beta^{m-k/2})^2 \\
&= (a^2\alpha^{2m} + b^2\beta^{2m} + a^2\alpha^{2m-k}\beta^k + b^2\beta^{2m-k}\alpha^k + 2ab\alpha^{m+k/2}\beta^{m-k/2} + 2ab\alpha^{m-k/2}\beta^{m+k/2}) \\
&= (W_m^2 - 2ab\alpha^m\beta^m + a^2\alpha^{2m-2k} + b^2\beta^{2m-2k} + 2ab\alpha^{m+k/2}\beta^{m-k/2} + 2ab\alpha^{m-k/2}\beta^{m+k/2}).
\end{aligned}$$

since $\alpha^k = (-1)^k\beta^{-k}$. Thus,

$$\begin{aligned}
L_k W_{m-\frac{k}{2}}^2 &= (W_m^2 - 2ab\alpha^m\beta^m + W_{m-k}^2 - 2ab\alpha^{m-k}\beta^{m-k} + 2ab(-1)^{m-k/2}(\alpha^k + \beta^k)) \\
&= (W_m^2 + W_{m-k}^2 - \alpha^{m-3k/2}\beta^{m-3k/2}(4ab\alpha^{m-k}\beta^{m-k}) + 2ab(-1)^{m-k/2}(\alpha^k + \beta^k)) \\
&= W_m^2 + W_{m-k}^2 + 2ab(-1)^{m-k/2}(\alpha^k - 2\alpha^{k/2}\beta^{k/2} + \beta^k) \\
&= W_m^2 + W_{m-k}^2 + 10ab(-1)^{m-k/2}F_{\frac{k}{2}}^2.
\end{aligned}$$

□

The next section expands these results to sums of fourth powers of Fibonacci and Lucas numbers. Again we begin by exploring sums of fourth powers of Fibonacci and Lucas numbers that are a fixed distance from each other.

4. Sums of Fourth Powers of Fibonacci and Lucas Numbers

In Lemmas 2 and 3, we look at when the indices differ by 2 for Fibonacci and Lucas numbers respectively and in Theorem 4 we generalize to when the indices differ by k, a positive multiple of 4. The general closed form of sums of fourth powers of Fibonacci and Lucas numbers, presented in Theorem 4, is unique to this paper.

Lemma 2. *Index Difference of Two for Fourth Powers of Fibonacci Numbers*

$$F_m^4 + F_{m-2}^4 = 7F_{m-1}^4 + 8(-1)^{m-1}F_{m-1}^2 + 2$$

for all integers $m > 2$.

Proof. Recall that by the definition of the Fibonacci sequence,
$F_m^2 = (F_{m-1} + F_{m-2})^2$ and $F_{m-2}^2 = (F_m - F_{m-1})^2$ and thus,

$$\begin{aligned} F_m^4 + F_{m-2}^4 &= F_m^2(F_{m-1} + F_{m-2})^2 + F_{m-2}^2(F_m - F_{m-1})^2 \\ &= 2F_m^2F_{m-2}^2 + (F_m^2 + F_{m-2}^2)F_{m-1}^2 + 2F_mF_{m-1}F_{m-2}(F_m - F_{m-2}) \\ &= 2F_mF_{m-2}(F_{m-1}^2 + F_mF_{m-2}) + (F_m^2 + F_{m-2}^2)F_{m-1}^2 \end{aligned}$$

From Lemma 1, $F_m^2 + F_{m-2}^2 = 3F_{m-1}^2 - 2(-1)^m$ and Cassini's Identity $F_mF_{m-2} = F_{m-1}^2 + (-1)^m$. Making these substitution,

$$\begin{aligned} F_m^4 + F_{m-2}^4 &= F_{m-1}^2(2F_{m-1}^2 - 2(-1)^m + 3F_{m-1}^2 - 2(-1)^m) + 2(F_mF_{m-2})^2 \\ &= F_{m-1}^2(5F_{m-1}^2 - 4(-1)^m) + 2(F_{m-1}^2 - (-1)^m)^2 \\ &= 7F_{m-1}^4 + 8(-1)^{m-1}F_{m-1}^2 + 2. \end{aligned}$$

□

Similarly,

Lemma 3. *Index Difference of Two for Fourth Powers of Lucas Numbers*

$$L_m^4 + L_{m-2}^4 = 7L_{m-1}^4 - 40(-1)^{m-1}L_{m-1}^2 + 50$$

for all integers $m > 2$.

In general,

Theorem 4. *For integer m and even integer k, where* $k \equiv 0 \mod 4$,

$$F_m^4 + F_{m-k}^4 = L_{2k} F_{m-\frac{k}{2}}^4 + 4(-1)^{m-\frac{k}{2}} F_{\frac{k}{2}} F_{\frac{3k}{2}} F_{m-\frac{k}{2}}^2 + 2F_{\frac{k}{2}}^4,$$

$$L_m^4 + L_{m-k}^4 = L_{2k} L_{m-\frac{k}{2}}^4 - 20(-1)^{m-\frac{k}{2}} F_{\frac{k}{2}} F_{\frac{3k}{2}} L_{m-\frac{k}{2}}^2 + 50F_{\frac{k}{2}}^4,$$

and

$$W_m^4 + W_{m-k}^4 = L_{2k} W_{m-\frac{k}{2}}^4 + 20ab(-1)^{m-\frac{k}{2}} F_{\frac{k}{2}} F_{\frac{3k}{2}} W_{m-\frac{k}{2}}^2 + 100(ab)^2 F_{\frac{k}{2}}^4.$$

