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Abstract. A 2011 paper by Blanco and Rosales describes an algorithm for constructing a directed tree graph of irreducible numerical semigroups of fixed Frobenius numbers. In this paper we show that for odd Frobenius numbers the corresponding tree has a unique longest branch.

1. Introduction

In [1] , Blanco and Rosales present an algorithm so that given any Frobenius number, one can construct a directed tree graph whose vertices are the irreducible numerical semigroups with that Frobenius number. The authors of this paper showed in [3] that for each odd Frobenius number greater than 11 the corresponding tree has a branch that is always constructed the same way. Based on many examples, the authors conjectured that this branch was the unique longest branch in the tree. In this paper, we verify this conjecture and therefore determine a formula for the height of any such tree. In Section 2 we present basic information about numerical semigroups. Section 3 provides an overview of the algorithm of Blanco and Rosales and Section 4 introduces new results.

2. Numerical Semigroups

We begin by presenting some basic information associated with numerical semigroups.

Let $\mathbb{N}$ denote the set of nonnegative integers. A *numerical semigroup* is a set $S \subset \mathbb{N}$ such that $0 \in S$, S is closed under addition, and the complement of S is finite. That is, a numerical semigroup is an additive submonoid S of $\mathbb{N}$ such that $\mathbb{N} \setminus S$ is finite. We say $\{a_1, a_2, \ldots, a_n\}$ is a *generating set* for S if $S = \{k_1 a_1 + k_2 a_2 + \ldots + k_n a_n | k_1, k_2, \ldots, k_n \in \mathbb{N}\}$, and we call each a_i a *generator* of S. If no proper subset is a generating set for S, we say $\{a_1, a_2, \ldots, a_n\}$ is the *minimal generating set* and we write $S = \langle a_1, a_2, \ldots, a_n \rangle$, $0 < a_1 < a_2 < \ldots < a_n$. The elements of the minimal generating set are called *minimal generators.* It is well known in the literature that every numerical semigroup has a unique minimal generating

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set. The *Frobenius number* of a numerical semigroup S, denoted $F(S)$, is the largest integer not in S. The *multiplicity* of S, denoted $m(S)$, is the smallest positive element of S.

Example 2.1. Consider the set of nonnegative integers $\{0,5,7,8,10,12,13,14,15\ldots\}$ This is a numerical semigroup with minimal generating set $\langle 5,7,8\rangle$. Note that $F(S) = 11$ and $m(S) = 5$.

We will be looking at the two following types of numerical semigroups.

Definition 2.2. A numerical semigroup S is said to be *symmetric* if $F(S)$ is odd and if $x \in \mathbb{Z}\setminus S$, then $F(S) - x \in S$.

Definition 2.3. A numerical semigroup S is said to be *pseudo-symmetric* if $F(S)$ is even and if $x \in \mathbb{Z}\setminus S$, then either $x = \frac{F(S)}{2}$ or $F(S) - x \in S$.

Example 2.4. Consider, $S = \langle 5,7,8,9\rangle = \{0,5,7,8,9,10,12,\rightarrow\}$. Here $F(S) = 11$, which is odd, as required. In addition, for every integer $x \notin S$ we have $F(S) - x \in S$ as depicted in Figure 1 below. For example, note that $6 \notin S$, and that we have $11 - 6 = 5 \in S$. We leave it to the reader to verify that for every $x \in \mathbb{Z}\setminus S$, $F(S) - x \in S$. Hence S is symmetric. Figure 1 provides visual motivation for the use of the word symmetric to describe these types of numerical semigroups.

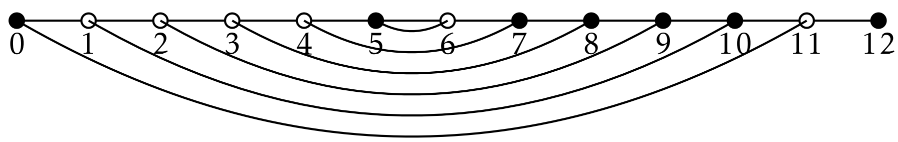

FIGURE 1

Example 2.5. Consider $S = \langle 4,7,9\rangle = \{0,4,7,8,9,11,13 \rightarrow\}$ and note that $F(S) = 10$, which is even. As indicated in Figure 2, for every integer $x \notin S$ we have either $x = \frac{F(S)}{2}$ or $F(S)-x \in S$, and so S is pseudo-symmetric.

