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Kathryn Lezark* and Mindy Capaldi

Valparaiso University

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# New Findings in Old Geometry: using triangle centers to create similar or congruent triangles

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Abstract. The Euler line of a triangle passes through several important points, including three specific triangle centers: the centroid, orthocenter, and circumcenter. Each of these centers is the intersection of lines related to the triangle, mainly its medians, altitudes, and perpendicular bisectors, respectively. We present three theorems which initially share a similar construction. Each involves starting with a triangle and a point. After connecting the triangle's vertices to that point, creating additional triangles, we establish connections to either the centroids, orthocenters, or circumcenters of the new triangles.

## 1. Introduction

Geometry is a classic mathematics subject, both in the sense that there were important geometric developments in classical times and in the fact that most students are introduced to geometry in high school. If we consider the geometrical subfields which are taught and researched in graduate schools as modern geometry, then there is currently great work being done in those areas. The focus of this paper follows in the footsteps of older geometry. Mathematicians in "premodern" geometry returned to, and expanded upon, Euclidean geometry. This paper returns to premodern geometry and expands upon it.

To set the stage, we turn to Leonhard Euler, one of the greatest influencers of mathematics. His reach extended to geometry, as seen through the founding of the subfields topology and graph theory. He also made his mark on the geometry of the classic Euclidean triangle. The Euler line of a triangle passes through three main triangle centers: the centroid, circumcenter, and orthocenter. These points are defined as follows:

**Definition 1.1.** The *centroid* of a triangle is the intersection of the triangle's medians (lines connecting vertices to each side's midpoint).

**Definition 1.2.** The *circumcenter* of a triangle is the intersection of the triangle's perpendicular bisectors (lines perpendicular to, and through the midpoint of, each side).

**Definition 1.3.** The *orthocenter* of a triangle is the intersection of the triangle's altitudes (lines through vertices which are perpendicular to the opposite sideline).

* *Corresponding author*

Euler noted that the three centers are collinear. Also, the centroid is between the circumcenter and the orthocenter and is twice as far from the orthocenter as it is from the circumcenter [5].

Every so often the work of Euler is resurrected with new results. Mathematicians have since found significant points that also reside on the Euler line, such as de Longchamp's point and more recently, in 1985, the Schiffler point [2, 9]. Also in the 20th Century, Euler's conclusion about the line was reproven using position vectors [4]. Oldknow (1996) investigated these centers, along with the incenter and associated circles [8]. In 2000, Longuet-Higgens wrote about another interesting point that lies on the Euler line [7]. A year later, we saw a conclusion about the concurrency of four Euler lines [6]. Euler's work has even been extended by Villiers et. al. ([3], 2014) to a quasi-Euler line of a quadrilateral. This type of geometry continues to hold mathematicians' interests and is not the finished field that it is sometimes perceived to be.

The definitions of centroid, circumcenter, and orthocenter are central to this paper, but there are a few more topics that recur in multiple sections. We recall these, which the reader has likely seen before.

Two line segments are congruent if they have the same length, and two angles are congruent if they have the same measure. Two triangles are *congruent* if their corresponding sides and angles are congruent. Corresponding parts of congruent figures are congruent, a theorem that is used often in the proofs of this paper. Congruency of triangles is denoted $\triangle ABC \cong \triangle XYZ$.

A less strong, but frequently useful, statement to make about triangles is that they are similar. Two triangles are *similar* if the three angles of one are congruent to the three angles of the other, and if corresponding sides are proportional. Similarity is denoted $\triangle ABC \sim \triangle XYZ$.

For a thorough introduction to Euclidean geometry, we direct the reader to [1]. To explore more advanced topics, especially those involving the aforementioned triangle centers, see [10].

This paper is split into three sections, one each on the centroid, orthocenter, and circumcenter. Our three main theorems were developed similarly, beginning with an arbitrary triangle and a point (not necessarily arbitrary). By constructing centroids, orthocenters, or circumcenters, respectively, of three resulting triangles, we were led to unique and interesting conclusions. In the next section, centroids will be used to create a new triangle that is similar to the given triangle. We also prove another set of collinear points that have a two-thirds distance relationship. Section 3 demonstrates how orthocenters lead to a triangle congruent to the original. Finally, in Section 4 we show a connection between a triangle, a circumcircle, and three created circumcenters.

## 2. Centroid Theorem

We initially consider the centroid of a triangle, which is the most well-known center of a triangle. Centroids, or centers of mass, are often considered in other courses and topics, and are certainly not restricted to triangles. The proofs of Theorem 2.7 and Corollary

2.10, our main centroid theorems, were completed through the use a few simple lemmas, mostly relating to similar triangles. These were likely theorems seen in high school geometry classes. We include them here for reference.