Proof.

$$\begin{aligned} W_m^4 + W_{m-k}^4 &= (W_m^2 + W_{m-k}^2)^2 - 2W_m^2 W_{m-k}^2 \\ &= (L_k W_{m-\frac{k}{2}}^2 - 10abF_{\frac{k}{2}}^2 (-1)^{m-\frac{k}{2}})^2 - 2W_m^2 W_{m-k}^2 \end{aligned}$$

from Theorem 3. Thus, since $L_k^2 = L_{2k} + 2$ and $F_{\frac{k}{2}} L_k = F_{\frac{3k}{2}} - F_{\frac{k}{2}}$,

$$\begin{aligned} W_m^4 + W_{m-k}^4 &= (L_k^2 W_{m-\frac{k}{2}}^4 - 20ab(-1)^{m-\frac{k}{2}} F_{\frac{k}{2}}^2 L_k W_{m-\frac{k}{2}}^2 + 100(ab)^2 F_{\frac{k}{2}}^4) - 2W_m^2 W_{m-k}^2 \\ &= L_{2k} W_{m-\frac{k}{2}}^4 - 20ab(-1)^{m-\frac{k}{2}} F_{\frac{k}{2}} F_{\frac{3k}{2}} W_{m-\frac{k}{2}}^2 + 50(ab)^2 F_{\frac{k}{2}}^4 - 2W_m^2 W_{m-k}^2 \\ &\quad + 50(ab)^2 F_{\frac{k}{2}}^4 + 2W_{m-\frac{k}{2}}^4 + 20ab(-1)^{m-\frac{k}{2}} F_{\frac{k}{2}}^2 W_{m-\frac{k}{2}}^2. \end{aligned}$$

Denote $\tilde{W}_m = a^2\alpha^m + b^2\beta^m$, then $W_m W_{m-k} = (a\alpha^m + b\beta^m)(a\alpha^{m-k} + b\beta^{m-k}) = a^2\alpha^{2m-k} + b^2\beta^{2m-k} + ab(\alpha^m\beta^{m-k} + \alpha^{m-k}\beta^m)$. One can see that $\alpha^m\beta^m = (-1)^m = (-1)^{m-k}$ since k is even. Therefore, $W_m W_{m-k} = \tilde{W}_{2m-k} + ab(-1)^{m-k} L_k$ and

$$-2(W_m W_{m-k})^2 = -2\tilde{W}_{2m-k}^2 - 4ab(-1)^{m-k} L_k \tilde{W}_{2m-k} - 2(ab)^2 L_k^2.$$

Similarly,

$$2(W_{m-k/2})^4 = 2\tilde{W}_{2m-k}^2 + 8ab(-1)^{m-k}\tilde{W}_{2m-k} + 8(ab)^2.$$

Applying the Fundamental Identity,

$$\begin{aligned} 20ab(-1)^{m-k/2} F_{k/2}^2 W_{m-k/2}^2 &= 4ab(-1)^{m-k/2}(L_k - 2)(\tilde{W}_{2m-k} + 2ab(-1)^{m-k/2}) \\ &= 4ab(-1)^{m-k/2} L_k \tilde{W}_{2m-k} - 8ab(-1)^{m-k/2}\tilde{W}_{2m-k} \\ &\quad + 8(ab)^2 L_k - 16(ab)^2. \end{aligned}$$

And lastly,

$$\begin{aligned} 50(ab)^2 F_{\frac{k}{2}}^4 &= 2(ab)^2 (L_k - 2)^2 \\ &= 2(ab)^2 L_k^2 - 8(ab)^2 L_k + 8(ab)^2 \end{aligned}$$

It is important for k to be divisible by 4, so that $(-1)^{m-k} = (-1)^{m-k/2}$. With this assumption, we can combine the above results to determine that

$$W_m^4 + W_{m-k}^4 = L_{2k} W_{m-\frac{k}{2}}^4 - 20ab(-1)^{m-\frac{k}{2}} F_{\frac{k}{2}} F_{\frac{3k}{2}} W_{m-\frac{k}{2}}^2 + 50(ab)^2 F_{\frac{k}{2}}^4.$$

□

5. Higher powers

In this section, we pursue a few additional results related to this idea in relation to Lucas Numbers. In the future, we hope to generate a general statement related to Theorems 5 and 6 as well as a general statement for all sums of even powers of Fibonacci numbers. Some important identities that will be helpful in the proofs of Theorems 5 and 6 are included in Lemma 4. Although the proofs in this section are particularly cumbersome, the closed forms presented in Theorems 5 and 6 are unique.