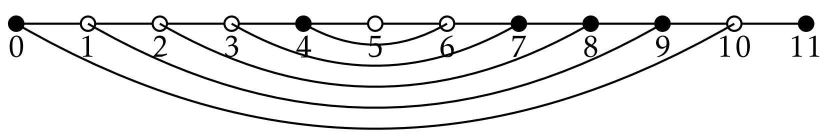

FIGURE 2

Irreducible Numerical Semigroups. We are now prepared to formally define the specific family of numerical semigroups that we have been investigating.

Definition 2.6. A numerical semigroup S is *irreducible* if it cannot be expressed as an intersection of two numerical semigroups that properly contain S. For a given Frobenius number F, the set of all irreducible numerical semigroups with Frobenius number F is denoted $I(F)$.

It has been shown in [4] that a numerical semigroup is irreducible if and only if it is symmetric or pseudo-symmetric. Given a positive integer F, the following theorem of Blanco and Rosales defines a particular element of $I(F)$. This irreducible numerical semigroup plays an important role in the remainder of this paper.

Theorem 2.7. *For every positive integer F, there exists a unique irreducible numerical semigroup $C(F)$ whose generators are all larger than $\frac{F}{2}$. Moreover,*

$$C(F) = \begin{cases} \{0, \frac{F+1}{2}, \rightarrow\} \setminus \{F\} & \text{if } F \text{ is odd,} \\ \{0, \frac{F}{2}+1, \rightarrow\} \setminus \{F\} & \text{if } F \text{ is even.} \end{cases}$$
$$= \begin{cases} \langle \frac{F+1}{2}, \frac{F+3}{2}, \ldots, F-1 \rangle & \text{if } F \text{ is odd,} \\ \langle \frac{F}{2}+1, \frac{F}{2}+2, \ldots, F-1, F+1 \rangle & \text{if } F \text{ is even.} \end{cases}$$

Example 2.8. Using the above theorem we will now construct the unique irreducible numerical semigroup for Frobenius number 17.

$$\begin{aligned} C(17) &= \{0, \frac{17+1}{2}, \rightarrow\} \setminus \{17\} \\ &= \{0, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, \rightarrow\} \setminus \{17\} \\ &= \{0, 9, 10, 11, 12, 13, 14, 15, 16, 18, \rightarrow\} \\ &= \langle 9, 10, 11, 12, 13, 14, 15, 16 \rangle \end{aligned}$$

3. Trees Of Irreducible Numerical Semigroups

The following is the algorithm introduced by Blanco and Rosales in [1] for finding all irreducible numerical semigroups with a given Frobenius number.

Theorem 3.1. *Let F be a positive integer. Then the elements of I(F) comprise a directed tree graph, denoted G(I(F)), with root C(F). If S is an element of I(F), then the children of S are $S \setminus \{x_1\} \cup \{F - x_1\}$, $S \setminus \{x_2\} \cup \{F - x_2\}$, ..., $S \setminus \{x_r\} \cup \{F - x_r\}$), where $\{x_1, \ldots, x_r\}$ is the set of minimal generators of S such that for each $x \in \{x_1, \ldots, x_r\}$ the following conditions are satisfied:*

(1) $\frac{F}{2} < x < F$

(2) $2x - F \notin S$

(3) $3x \neq 2F$

(4) $4x \neq 3F$

(5) $F - x < m(s)$

When a minimal generator, x, of $S \in G(I(F)$ satisfies the five conditions, we say that x *spawns* a child of S. A minimal generator of a given vertex that does not spawn a child is said to be *nonspawning* . Note that condition 5 implies that the spawning generators of a branch must be increasing.