**Lemma 2.1.** *Given two triangles $\triangle ABC$ and $\triangle DEF$, $\angle ABC \cong \angle DEF$ and $\angle BAC \cong \angle EDF$ (or any two corresponding angles) if and only if $\triangle ABC \sim \triangle DEF$.*

**Lemma 2.2.** *(SSS for Similar Triangles): If two triangles have all three pairs of corresponding sides in the same ratio, then the triangles are similar.*

**Definition 2.3.** A *transversal* is a line that crosses at least two other lines.

**Definition 2.4.** Suppose two lines are cut by a transversal. The pairs of angles on opposite sides of the transversal but inside the two lines are called *alternate interior angles*.

**Lemma 2.5.** *(Alternate Interior Angle Theorem, AIAT): If two parallel lines are cut by a transversal, then the resulting pairs of alternate interior angles are congruent.*

One more well-known theorem is needed. This last lemma might be new to someone who has not taken a college geometry course.

**Lemma 2.6.** *The centroid is $\frac{2}{3}$ from any triangle vertex along the length of the median from that vertex.*

To set up our main centroid theorem, we need to define a number of points. See Figure 1 for a visual representation. Let there be an arbitrary point $L$ and $\triangle ABC$ with medians $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. The medians intersect at centroid $M$. Also, let there be segments $\overline{LA}$ with midpoint $P$, $\overline{LB}$ with midpoint $R$, and $\overline{LC}$ with midpoint $Q$. The centroid of $\triangle ABL$ is $Z$, the centroid of $\triangle BCL$ is $X$, and the centroid of $\triangle ACL$ is $Y$. Finally, we construct $\triangle XYZ$ with centroid $N$ and medians $\overline{XI}$, $\overline{YJ}$, and $\overline{ZK}$.

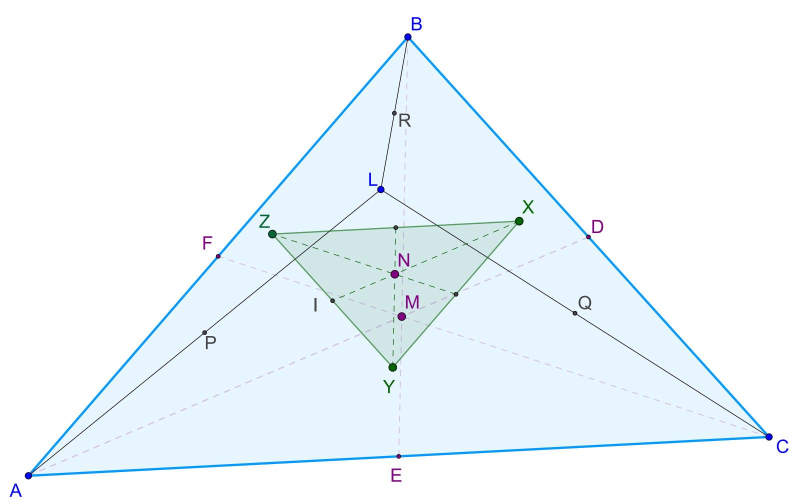



FIGURE 1. The initial set-up for Theorem 2.7.

**Theorem 2.7.** *Given $\triangle ABC$ and an arbitrary point $L$, with centroids $X, Y$, and $Z$ of $\triangle BCL$, $\triangle ACL$, and $\triangle ABL$, then $\triangle XYZ$ is similar to $\triangle ABC$. Also, the side lengths of $\triangle XYZ$ are $\frac{1}{3}$ the length of $\triangle ABC$.*

*Proof.* First, we will show similarity by proving the ratio between the sides of $\triangle XYZ$ and $\triangle ABC$ is one to three. Thus, we show that each of sides $\overline{XY}, \overline{ZX}$, and $\overline{YZ}$ are $\frac{1}{3}$ the lengths of the corresponding sides of $\triangle ABC$.

Consider $\triangle ACR$ in Figure 2. Recall that $R$ is the midpoint of the segment connecting the arbitrary $L$ with the main triangle vertex $B$. We want to show that this triangle is similar to $\triangle ZXR$, where $X$ is the centroid of $\triangle BCL$ and $Z$ is the centroid of $\triangle ABL$. We cannot assume that $\overline{ZX}$ and $\overline{AC}$ are parallel. Because $\overline{AR}$ is a median of $\triangle ABL$, by Lemma 2.6 $AZ = \frac{2}{3}AR$. Similarly, $CX = \frac{2}{3}CR$ since $\overline{CR}$ is a median of $\triangle CBL$. Also, $\angle CRA \cong \angle XRZ$ by the reflexive property. Then, by SAS for Similar Triangles, $\triangle ACR \sim \triangle ZXR$.