Lemma 4. *For positive integer k,*

$$\begin{aligned}
L_{2k}L_k &= L_{3k}+L_k \\
L_kF_{\frac{k}{2}} &= F_{\frac{3k}{2}}-F_{\frac{k}{2}}, \\
L_kF_{\frac{3k}{2}} &= F_{\frac{5k}{2}}+F_{\frac{k}{2}}, \\
L_{2k}F_{\frac{k}{2}} &= F_{\frac{5k}{2}}-F_{\frac{3k}{2}}, \\
L_{2k}F_{\frac{3k}{2}} &= F_{\frac{7k}{2}}-F_{\frac{k}{2}}, \\
L_{3k}F_{\frac{k}{2}} &= F_{\frac{7k}{2}}-F_{\frac{5k}{2}}, \\
F^2_{\frac{k}{2}} &= \frac{1}{5}(L_k-2), \\
F^2_{\frac{3k}{2}} &= \frac{1}{5}(L_{3k}-2), \\
F_{\frac{3k}{2}}F_{\frac{k}{2}} &= \frac{1}{5}(L_{2k}-L_k), \\
L^2_{m-\frac{k}{2}} &= L_{2m-k}+2(-1)^{m-k/2}.
\end{aligned}$$

Theorem 5. *For integers m, and* $k\equiv 0 \mod 4$*,*

$$L_m^6+L_{m-k}^6 = L_{3k}L^6_{m-\frac{k}{2}}-30(-1)^{m-\frac{k}{2}}F_{\frac{5k}{2}}F_{\frac{k}{2}}L^4_{m-\frac{k}{2}}+(45F_{\frac{5k}{2}}-75F_{\frac{3k}{2}})F_{\frac{k}{2}}L^2_{m-\frac{k}{2}}-250(-1)^{m-\frac{k}{2}}F^6_{\frac{k}{2}}.$$

Proof. From Theorems 3 and 4,

$$\begin{aligned}
L_m^6+L_{m-k}^6 &= (L_m^4+L_{m-k}^4)(L_m^2+L_{m-k}^2)-L_m^2L_{m-k}^4-L_m^4L_{m-k}^2 \\
&= (L_{2k}L^4_{m-\frac{k}{2}}-20(-1)^{m-\frac{k}{2}}F_{\frac{3k}{2}}F_{\frac{k}{2}}L^2_{m-\frac{k}{2}}+50F^4_{\frac{k}{2}})(L_kL^2_{m-\frac{k}{2}}-10F^2_{\frac{k}{2}}(-1)^{m-\frac{k}{2}}) \\
&- L_m^2L_{m-k}^4-L_m^4L_{m-k}^2 \\
&= L_kL_{2k}L^6_{m-\frac{k}{2}}-20(-1)^{m-\frac{k}{2}}L_kF_{\frac{3k}{2}}F_{\frac{k}{2}}L^4_{m-\frac{k}{2}}+50L_kL^2_{m-\frac{k}{2}}F^4_{\frac{k}{2}} \\
&- 10(-1)^{m-\frac{k}{2}}L_{2k}F^2_{\frac{k}{2}}L^4_{m-\frac{k}{2}}+200F_{\frac{3k}{2}}F^3_{\frac{k}{2}}L^2_{m-\frac{k}{2}}-500(-1)^{m-\frac{k}{2}}F^6_{\frac{k}{2}} \\
&- L_m^2L_{m-k}^4-L_m^4L_{m-k}^2.
\end{aligned}$$

From Lemma 4, since $L_kL_{2k} = L_{3k} + L_k, L_kF_{\frac{3k}{2}} = F_{\frac{5k}{2}} + F_{\frac{k}{2}}$ and $L_{2k}F_{\frac{k}{2}} = F_{\frac{5k}{2}} - F_{\frac{3k}{2}}$,

$$\begin{aligned}
L_m^6 + L_{m-k}^6 &= L_{3k}L_{m-\frac{k}{2}}^6 - 20(-1)^{m-\frac{k}{2}}L_kF_{\frac{3k}{2}}F_{\frac{k}{2}}L_{m-\frac{k}{2}}^4 + 50L_kF_{\frac{k}{2}}^4L_{m-\frac{k}{2}}^2 \\
&\quad -10(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^2L_{2k}L_{m-\frac{k}{2}}^4 + 200F_{\frac{3k}{2}}F_{\frac{k}{2}}^3L_{m-\frac{k}{2}}^2 - 500(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^6 \\
&\quad -L_m^2L_{m-k}^4 - L_m^4L_{m-k}^2 + L_kL_{m-\frac{k}{2}}^6 \\
&= L_{3k}L_{m-\frac{k}{2}}^6 - 10(-1)^{m-\frac{k}{2}}(3F_{\frac{5k}{2}} - F_{\frac{3k}{2}} + 2F_{\frac{k}{2}})F_{\frac{k}{2}}L_{m-\frac{k}{2}}^4 + 50L_kL_{m-\frac{k}{2}}^2F_{\frac{k}{2}}^4 \\
&\quad +200F_{\frac{3k}{2}}F_{\frac{k}{2}}^3L_{m-\frac{k}{2}}^2 - 500(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^6 - L_m^2L_{m-k}^4 - L_m^4L_{m-k}^2 + L_kL_{m-\frac{k}{2}}^6
\end{aligned}$$

Making similar substitutions with the $L_{m-\frac{k}{2}}^2$ terms,

$$\begin{aligned}
L_m^6 + L_{m-k}^6 &= L_{3k}L_{m-\frac{k}{2}}^6 - 10(-1)^{m-\frac{k}{2}}(3F_{\frac{5k}{2}} - F_{\frac{3k}{2}} + 2F_{\frac{k}{2}})F_{\frac{k}{2}}L_{m-\frac{k}{2}}^4 \\
&\quad +10(5F_{\frac{5k}{2}} - 11F_{\frac{3k}{2}} + 8F_{\frac{k}{2}})F_{\frac{k}{2}}L_{m-\frac{k}{2}}^2 - 500(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^6 - L_m^2L_{m-k}^4 \\
&\quad -L_m^4L_{m-k}^2 + L_kL_{m-\frac{k}{2}}^6.
\end{aligned}$$