In a directed tree graph, a vertex with no children is called a *leaf*. A *branch* is the shortest path from the root vertex to a leaf. The vertices that appear in a branch will be referred to as the children in the branch. We will sometimes refer to a specific child in a branch by its position in the path. The first child in a branch is the first vertex that appears in the path after the root vertex. In the context of this paper an *odd branch* is a branch whose children are all spawned by odd minimal generators.

Example 3.2. To build $G(I(15))$ we begin by constructing the root, $C(15)$:

$$\begin{aligned} C(11) = \{0, \frac{F(S)+1}{2}, \rightarrow\} \setminus \{F\} &= \{0,8,9,10,11,12,13,14,16,\rightarrow\} \\ &= \langle 8,9,10,11,12,13,14 \rangle. \end{aligned}$$

Using Theorem 3.1, we find that the following minimal generators spawn children of the root $C(15)$ in $G(I(15))$.

$$\begin{aligned} x_1 = 8 &: (\langle 8,9,10,11,12,13,14\rangle \setminus \{8\}) \cup \{7\} = \{0,7,9,10,11,12,13,14,16,\rightarrow\} = \langle 7,9,10,11,12,13\rangle \\ x_2 = 9 &: (\langle 8,9,10,11,12,13,14\rangle \setminus \{9\}) \cup \{6\} = \{0,6,8,10,11,12,13,14,16 \rightarrow\} = \langle 6,8,10,11,13\rangle \\ x_3 = 11 &: (\langle 8,9,10,11,12,13,14\rangle \setminus \{11\}) \cup \{4\} = \{0,4,8,9,10,12,13,14,16,\rightarrow\} = \langle 4,9,10\rangle \end{aligned}$$

Continuing the algorithm in this manner we find that $\langle 7,9,10,11,12,13\rangle$ has a child $\langle 6,7,10,11\rangle$, spawned by 9. Then $\langle 4,6,13\rangle$ is a child of $\langle 6,8,10,11,13\rangle$ spawned by 13, and $\langle 2,17\rangle$ is a child of $\langle 4,6,13\rangle$ spawned by 13. The entire tree $G(I(15))$ is shown in Figure 3 below, with vertices and edges labeled accordingly.

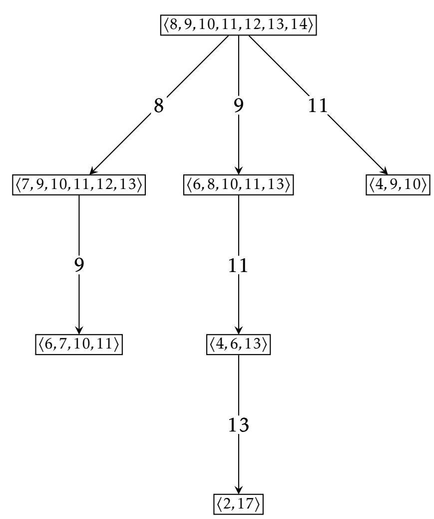

Figure 3

In order to study these trees in more detail, the authors used code in Mathematica to run through the algorithm. Images of the resulting trees were generated using QtikZ and are shown below in Figures 4 – 7.