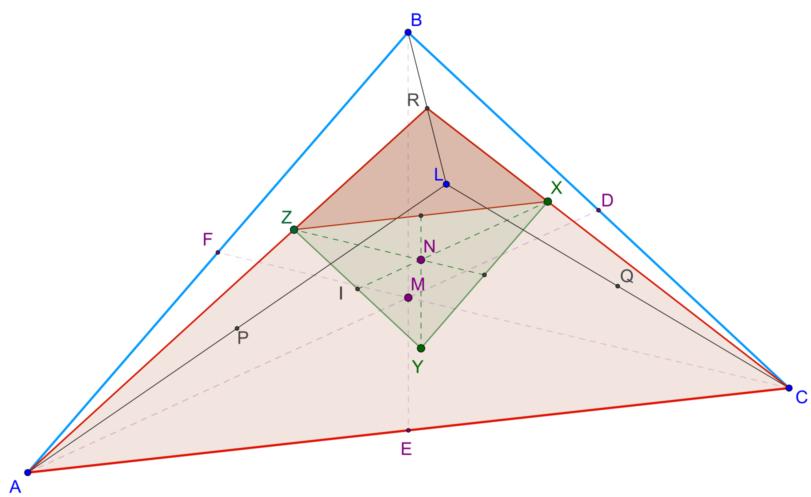



FIGURE 2. Similar triangles in the proof of Theorem 2.7.

By triangle similarity, $\frac{ZR}{AR} = \frac{ZX}{AC} = \frac{1}{3}$. Then $\overline{ZX}$ is $\frac{1}{3}$ the size of $\overline{AC}$. Also, by AIAT, $\overline{ZX} \parallel \overline{AC}$. We have our first side comparison: $ZX = \frac{1}{3}AC$ and $\overline{ZX} \parallel \overline{AC}$.

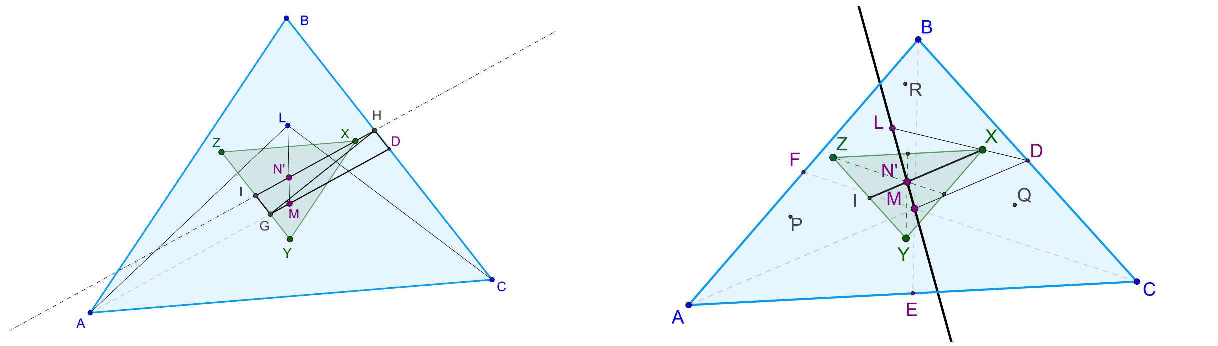
Now consider the point $Y$, which is the centroid of $\triangle ACL$. By equivalent reasoning, $\overline{YX}$ and $\overline{YZ}$ are parallel to, and $\frac{1}{3}$ the size of, $\overline{AB}$ and $\overline{CB}$, respectively. Thus, all the side lengths of $\triangle XYZ$ are $\frac{1}{3}$ the lengths of the corresponding sides of $\triangle ABC$.

Finally, to show similarity we simply apply SSS for Similar Triangles and get $\triangle XYZ \sim \triangle ABC$. $\square$

The proof of Corollary 2.10 will introduce quadrilaterals, denoted $\square ABCD$, so we include a lemma connected to that shape:

**Lemma 2.8.** *If opposite angles of a quadrilateral are congruent, then it is a parallelogram.*

We will also make use of the following fact about the medians of similar triangles.