From Theorem 3,

$$\begin{aligned}
-L_m^2L_{m-k}^4 - L_m^4L_{m-k}^2 + L_kL_{m-\frac{k}{2}}^6 &= -L_m^2L_{m-k}^2(L_m^2 + L_{m-k}^2) + L_kL_{m-\frac{k}{2}}^6 \\
&= -(L_{2m-k} + (-1)^{m-\frac{k}{2}}L_k)^2(L_kL_{m-\frac{k}{2}}^2 - 10F_{\frac{k}{2}}^2(-1)^{m-\frac{k}{2}}) + L_kL_{m-\frac{k}{2}}^6.
\end{aligned}$$

Since $L_{m-\frac{k}{2}}^2 = L_{2m-k} + 2(-1)^{m-\frac{k}{2}}$,

$$\begin{aligned}
&-L_m^2L_{m-k}^4 - L_m^4L_{m-k}^2 + L_kL_{m-\frac{k}{2}}^6 \\
&= -(L_{m-\frac{k}{2}}^2 - 2(-1)^{m-\frac{k}{2}} + (-1)^{m-\frac{k}{2}}L_k)^2(L_kL_{m-\frac{k}{2}}^2 - 10F_{\frac{k}{2}}^2(-1)^{m-\frac{k}{2}}) + L_kL_{m-\frac{k}{2}}^6 \\
&= -(L_{m-\frac{k}{2}}^4 - 4(-1)^{m-\frac{k}{2}}L_{m-\frac{k}{2}}^2 + 4 + 2(-1)^{m-\frac{k}{2}}L_kL_{m-\frac{k}{2}}^2 + L_k^2 - 4L_k)(L_kL_{m-\frac{k}{2}}^2 - 10(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^2) + L_kL_{m-\frac{k}{2}}^6 \\
&= 10(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^2L_{m-\frac{k}{2}}^4 + 4(-1)^{m-\frac{k}{2}}L_kL_{m-\frac{k}{2}}^4 - 40F_{\frac{k}{2}}^2L_{m-\frac{k}{2}}^2 - 4L_kL_{m-\frac{k}{2}}^2 + 40(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^2 \\
&\qquad -2(-1)^{m-\frac{k}{2}}L_k^2L_{m-\frac{k}{2}}^4 + 20L_kF_{\frac{k}{2}}^2L_{m-\frac{k}{2}}^2 - L_k^3L_{m-\frac{k}{2}}^2 + 10(-1)^{m-\frac{k}{2}}L_k^2F_{\frac{k}{2}}^2 + 4L_k^2L_{m-\frac{k}{2}}^2 \\
&\qquad -40(-1)^{m-\frac{k}{2}}L_kF_{\frac{k}{2}}^2.
\end{aligned}$$

From Lemma 4, $L_kF_{\frac{k}{2}} = F_{\frac{3k}{2}}F_{\frac{k}{2}} - F_{\frac{k}{2}}^2$, and thus

$$\begin{aligned}
& -L_m^2L_{m-k}^4 - L_m^4L_{m-k}^2 + L_kL_{m-\frac{k}{2}}^6 \\
= \; & 20(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^2L_{m-\frac{k}{2}}^4 - 10(-1)^{m-\frac{k}{2}}F_{\frac{3k}{2}}F_{\frac{k}{2}}L_{m-\frac{k}{2}}^4 - 40F_{\frac{k}{2}}^2L_{m-\frac{k}{2}}^2 - 4L_kL_{m-\frac{k}{2}}^2 \\
& -40(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^2 + 20L_kF_{\frac{k}{2}}^2L_{m-\frac{k}{2}}^2 - L_k^3L_{m-\frac{k}{2}}^2 + 10(-1)^{m-\frac{k}{2}}L_k^2F_{\frac{k}{2}}^2 + 4L_k^2L_{m-\frac{k}{2}}^2 \\
& -40(-1)^{m-\frac{k}{2}}L_kF_{\frac{k}{2}}^2.
\end{aligned}$$

By writing $L_k^3L_{m-\frac{k}{2}}^2$ as $(5F_{\frac{k}{2}}^2+2)(L_{2k}+2)L_{m-\frac{k}{2}}^2$,