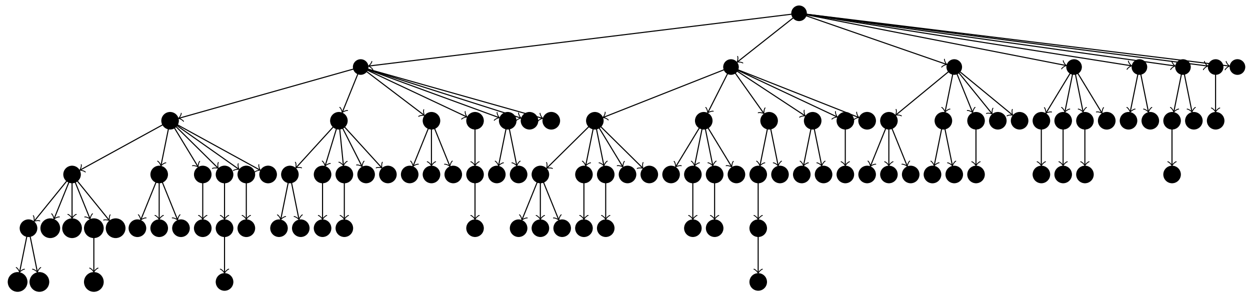

Figure 4. Tree of Frobenius Number 34

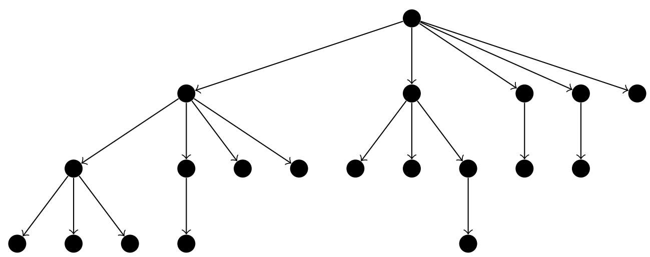

Figure 5. Tree of Frobenius Number 22

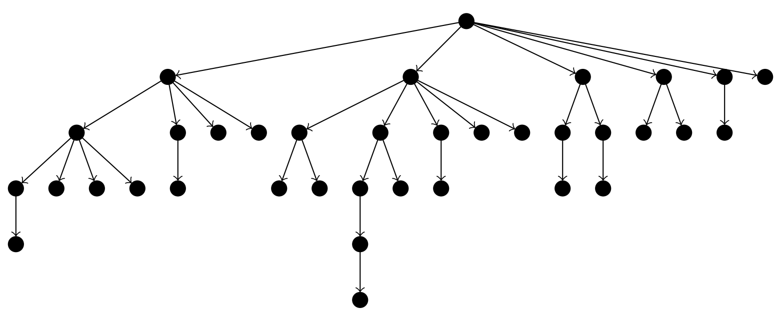

Figure 6. Tree of Frobenius Number 23

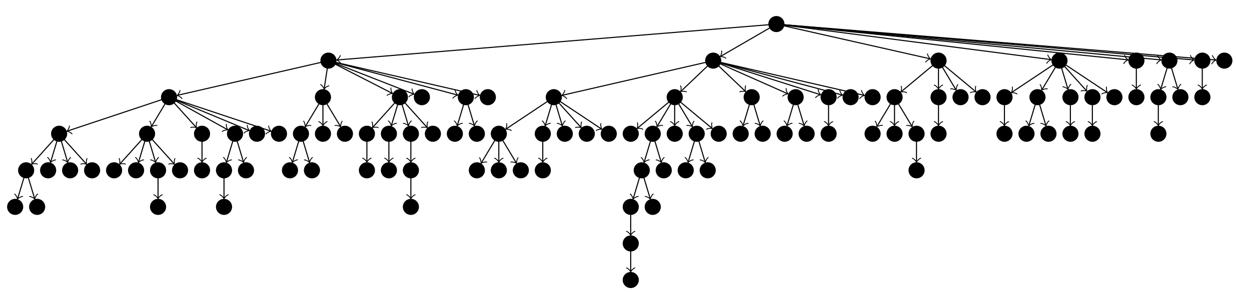

Figure 7. Tree of Frobenius Number 31

We have found that trees associated with even and odd Frobenius numbers - which we will refer to as "even trees" and "odd trees," respectively, differ in structure. Namely, the odd trees (with Frobenius number greater than 11) always have a unique longest branch (this is proven in the next section) and the even trees do not appear to have a longest branch (in the non trivial cases). As mentioned in [3], this infinite family of trees is rather difficult to analyze: as the Frobenius numbers increase, the number of vertices in the trees gets large very quickly, and yet the number of vertices does not monotonically increase. However, when the trees are categorized according to even and odd Frobenius numbers the height of the trees appear to follow a predictable pattern. This is explored further in the next section.