(A) Used for part one, to show two-thirds length. (B) Used for part two, to show that $N'$ is the same as $N$.

Figure 3. Figures used in the proof of Corollary 2.10.

**Lemma 2.9.** *The ratio of the areas of similar triangles is equal to the ratio of their corresponding medians.*

**Corollary 2.10.** *The centroid, $M$, of $\triangle ABC$, and the centroid, $N$, of $\triangle XYZ$, are collinear with $L$ and $LN = \frac{2}{3}LM$.*

*Proof.* Recall that point $I$ is the midpoint of $\overline{ZY}$, with $\overline{XI}$ being a median. Our proof necessitates two cases: one where $\overleftrightarrow{LM}$ and $\overleftrightarrow{XI}$ intersect and another in which they do not intersect in a single point. We first give the proof of single intersection case.

Part One: In this part, we show the two-thirds length relationship. See Figure 3a for a visual representation. Assume that the intersection of $\overleftrightarrow{LM}$ and $\overleftrightarrow{XI}$ exists; call it $N'$. From the proof of Theorem 2.7, we know that $\triangle XYZ \sim \triangle ABC$ and they have parallel corresponding sides. By SAS similarity, $\triangle YIX \sim \triangle BDA$, where $D$ is the midpoint of $\overline{BC}$. Therefore, $\angle YIX \cong \angle BDA$. Let $H$ be the intersection of $\overleftrightarrow{XI}$ and $\overleftrightarrow{BC}$, and $G$ the intersection of $\overleftrightarrow{YZ}$ and $\overleftrightarrow{AD}$. Then by AIAT, the alternate interior angles $\angle IGH$ and $\angle DHG$ are congruent. Since $\overline{GH} \cong \overline{GH}$, then AAS gives congruency of $\triangle IGH$ and $\triangle DHG$. Lemma 2.8, along with $\overline{IG} \cong \overline{HD}$ and $\overline{GD} \cong \overline{IH}$ implies $\square IGDH$ is a rectangle and $\overline{XI} \parallel \overline{DA}$. By AIAT, $\angle LN'X \cong \angle LMD$ and $\angle LXN' \cong \angle LDM$. By Lemma 2.1, $\triangle LN'X \sim \triangle LMD$. By the definition of similarity $\frac{LX}{LD} = \frac{LN'}{LM} = \frac{2}{3}$. Then $LN' = \frac{2}{3}LM$.

Part Two: Show that $N = N'$. See Figure 3b for a visual representation.

Again, by similarity for $\triangle LN'X$ and $\triangle LMD$, $\frac{LX}{LD} = \frac{N'X}{MD} = \frac{2}{3}$. Then we have

$$MD = \frac{3}{2}N'X. \tag{1}$$

By Theorem 2.7, the sides of $\triangle XYZ$ are $\frac{1}{3}$ the length of the corresponding sides of $\triangle ABC$. Then by Lemma 2.9, we have $AD = 3XI$. Also, since $M$ is the centroid on median $\overline{AD}$, we have $AD = 3MD$. This implies $3MD = 3XI$ and $MD = XI$.

Substituting equation (1), we see

$$\begin{aligned}\frac{3}{2}N'X &= XI \\ N'X &= \frac{2}{3}XI.\end{aligned}$$

Because $N$ is the centroid, by Lemma 2.6 we know that $XN = \frac{2}{3}XI$. It follows that $XN = \frac{2}{3}XI = XN'$. Thus, we know that $N = N'$ and $L$, $N$, and $M$ are collinear with $LN = \frac{2}{3}LM$.

We now give an abbreviated proof of the case where $\overleftrightarrow{LM}$ and $\overleftrightarrow{XI}$ do not intersect at a single point. Assuming that they do not intersect, then $\overleftrightarrow{XI} \parallel \overleftrightarrow{LM}$. Then $L$ is on $\overleftrightarrow{AD}$ since $M$ is on $\overleftrightarrow{AD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{XI}$. Since $\overleftrightarrow{LD}$ is a median of $\triangle BCL$, and $X$ is the centroid of that triangle, then $X$ must lie on $\overleftrightarrow{LD}$. Thus $\overleftrightarrow{AD}$, $\overleftrightarrow{LD}$, $\overleftrightarrow{XD}$ are coincident lines and $\overleftrightarrow{XD} \parallel \overleftrightarrow{XI}$. Of course, this means that $\overleftrightarrow{XD}$ and $\overleftrightarrow{XI}$ are coincident. Then medians $\overline{LD}$ and $\overline{AD}$ are on the same line, and since $M$ is on $\overleftrightarrow{AD}$, then $L$ and $M$ are collinear. Since they are on $\overleftrightarrow{XI}$, then they are also collinear with $N$.