$$\begin{aligned}
& -L_m^2L_{m-k}^4 - L_m^4L_{m-k}^2 + L_kL_{m-\frac{k}{2}}^6 \\
= \; & 20(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^2L_{m-\frac{k}{2}}^4 - 10(-1)^{m-\frac{k}{2}}F_{\frac{3k}{2}}F_{\frac{k}{2}}L_{m-\frac{k}{2}}^4 + 20F_{\frac{3k}{2}}F_{\frac{k}{2}}L_{m-\frac{k}{2}}^2 - 80F_{\frac{k}{2}}^2L_{m-\frac{k}{2}}^2 \\
& -8L_{m-\frac{k}{2}}^2 - (5F_{\frac{k}{2}}^2+2)(L_{2k}+2)L_{m-\frac{k}{2}}^2 + 80(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^2 + 10(-1)^{m-\frac{k}{2}}L_k^2F_{\frac{k}{2}}^2 \\
& +4L_k^2L_{m-\frac{k}{2}}^2 - 40(-1)^{m-\frac{k}{2}}F_{\frac{3k}{2}}F_{\frac{k}{2}} \\
= \; & 20(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^2L_{m-\frac{k}{2}}^4 - 10(-1)^{m-\frac{k}{2}}F_{\frac{3k}{2}}F_{\frac{k}{2}}L_{m-\frac{k}{2}}^4 + 35F_{\frac{3k}{2}}F_{\frac{k}{2}}L_{m-\frac{k}{2}}^2 - 90F_{\frac{k}{2}}^2L_{m-\frac{k}{2}}^2 \\
& -5F_{\frac{5k}{2}}F_{\frac{k}{2}}L_{m-\frac{k}{2}}^2 + 80(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^2 + 10(-1)^{m-\frac{k}{2}}L_k^2F_{\frac{k}{2}}^2 + 4(L_k-2)L_{m-\frac{k}{2}}^2 - 40(-1)^{m-\frac{k}{2}}F_{\frac{3k}{2}}F_{\frac{k}{2}} \\
= \; & 20(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^2L_{m-\frac{k}{2}}^4 - 10(-1)^{m-\frac{k}{2}}F_{\frac{3k}{2}}F_{\frac{k}{2}}L_{m-\frac{k}{2}}^4 + 35F_{\frac{3k}{2}}F_{\frac{k}{2}}L_{m-\frac{k}{2}}^2 - 80F_{\frac{k}{2}}^2L_{m-\frac{k}{2}}^2 \\
& -5F_{\frac{5k}{2}}F_{\frac{k}{2}}L_{m-\frac{k}{2}}^2 + 40(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^2 + 10(-1)^{m-\frac{k}{2}}L_k^2F_{\frac{k}{2}}^2 - 80(-1)^{m-\frac{k}{2}}F_{\frac{3k}{2}}F_{\frac{k}{2}}.
\end{aligned}$$

It can be shown that $250F_{\frac{k}{2}}^6 = 10L_k^2F_{\frac{k}{2}}^2 - 40F_{\frac{3k}{2}}F_{\frac{k}{2}} + 80F_{\frac{k}{2}}^2$. Thus

$$\begin{aligned}
-L_m^2L_{m-k}^4 - L_m^4L_{m-k}^2 + L_kL_{m-\frac{k}{2}}^6 = \; & 20(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^2L_{m-\frac{k}{2}}^4 - 10(-1)^{m-\frac{k}{2}}F_{\frac{3k}{2}}F_{\frac{k}{2}}L_{m-\frac{k}{2}}^4 \\
& +35F_{\frac{3k}{2}}F_{\frac{k}{2}}L_{m-\frac{k}{2}}^2 - 80F_{\frac{k}{2}}^2L_{m-\frac{k}{2}}^2 \\
& -5F_{\frac{5k}{2}}F_{\frac{k}{2}}L_{m-\frac{k}{2}}^2 - 250(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^6
\end{aligned}$$

and

$$L_m^6 + L_{m-k}^6 = L_{3k}L_{m-\frac{k}{2}}^6 - 30(-1)^{m-\frac{k}{2}}F_{\frac{5k}{2}}F_{\frac{k}{2}}L_{m-\frac{k}{2}}^4 + (45F_{\frac{5k}{2}} - 75F_{\frac{3k}{2}})F_{\frac{k}{2}}L_{m-\frac{k}{2}}^2 - 250(-1)^{m-\frac{k}{2}}F_{\frac{k}{2}}^6.$$

□

Theorem 6. *For integers m, and* $k \equiv 0 \mod 4$,

$$\begin{aligned}
L_m^8 + L_{m-k}^8 = \; & L_{4k}L_{m-\frac{k}{2}}^8 - 40(-1)^{m-\frac{k}{2}}F_{\frac{7k}{2}}F_{\frac{k}{2}}L_{m-\frac{k}{2}}^6 + (100F_{\frac{7k}{2}} - 140F_{\frac{5k}{2}})F_{\frac{k}{2}}L_{m-\frac{k}{2}}^4 \\
& +(-1)^{m-\frac{k}{2}}(-80F_{\frac{7k}{2}} + 280F_{\frac{5k}{2}} - 280F_{\frac{3k}{2}})F_{\frac{k}{2}}L_{m-\frac{k}{2}}^2 + 1250F_{\frac{k}{2}}^8.
\end{aligned}$$

Proof. From Theorem 4,

$$\begin{aligned} L_m^8 + L_{m-k}^8 &= (L_m^4 + L_{m-k}^4)^2 - 2L_m^4 L_{m-k}^4 \\ &= (L_{2k}L_{m-\frac{k}{2}}^4 - 20(-1)^{m-\frac{k}{2}} F_{\frac{3k}{2}} F_{\frac{k}{2}} L_{m-\frac{k}{2}}^2 + 50F_{\frac{k}{2}}^4)^2 - 2L_m^4 L_{m-k}^4 \\ &= L_{2k}^2 L_{m-\frac{k}{2}}^8 + 400F_{\frac{3k}{2}}^2 F_{\frac{k}{2}}^2 L_{m-\frac{k}{2}}^4 + 2500F_{\frac{k}{2}}^8 \\ &\qquad -40(-1)^{m-\frac{k}{2}} L_{2k} F_{\frac{3k}{2}} F_{\frac{k}{2}} L_{m-\frac{k}{2}}^6 + 100L_{2k} F_{\frac{k}{2}}^4 L_{m-\frac{k}{2}}^4 \\ &\qquad -2000(-1)^{m-\frac{k}{2}} F_{\frac{3k}{2}} F_{\frac{k}{2}}^5 L_{m-\frac{k}{2}}^2 - 2L_m^4 L_{m-k}^4. \end{aligned}$$