4. New Results

In [3] it is shown that for each odd Frobenius number F, the tree of irreducible semigroups $G(I(F))$ has a branch whose children are spawned by all of the odd minimal generators of $C(F)$. The following results show that this branch is the unique longest branch in $G(I(F))$. We will refer to this branch as the *odd branch* .

Lemma 4.1. *Let $F = 2k+1, k > 5$. For each minimal generator of $C(F)$ that spawns a child in a branch there is a unique minimal generator that cannot spawn a child in the same branch.*

Proof. Let $F = 2k+1$ for some $k > 5$. Then we have $C(F) = \langle k+1, \dots, 2k\rangle$. Let B denote a branch of $G(I(F))$. There are two cases: either $k+1$ spawns the first child in B, or $k+1$ does not spawn a child in B.
Suppose $k+1$ spawns the first child in B. Then $F-(k+1) = k$ is an element of the remaining children in B. Note that each integer $k+1, \dots, 2k$ can be written as the sum of k and exactly one of $1, \dots, k$. Let $1 < n \le k$ and suppose $k+n$ spawns a child in B. Then the integer $F-(k+n) = k+1-n$ is an element of the remaining children in B. Note that $1 \le k+1-n < k$, so $k+1 \le k+(k+1-n) < 2k$, and hence, one of $k+2, \dots, 2k$ is no longer a minimal generator in the remaining children of B. Note that the mapping

$$k+m \mapsto 2k+1-m, \ for \ m \in \{1, \dots, k\}$$

is a bijection from the set of minimal generators to itself. Thus, each spawning minimal generator $k+m$ in a branch has a unique nonspawning minimal generator partner $k+1-m$.
For the second case, suppose $k+1$ does not spawn a child in B. Then every child in B contains $k+1$. Each of $k+2, \dots, 2k$ can be written as the sum of $k+1$ and exactly one of $1, \dots, k-1$. Let $n \in \{1, \dots, k-1\}$ and note that when $k+n \in \{k+2, \dots, 2k\}$ spawns a child in B, then $k+1-n \in \{1, \dots, k-1\}$ is an element of the rest of the children that form B. Let $n \in \{2, \dots, k\}$. Then the mapping

$$k+n \mapsto 2k+2-n$$

is a bijection from $\{k+2, \dots, 2k\}$ to itself. Thus each spawning minimal generator has a unique nonspawning minimal generator partner. □

Corollary 4.2. *Let $F = 2k+1$ for some $k > 5$. Let l be the length of the longest branch of $G(I(F))$. Then $l \le \lfloor \frac{k}{2} \rfloor$.*

Proof. Note that $C(F)$ has k minimal generators. If a minimal generator is its own non-spawning partner, then it cannot spawn. Thus, by the previous lemma, the length of a branch of $G(I(F))$ must be less than or equal to $\lfloor \frac{k}{2} \rfloor$. □

For brevity, we introduce another term. If a minimal generator is made nonspawning in a branch then we say that it is *sterilized* .

Theorem 4.3. *Let $F = 2k+1, k > 5$. The odd branch is the unique longest branch of $G(I(F))$ with length $\lfloor \frac{k}{2} \rfloor$.*

Proof. To prove that the odd branch is the unique longest branch we must show that all other branches have length less that $\lfloor \frac{k}{2} \rfloor$. Let B be a branch of $G(I(F))$. We have two cases to consider: either $k+1$ spawns the fist child in B, or $k+1$ does not spawn a child in B.

Case 1: Suppose $k+1$ spawns the first child in B. Assume $k+2$ also spawns a child in the B. Then both k and $k-1$ are in the remaining children in B. Let $1 < n < \lfloor \frac{k}{2} \rfloor$. Note that the nonspawning minimal generator partner of $k+n$ is $2k+1-n$. Since $k+2$ has spawned a child in B then $2k-n$ is also sterilized when $k+n$ spawns a child. Note that $2k-n$ is the minimal generator partner of $k+n+1$. Hence, if $k+n+1$ does not spawn a child then B has length less than $\lfloor \frac{k}{2} \rfloor$. If $k+n+1$ spawns a child, then the minimal generator partner of $k+n+2$ is sterilized. If the next consecutive minimal generator does not spawn a child then the length of B is less that $\lfloor \frac{k}{2} \rfloor$. Note that if $n = \lfloor \frac{k}{2} \rfloor$, then $2k-n = k+\lfloor \frac{k}{2} \rfloor$. Thus, $k+\lfloor \frac{k}{2} \rfloor$ is nonspawning. Hence, the length of B is less than $\lfloor \frac{k}{2} \rfloor$.