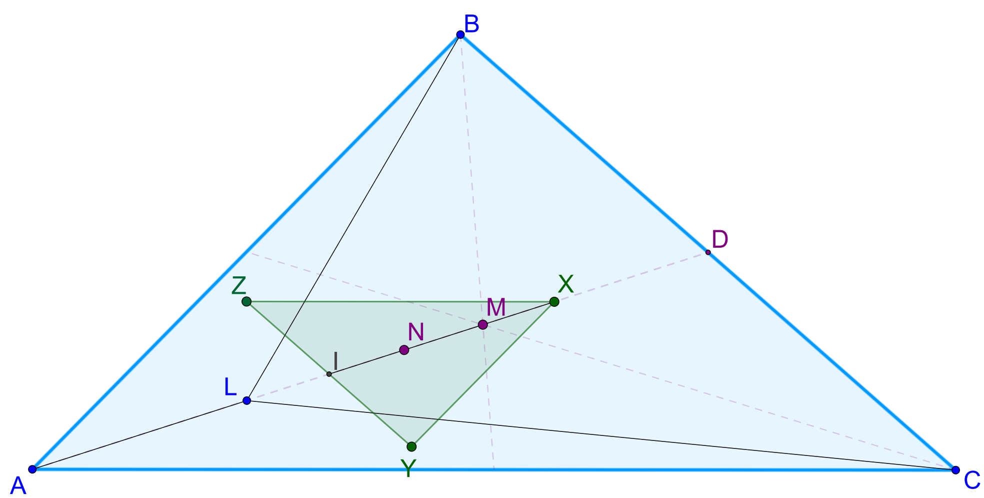



Figure 4. Collinearity case of Corollary 2.10.

Figure 4 shows one example of this collinearity. Depending on the placement of $L$, the order of the points may change. For clarity, we use the placement as seen in the figure. However, other than sign changes, the following proof sketch holds for any position of $L$. We need to show that $\overline{LN} = \frac{2}{3}\overline{LM}$. The following equalities use triangle similarity, proportionality, and the two-thirds centroid relationship.

First,

$$LN = LX - NX = \frac{2}{3}LD - NX = \frac{2}{3}LD - \frac{2}{3}IX = \frac{2}{3}(LD - IX) = \frac{2}{3}(XD + LI). \tag{2}$$

Also,

$$XD = MD - MX = \frac{1}{2}AM - MX = \frac{1}{2}(3XN) - MX = IX - MX$$
$$= (IM + MX) - MX = IM. \tag{3}$$

Substituting the conclusion of equation (3) into equation (2) gives

$$LN = \frac{2}{3}(IM + LI) = \frac{2}{3}(LM).$$

□

## 3. Orthocenter Theorem

Section 2 was devoted mainly to centroids. We now move along the Euler line to another triangle center, the orthocenter. Our main orthocenter theorem uses circles and triangles, so we begin by presenting the associated lemmas that will be helpful later:

**Lemma 3.1.** *(SSS): If two triangles have three pairs of congruent corresponding sides, then the triangles are congruent.*

**Definition 3.2.** A *central angle* is an angle whose vertex is the center of a circle and whose legs, or sides, are radii intersecting the circle in two distinct points.

**Definition 3.3.** An *inscribed angle* is an angle formed by two chords in a circle which have a common endpoint. This common endpoint forms the vertex of the inscribed angle.

**Lemma 3.4.** *An inscribed angle is half the central angle of that arc.*

**Lemma 3.5.** *(Vertical Angle Theorem) Vertical angles are congruent.*

**Lemma 3.6.** *The interior angles of a quadrilateral sum to* 360°.

Theorem 2.7 began with an arbitrary triangle and point. The following orthocenter theorem is initially similar except that it also requires a circumscribed circle of the triangle, which informs the selection of the extra point. The circumscribed circle passes through the vertices of the triangle and has as its center the circumcenter of the triangle.