Note that $L_{2k}^2 = L_{4k} + 2$ and $L_{2k}F_{\frac{3k}{2}} = F_{\frac{7k}{2}} - F_{\frac{k}{2}}$. These substitutions as well as substitutions for $F_{\frac{3k}{2}}^2$ and $F_{\frac{k}{2}}^2$ can be made so that

$$\begin{aligned} L_m^8 + L_{m-k}^8 &= L_{4k}L_{m-\frac{k}{2}}^8 + 80(L_{3k} - 2)F_{\frac{k}{2}}^2 L_{m-\frac{k}{2}}^4 + 2500F_{\frac{k}{2}}^8 \\ &\qquad -40(-1)^{m-\frac{k}{2}}(F_{\frac{7k}{2}} F_{\frac{k}{2}} - F_{\frac{k}{2}}^2) L_{m-\frac{k}{2}}^6 + 20L_{2k}(L_k - 2)F_{\frac{k}{2}}^2 L_{m-\frac{k}{2}}^4 \\ &\qquad -2000(-1)^{m-\frac{k}{2}} F_{\frac{3k}{2}} F_{\frac{k}{2}}^5 L_{m-\frac{k}{2}}^2 - 2L_m^4 L_{m-k}^4 + 2L_{m-\frac{k}{2}}^8 \\ &= L_{4k}L_{m-\frac{k}{2}}^8 + 80(F_{\frac{7k}{2}} - F_{\frac{5k}{2}} - 2F_{\frac{k}{2}})F_{\frac{k}{2}} L_{m-\frac{k}{2}}^4 + 2500F_{\frac{k}{2}}^8 \\ &\qquad -40(-1)^{m-\frac{k}{2}}(F_{\frac{7k}{2}} F_{\frac{k}{2}} - F_{\frac{k}{2}}^2) L_{m-\frac{k}{2}}^6 + 20L_{2k}(L_k - 2)F_{\frac{k}{2}}^2 L_{m-\frac{k}{2}}^4 \\ &\qquad -2000(-1)^{m-\frac{k}{2}} F_{\frac{3k}{2}} F_{\frac{k}{2}}^5 L_{m-\frac{k}{2}}^2 - 2L_m^4 L_{m-k}^4 + 2L_{m-\frac{k}{2}}^8. \end{aligned}$$

Expanding $-2L_m^4 L_{m-k}^4$,

$$\begin{aligned} -2L_m^4 L_{m-k}^4 &= -2(L_{2m-k} + (-1)^{m-\frac{k}{2}} L_k)^4 \\ &= -2(L_{2m-k}^4 + 4(-1)^{m-\frac{k}{2}} L_{2m-k}^3 L_k + 6L_{2m-k}^2 L_k^2 + 4(-1)^{m-\frac{k}{2}} L_{2m-k} L_k^3 + L_k^4) \end{aligned}$$

Again, since $L_{m-\frac{k}{2}}^2 = L_{2m-k} + 2(-1)^{m-\frac{k}{2}}$,

$$\begin{aligned} -2L_m^4 L_{m-k}^4 &= -2((L_{m-\frac{k}{2}}^2 - 2(-1)^{m-\frac{k}{2}})^4 + 4(-1)^{m-\frac{k}{2}}(L_{m-\frac{k}{2}}^2 - 2(-1)^{m-\frac{k}{2}})^3 L_k \\ &\qquad +6(L_{m-\frac{k}{2}}^2 - 2(-1)^{m-\frac{k}{2}})^2 L_k^2 + 4(-1)^{m-\frac{k}{2}}(L_{m-\frac{k}{2}}^2 - 2(-1)^{m-\frac{k}{2}})L_k^3 + L_k^4) \\ &= -2(L_{m-\frac{k}{2}}^8 + 4(-1)^{m-\frac{k}{2}} L_{m-\frac{k}{2}}^6 (L_k - 2) + (6L_k^2 - 24L_k + 24)L_{m-\frac{k}{2}}^4 \\ &\qquad +(4L_k^3 - 24L_k^2 + 48L_k - 32)(-1)^{m-\frac{k}{2}} L_{m-\frac{k}{2}}^2 + (L_k^4 - 8L_k^3 + 24L_k^2 - 32L_k + 16)). \end{aligned}$$