Assume $k+2$ is nonspawning in B. The minimal generator partner of $k+2$ is $2k-1$. Note that $2k-1$ can spawn a child in B if no evens have spawned children. This is because $F-(2k-1) = 2$, and so if an even generator spawns then both $k+2$ and $2k-1$ are sterilized and by the lemma the length of B must be strictly less than $\lfloor \frac{k}{2} \rfloor$. (If no even minimal generators spawn children in B then either B is the odd branch or B is some smaller branch).

It follows that all branches in which $k+1$ spawns a child have length less than $\lfloor \frac{k}{2} \rfloor$, except for the odd branch which has length $\lfloor \frac{k}{2} \rfloor$.

Case 2: Assume $k+1$ does not spawn a child in B. Note that there are then $k-1$ minimal generators that may spawn children. By the Lemma we find that the length of B must be less than or equal to $\lfloor \frac{k-1}{2} \rfloor < \frac{k}{2}$. So the length of a branch in which $k+1$ spawns no children must be less than $\lfloor \frac{k}{2} \rfloor$.

Thus, the odd branch is the unique longest branch of the tree $G(I(F))$. □

Example 4.4. The result of the previous theorem is illustrated in Figure 8, which shows the tree $G(I(23))$ with the children and spawning generators of the longest branch labeled. The root of the tree is $\langle 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22 \rangle$. Note how each of the odd minimal generators of the root spawns a child in the longest branch.

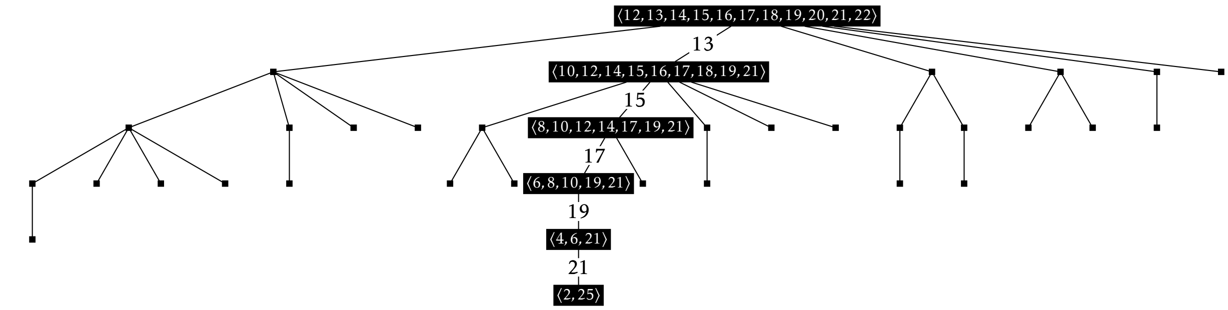

Figure 8

5. Open Questions

An obvious question to ask is "can one produce a formula for the number of vertices in a given tree?" This is, of course, simply a rephrasing of a problem in commutative algebra, namely, determining the number of irreducible numerical semigroups with a given Frobenius number. It has been shown in [2] that the number of irreducible numerical semigroups with odd Frobenius number F has a lower bound of $2^{[F/8]}$ where $[F/8]$ is the integer part of $\frac{F}{8}$.

By examining many examples of the even trees we have found that those trees do not have a unique longest branch. Numerous examples strongly suggest that a formula for the height of the even trees does exist but this has yet to be proven. The interested reader may refer to the last section of [3] for more information.

6. Acknowledgments

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