**Theorem 3.7.** *Given $\triangle ABC$ and a circumscribed circle $\beta$. Let point $D$ be the intersection of a diameter through one of the vertices with $\beta$. Connect each vertex of $\triangle ABC$ to $D$ with segments $\overline{AD}, \overline{BD}$,and $\overline{CD}$. The orthocenters of $\triangle ABD, \triangle ACD$, and $\triangle BCD$ create a triangle that is congruent to $\triangle ABC$.*

*Proof.* Although Figure 5 demonstrates one case of this proof, the reader may refer to that figure for the following general set-up. Begin with $\triangle ABC$ and a circumscribed circle $\beta$. Without loss of generality, use vertex $A$ to create point $D$ on $\beta$ such that $\overline{AD}$ is a diameter of $\beta$. Connect the vertices of $\triangle ABC$ to $D$. We then have $\triangle ABD, \triangle ACD$, and $\triangle BCD$. Because $\overline{AD}$ is a diameter of $\beta$, the arc length of $\widehat{\text{ACD}}$ is 180°. Then by Lemma 3.4, $m\angle ACD$ is 90°.

Since $\triangle ACD$ has right angle $\angle ACD$, and the orthocenter is the intersection of altitudes, its orthocenter is on vertex $C$. Similarly, $\triangle ABD$ has an orthocenter on $B$. Call the orthocenter of $\triangle CBD$ point $Z$. Next, let the intersection of the altitude from $C$ with $\overline{BD}$ be point $Q$. Let the intersection of the altitude from $B$ with $\overline{CD}$ be point $R$.

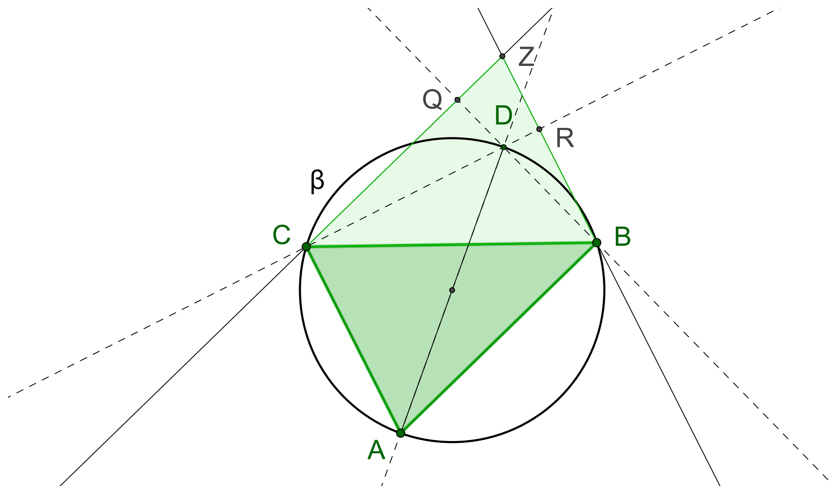



FIGURE 5. Diagram representing Case 1: $\triangle ABC$ is acute.

Case 1: $\triangle ABC$ is acute.

We have $m\angle CQD = 90° = m\angle BRD$ because $\overline{CQ}$ and $\overline{BR}$ are altitudes. Also, $\angle CDQ \cong \angle BDR$ by the Vertical Angle Theorem. By Lemma 2.1 (AA Similarity), $\triangle CDQ \sim \triangle BDR$ and $\angle DCQ \cong \angle DBR$. Call this angle measure $\theta$. Then $m\angle ACZ = 90° + \theta = m\angle ABZ$. Also, because $\overline{BQ}$ is an altitude, $m\angle CQB = 90°$. Similarly, $m\angle CRB = 90°$. In addition, $\angle CDB \cong \angle QDR$ by the Vertical Angle Theorem.

Consider $\square ACDB$ and $\square DRZQ$. From Lemma 3.6, we know

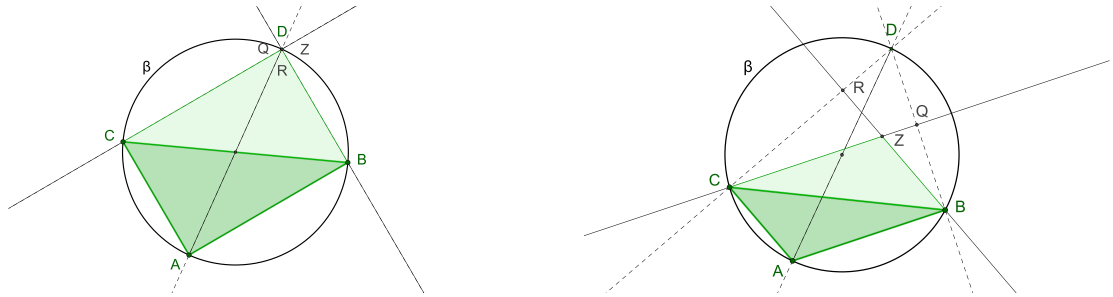
$$m\angle DQZ + m\angle QZR + m\angle ZRD + m\angle RDQ = 360° = m\angle ACD + m\angle CDB + m\angle DBA + m\angle BAC.$$