Recall that $F_{\frac{k}{2}}^2 = \frac{1}{5}(L_k - 2)$ and thus,

$$\begin{aligned}
-2L_m^4 L_{m-k}^4 &= -2(L_{m-\frac{k}{2}}^8 + 20(-1)^{m-\frac{k}{2}} F_{\frac{k}{2}}^2 L_{m-\frac{k}{2}}^6 + (30F_{\frac{k}{2}}^2 L_k - 60F_{\frac{k}{2}}^2) L_{m-\frac{k}{2}}^4 \\
&\quad +((20F_{\frac{k}{2}}^2 + 8)L_k^2 - 120F_{\frac{k}{2}}^2 L_k - 32)(-1)^{m-\frac{k}{2}} L_{m-\frac{k}{2}}^2 + (L_k^4 - 8L_k^3 + 24L_k^2 - 32L_k + 16)) \\
&= -2(L_{m-\frac{k}{2}}^8 + 20(-1)^{m-\frac{k}{2}} F_{\frac{k}{2}}^2 L_{m-\frac{k}{2}}^6 + (30F_{\frac{k}{2}}^2 L_k - 60F_{\frac{k}{2}}^2) L_{m-\frac{k}{2}}^4 \\
&\quad +(20(F_{\frac{5k}{2}} - F_{\frac{3k}{2}})F_{\frac{k}{2}} + 120F_{\frac{k}{2}}^2 - 80F_{\frac{k}{2}}^2 L_k)(-1)^{m-\frac{k}{2}} L_{m-\frac{k}{2}}^2 \\
&\quad +((5F_{\frac{k}{2}}^2 + 2)^4 - 8(5F_{\frac{k}{2}}^2 + 2)^3 + 24(5F_{\frac{k}{2}}^2 + 2)^2 - 32(5F_{\frac{k}{2}}^2 + 2) + 16)) \\
&= -2(L_{m-\frac{k}{2}}^8 + 20(-1)^{m-\frac{k}{2}} F_{\frac{k}{2}}^2 L_{m-\frac{k}{2}}^6 + (30F_{\frac{3k}{2}} - 90F_{\frac{k}{2}}^2) L_{m-\frac{k}{2}}^4 \\
&\quad +500F_{\frac{k}{2}}^6 (-1)^{m-\frac{k}{2}} L_{m-\frac{k}{2}}^2 + 625F_{\frac{k}{2}}^8).
\end{aligned}$$

Combining this result with the expanded form of $L_m^8 + L_{m-k}^8$ we get that

$$\begin{aligned}
L_m^8 + L_{m-k}^8 &= L_{4k} L_{m-\frac{k}{2}}^8 + 80(F_{\frac{7k}{2}} - F_{\frac{5k}{2}} - 2F_{\frac{k}{2}})F_{\frac{k}{2}} L_{m-\frac{k}{2}}^4 + 2500F_{\frac{k}{2}}^8 \\
&\quad -40(-1)^{m-\frac{k}{2}}(F_{\frac{7k}{2}} F_{\frac{k}{2}} - F_{\frac{k}{2}}^2) L_{m-\frac{k}{2}}^6 + 20L_{2k}(L_k - 2)F_{\frac{k}{2}}^2 L_{m-\frac{k}{2}}^4 \\
&\quad -2000(-1)^{m-\frac{k}{2}} F_{\frac{3k}{2}} F_{\frac{k}{2}}^5 L_{m-\frac{k}{2}}^2 - 1000(-1)^{m-\frac{k}{2}} F_{\frac{k}{2}}^6 L_{m-\frac{k}{2}}^2 + 2L_{m-\frac{k}{2}}^8 \\
&\quad -2(L_{m-\frac{k}{2}}^8 + 20(-1)^{m-\frac{k}{2}} F_{\frac{k}{2}}^2 L_{m-\frac{k}{2}}^6 + (30F_{\frac{3k}{2}} - 90F_{\frac{k}{2}}^2) L_{m-\frac{k}{2}}^4 + 625F_{\frac{k}{2}}^8 \\
&= L_{4k} L_{m-\frac{k}{2}}^8 - 40(-1)^{m-\frac{k}{2}} F_{\frac{7k}{2}} F_{\frac{k}{2}} L_{m-\frac{k}{2}}^6 + 80(F_{\frac{7k}{2}} - F_{\frac{5k}{2}} - 2F_{\frac{k}{2}})F_{\frac{k}{2}} L_{m-\frac{k}{2}}^4 \\
&\quad +20(F_{\frac{7k}{2}} - 3F_{\frac{5k}{2}} + 3F_{\frac{3k}{2}} - F_{\frac{k}{2}})F_{\frac{k}{2}} L_{m-\frac{k}{2}}^4 - 2((30F_{\frac{3k}{2}} - 90F_{\frac{k}{2}}^2) L_{m-\frac{k}{2}}^4 \\
&\quad -40(-1)^{m-\frac{k}{2}}(L_{2k} - 4L_k + 6)(F_{\frac{k}{2}} + 2F_{\frac{3k}{2}})F_{\frac{k}{2}} L_{m-\frac{k}{2}}^2 + 1250F_{\frac{k}{2}}^8 \\
&= L_{4k} L_{m-\frac{k}{2}}^8 - 40(-1)^{m-\frac{k}{2}} F_{\frac{7k}{2}} F_{\frac{k}{2}} L_{m-\frac{k}{2}}^6 + 80(F_{\frac{7k}{2}} - F_{\frac{5k}{2}} - 2F_{\frac{k}{2}})F_{\frac{k}{2}} L_{m-\frac{k}{2}}^4 \\
&\quad +20(F_{\frac{7k}{2}} - 3F_{\frac{5k}{2}} + 3F_{\frac{3k}{2}} - F_{\frac{k}{2}})F_{\frac{k}{2}} L_{m-\frac{k}{2}}^4 - 2((30F_{\frac{3k}{2}} - 90F_{\frac{k}{2}}^2) L_{m-\frac{k}{2}}^4 \\
&\quad -40(-1)^{m-\frac{k}{2}}(2F_{\frac{7k}{2}} - 7F_{\frac{5k}{2}} + 7F_{\frac{3k}{2}})F_{\frac{k}{2}} L_{m-\frac{k}{2}}^2 + 1250F_{\frac{k}{2}}^8.
\end{aligned}$$