Since $m\angle ZRD = 90° = m\angle ACD$, $\angle RDQ \cong \angle CDB$, and $m\angle DQZ = 90° = m\angle DBA$, then $\angle QZR \cong \angle BAC$. Recall that $\angle ACZ \cong \angle ABZ$. By Lemma 2.8, since opposite angles are congruent then $\square ABZC$ is a parallelogram. Therefore, $\overline{CZ} \cong \overline{AB}$ and $\overline{AC} \cong \overline{ZB}$. Also, $\overline{CB} \cong \overline{CB}$. Then by SSS, $\triangle ABC \cong \triangle ZCB$.

Case 2: $\triangle ABC$ is right.

Refer to Figure 6a. Without loss of generality, assume $m\angle A = 90°$. Then Lemma 3.4 gives that $\overline{CB}$ is a diameter of $\beta$. Thus $m\angle BDC = 90°$, so the orthocenter of $\triangle BCD$ is at $D$. We know $m\angle ABD = m\angle BDC = m\angle DCA = m\angle CAB = 90°$. Then $\square ABDC$ is a rectangle, so $\overline{AC} \cong \overline{BD}$ and $\overline{AB} \cong \overline{CD}$. Also, $\overline{CB} \cong \overline{CB}$. Then by SSS, $\triangle ABC \cong \triangle DCB$.

Case 3: $\triangle ABC$ is obtuse.




(A) Used for case 2, when $\triangle ABC$ is right. (B) Used for case 3, with obtuse $\triangle ABC$.

FIGURE 6. Figures used in the proof of Corollary 3.7.

Refer to Figure 6b. Without loss of generality, assume the triangle is obtuse at $\angle A$. We know $m\angle ACD = 90° = m\angle ABD$. Additionally, $\angle CZR \cong \angle BZQ$ by the Vertical Angle Theorem. Then by AA, $\triangle CZR \sim \triangle BZQ$, so $\angle RCZ \cong \angle QBZ$. Call this angle measure $\theta$. Then $m\angle ACQ = 90° - \theta = m\angle ABR$. The remainder of the proof parallels case 1.

Therefore, in all cases the orthocenters of $\triangle ABD, \triangle ACD$, and $\triangle BCD$ create a triangle congruent to $\triangle ABC$, namely $\triangle ZCB$. $\square$

## 4. Circumcenter Theorem

Our final theorem uses triangle circumcenters and incenters. Although the incenter is not on the Euler line, it is a significant point in a triangle. We only need a few extra definitions and theorems for the proof of the circumcenter theorem:

**Definition 4.1.** The *incenter* of a triangle is the intersection of the angle bisectors of the triangle. It is also the center of the *incircle*, which is tangent to each side of the triangle.

**Lemma 4.2.** *(ASA) Triangles are congruent if two pairs of corresponding angles and a pair of opposite sides are congruent in both triangles.*

**Lemma 4.3.** *The chords of congruent inscribed angles in congruent circles are congruent.*

**Lemma 4.4.** *Inscribed angles subtended by the same arc are congruent.*

Lemmas 4.3 and 4.4 are a result of the Inscribed Angle Theorem, which states that the measure of an inscribed angle is half of the measure of the arc that it intersects.

**Theorem 4.5.** *Given $\triangle ABC$, its incenter $I$, and circumscribed circle $\alpha$, if lines are made connecting $A, B$, and $C$ to $I$, then the other intersections of those lines and $\alpha$ are the circumcenters of $\triangle CBI, \triangle ACI$, and $\triangle ABI$, respectively.*

*Proof.* Refer to Figure 7. Let the second intersections of $\overline{AI}, \overline{BI}$ and $\overline{CI}$ and $\alpha$ be $X$, $Y$, and $Z$, respectively. We will show that these intersections are the circumcenters of $\triangle CBI, \triangle ACI$, and $\triangle ABI$.