Finally, we conclude that

$$\begin{aligned}
L_m^8 + L_{m-k}^8 &= L_{4k} L_{m-\frac{k}{2}}^8 - 40(-1)^{m-\frac{k}{2}} F_{\frac{7k}{2}} F_{\frac{k}{2}} L_{m-\frac{k}{2}}^6 + (100F_{\frac{7k}{2}} - 140F_{\frac{5k}{2}})F_{\frac{k}{2}} L_{m-\frac{k}{2}}^4 \\
&\quad +(-1)^{m-\frac{k}{2}}(-80F_{\frac{7k}{2}} + 280F_{\frac{5k}{2}} - 280F_{\frac{3k}{2}})F_{\frac{k}{2}} L_{m-\frac{k}{2}}^2 + 1250F_{\frac{k}{2}}^8.
\end{aligned}$$

□

6. Future Work

In this paper, we focused on particular sums of powers of 2 for both Fibonacci and Lucas numbers. There are some obvious future research projects that can come directly from this work. First of all, it seems reasonable to think that Theorems 5 and 6 could be generalized in a similar way to what was done in Theorem 4. In addition, a generalization for all powers of two or all even powers could be explored. One might also explore odd powers as well.Throughout the majority of the higher power theorems, we also assumed $k \equiv 0 \mod 4$ so that $(-1)^{m-k} = (-1)^{m-\frac{k}{2}}$, however there may be similar results for $k \equiv 1,2,3 \mod 4$.

In our exploration of articles focusing on Fibonacci and Lucas number identities, we found many different and interesting proof techniques leading to intriguing results. Although in this paper we strictly use known properties and algebraic techniques to identify additional properties for higher powers, it would also be interesting to attempt to apply some of these other techniques, such as those combinatorical techniques found in [2] and [3], to these higher power problems.

References

[1] A. Benjamin and J. Quinn. *Recounting Fibonacci and Lucas Identities.* The College Mathematics Journal, Vol. 30, No. 5 (1999), 359-356.

[2] A. Benjamin and J. Quinn. *The Fibonacci Numbers - Exposed More Discretely*. Mathematics Magazine, Vol 76, No. 3 (2003), 182-192.

[3] A. T. Benjamin, T. A. Carnes and B. Cloitre, *Recounting the sum of cubes of Fibonacci numbers*, Congressus Numerantium, Proceedings of the Eleventh International Conference of Fibonacci Numbers and Their Applications, (W. Webb, ed.), Vol. 194 (2009), 45–51.

[4] S. Clary and P. D. Hemenway. *On Sums of Cubes of Fibonacci Numbers.* In Applications of Fibonacci Numbers 5:123-36. Ed. G. E. Bergum et al. Kluwer Academic Publishers. The Netherlands (1993).

[5] R. Dunlap. *The Golden Ratio and Fibonacci Numbers*. World Scientific Publishing. Singapore. 1997.

[6] D. Hathaway and S. Brown. *Fibonacci Powers and a Fascinating Triangle*. The College Mathematics Journal, Vol. 28, No. 2 (1997), 124-128.

[7] D. Kalman and R. Mena. *The Fibonacci Numbers: Exposed*. Mathematics Magazine, Vol. 76, No. 3 (2003), 167-181.

[8] R.S. Melham. *Alternating Sums of Fourth Powers of Fibonacci and Lucas Numbers.* The Fibonacci Quarterly 38.3 (2000): 254-259.

[9] R. S. Melham, *Some conjectures concerning sums of odd powers of Fibonacci and Lucas numbers*, The Fibonacci Quarterly, 46/47 (2008/2009), 312-315.AS Posamentier

[10] A. Posamentier and I. Lehmann, *The Fabulous Fibonnaci Numbers*. Prometheus Books. Amherst, NY. 2007.

[11] S. Robinowitz. *Algorithmic Manipulation of Fibonacci Identities.* In Applications of Fibonacci Numbers 6:389-408. Ed. G. E. Bergum, et al. Kluwer Academic Publishers. The Netherlands (1996).

[12] M. Spivey, *Fibonacci Identities via the Determinant Sum Property*, The College Mathematics Journal, Vol. 37, No. 4, (2006), 286-289.

[13] W. Tingting and Z. Wenpeng, *Some identities involving Fibonacci, Lucas Polynomials and their Applications* Bull. Math. Soc. Sci. Math. Roumanie Tome 55(103) No. 1, (2012), 95–103.

Student biographies

Burke Hrovat. (*Corresponding author:* bhrovat@elon.edu) Burke graduated from Elon University with a dual degree in finance and economics with a concentration in mathematical economics. He is currently pursuing a career in analytics in the Chicagoland area and plans on completing a master's degree in the near future.

Jacob Kelner. Jacob is currently a second year medical student pursuing his doctorate degree at Rowan University School of Osteopathic Medicine. He graduated cum laud from Elon University with a mathematics and biochemistry double major. He is interested in further research in mathematics within the medical field.