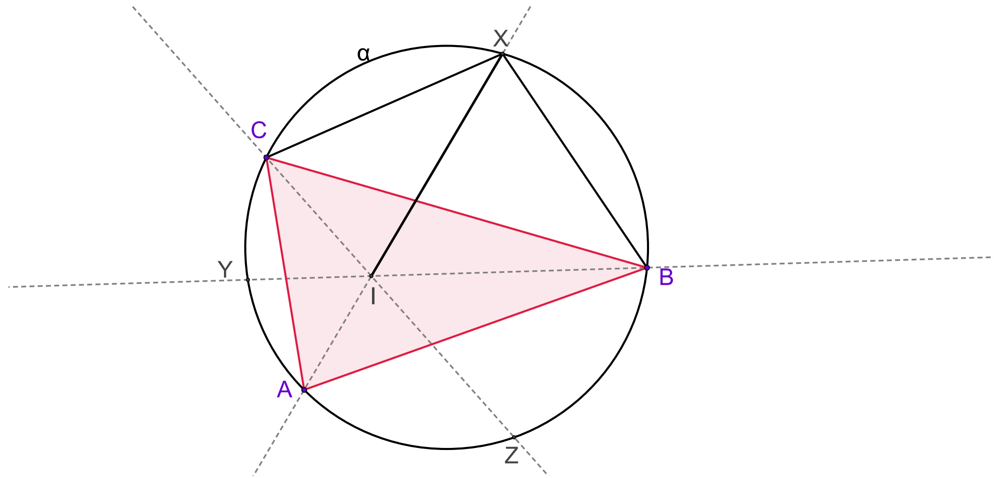



Figure 7. A demonstration of Theorem 4.5.

Consider $\triangle CBI$ and point $X$. Since $\overline{AI}$ is an angle bisector of $\angle CAB$, then $\angle CAX \cong \angle BAX$. By Lemma 4.3, the chords $\overline{CX}$ and $\overline{BX}$ are congruent.

We know that $m\angle AIC = 180° - m\angle ACI - m\angle CAI$. Thus, because $\angle AIC$ and $\angle CIX$ are supplementary angles,

$$\begin{aligned} m\angle CIX &= 180° - (180° - m\angle ACI - m\angle CAI) \\ &= m\angle ACI + m\angle CAI. \end{aligned} \tag{4}$$

Because $\angle BAX$ and $\angle BCX$ share chord $\overline{XB}$, by Lemma 4.4 $\angle BAX \cong \angle BCX$. Then

$$\angle BCX \cong \angle CAX \tag{5}$$

as $\overline{AX}$ is an angle bisector of $\angle BAC$. Also directly from an angle bisector,

$$\angle ACI \cong \angle ICB. \tag{6}$$

Substituting equations (5) and (6) into $m\angle ICX = m\angle ICB + m\angle BCX$ gives

$$m\angle ICX = m\angle ACI + m\angle CAX,$$

which we know from equation (4) is $\angle CIX$.

Thus $\angle ICX \cong \angle CIX$. Then $\triangle CIX$ is isosceles and $\overline{CX} \cong \overline{IX}$.

Finally, we now know that $\overline{IX} \cong \overline{CX} \cong \overline{BX}$. This means that $X$ is the center of a circle circumscribed around $\triangle CBI$, so $X$ is the circumcenter of $\triangle CBI$. Similarly, $Y$ is the circumcenter of $\triangle ACI$ and $Z$ is the circumcenter of $\triangle ABI$. □

## 5. Conclusion

Our journey down the Euler line took us to new discoveries related to the important points on that line. Some of these evoke connections to previously known theorems. Exploring the centroid led to a similar triangle which had sides a third the size of the original triangle. This result is reminiscent of the well-known fact that the medial triangle of $\triangle ABC$, made by connecting the midpoints of each side, is inversely similar to $\triangle ABC$

and creates four congruent triangles that are a fourth the size of the original. From Corollary 2.10 we saw two centroids which were found to be collinear, with one two-thirds the length to the other. This conclusion echoes the two-third distance of the centroid of a triangle along the segment connecting the circumcenter and orthocenter, as proven by Euler. Future studies could uncover additional interesting area and collinearity connections related to the extraordinary centroid.

As we saw in Theorem 3.7, associating the circumcircle of a triangle to orthocenters resulted in a remarkable triangle congruence. In Theorem 4.5, linking the circumcircle to the incenter resulted in other unanticipated circumcenter relations. Further research could explore what additional relationships there are between various centers of a triangle and intersections on the circumcircle. Mathematicians' fascination with this type of geometry has persevered for generations; it is common material in college geometry courses yet there is clearly still more to discover!

## 6. Acknowledgments

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## Student biographies

**Kathryn Lezark:** (*Corresponding author:* kathryn.lezark@valpo.edu) Katie will be a junior mathematics major with a Japanese minor at Valparaiso University in 2016–2017. She currently works as a peer tutor for math and as a volunteer tutor for 2nd and 3rd grade kids. She plans to enter the job market after graduation.