**$j$-invariant for Example 4.2.**

$$j(\mathcal{E}) = \frac{p(T)}{q(T)} = \frac{p_9\,T^9 + p_8\,T^8 + p_7\,T^7 + p_6\,T^6 + p_5\,T^5 + p_4\,T^4 + p_3\,T^3 + p_2\,T^2 + p_1\,T + p_0}{q_{10}\,T^{10} + q_9\,T^9 + q_8\,T^8 + q_7\,T^7 + q_6\,T^6 + q_5\,T^5 + q_4\,T^4 + q_3\,T^3 + q_2\,T^2 + q_1\,T + q_0}$$

$$p_9 = 16682776400034638205353357277029990400000000i + 17575652563096624654081015917110624256000000$$

$$p_8 = 532845708932067294181143672203276797041822400512000000i - 39996504015525953717021869319748743030664598860595200000$$

$$p_7 = -4305720390879118587546079614014698087918861397185987850671226880000i + 900424139374610416845658476454641943710667259547364453570628812800$$

$$\begin{aligned} p_6 = {} & 8367879412537156263636865160900429819994156204809944649204284277732256251904 00i \\ & + 780372143992097832843879075055363384523364203123765619484445454833105906958336 \end{aligned}$$

$$\begin{aligned} p_5 = {} & 98401656895804474856970242747210360183874579934980209616376583745921070910076315972403200i \\ & - 31140981822083812669719209950843943356251725597374910780656056616908950334064364655476736 0 \end{aligned}$$

$$\begin{aligned} p_4 = {} & 98401656895804474856970242747210360183874579934980209616376583745921070910076315972403200i \\ & - 31140981822083812669719209950843943356251725597374910780656056616908950334064364655476736 0 \end{aligned}$$

$$\begin{aligned} p_3 = {} & 1061758162355628438996339532970167802832113315427475184783037988700353078558499536744265686004312471467531960320 0i \\ & + 1082191718924355337255438891669234836352527942037888425622915560858967277767958049563536088561470606490493517824 0 \end{aligned}$$

$$\begin{aligned} p_2 = {} & 811618328791725820547213725043080885103203039451902675406030649485268654293433791095870866736107056357262838376065872691200i \\ & - 2452082866434256067117164456776684460534074989791465073951891300861934524196987475495609188255118078263513226979873786953728 \end{aligned}$$

$$\begin{aligned} p_1 = {} & -352108966477701036104998863672144784384124970392022007769929641272347337463391211760150311430248695157412380094720545438259614004019200i \\ & + 864139815314387534192660612051714707062602110590051002704665712847800991121909704173914006377328063988075084455268149261460345297305 60 \end{aligned}$$

$$\begin{aligned} p_0 = {} & 14504846888879728994624111945136839823036616281238283164881740620474547681087600177474028698719134410774568612533234283368234202803008818275942400i \\ & + 13056849314776370885812631688884315958794971090029577119737431152995003567524149857913911549101446422781851871128703483558883622177657903458025472 \end{aligned}$$

$$q_{10} = -127791999748249170739200000000$$

$$q_9 = -15803401435599770239582595343974400000000i + 20189185462796824692392053708947456000000$$

$$q_8 = 669706278010287200989273446964308869544463564800000000i + 1083737781469574275417466066183543379661003161600000 0$$

$$q_7 = -40685536485353139279830855210616705894595013500410947174400000 0i - 1868298452746889527687977508153279962879952470643487004426240000$$

$$q_6 = -6453189903488014392104492292547316903782153473293344420274004757472870400 00i + 4298958439674955257765078765710045067608639508510807132522388093434593280 0$$

$$\begin{aligned} q_5 = {} & 800401019625204455841067133899542891692808587609028837679715792294157735653598560256 00i \\ & + 515133644478441448650410649881561578678024071732004927216561250423434468522622378311 68 \end{aligned}$$

$$\begin{aligned} q_4 = {} & -5427258989669652136810186689856518512861081651382069756104061016994997062247012323630172340224000i \\ & - 3400892586046033183853114355239189100274752621385354392293919642047597112025170208465595941257216 \end{aligned}$$

$$\begin{aligned} q_3 = {} & 7342533986610993180539585746411710018238881305267836227553190025944573664130007211458927315577678594048000i \\ & - 1144686895520585156811782890109000971144859538553777918524668248524081233578557795413822575630443817142845 44 \end{aligned}$$

$$\begin{aligned} q_2 = {} & -495028688844584885299097852853103638894534448920018189200377580038017153563525003023710516130923398158907558854656000i \\ & + 774905686157502091131598730170547284671437872106760083858987712415783502098054109768769296555804595013301823910969344 \end{aligned}$$

$$\begin{aligned} q_1 = {} & -408585152159689129907580109426898097228385992946952966990179718715849860326553969273989248453692219119884442421730968862720000i \\ & - 261792733650399562980705704003886839775229678120001115649132840384100534141239347142169858288742692295139240138771931106115584 \end{aligned}$$

$$\begin{aligned} q_0 = {} & 27644522056760910828783482348413588496342717827428777984533895478758086161232232400259026812628790000264259333445552637663527466277273600i \\ & + 177722778390035552510523005701724460504482200488953111023831471747581851175590108636568002366940372906008167611754251464281254377155461 12 \end{aligned}$$

**$j$-invariant for Example 4.3.**

$$j(\mathcal{E}) = \frac{p(T)}{q(T)} = \frac{p_9\,T^9 + p_8\,T^8 + p_7\,T^7 + p_6\,T^6 + p_5\,T^5 + p_4\,T^4 + p_3\,T^3 + p_2\,T^2 + p_1\,T + p_0}{q_{10}\,T^{10} + q_9\,T^9 + q_8\,T^8 + q_7\,T^7 + q_6\,T^6 + q_5\,T^5 + q_4\,T^4 + q_3\,T^3 + q_2\,T^2 + q_1\,T + q_0}$$

$$p_9 = -1203674209337006199645159424\zeta_5^3 + 470942041084292914570780672\zeta_5^2 - 1034969760873268271839698944\zeta_5 - 272969531667632951696109568$$
$$p_8 = -141349726799227690194665472\zeta_5^3 - 615396352327917977680527 36\zeta_5^2 - 493253492332987208843919 36\zeta_5 - 148898570695226392280997888$$
$$p_7 = -30469600108423380020428 8\zeta_5^3 - 35865146889758980584284 16\zeta_5^2 + 20282754940316326284410 88\zeta_5 - 3774827173882125297340416$$
$$p_6 = 9019411224768227447603 2\zeta_5^3 - 786862635733299712819 2\zeta_5^2 + 606061054878014763991 04\zeta_5 + 4787440059685709591347 2$$
$$p_5 = -408942679378677116928\zeta_5^3 - 835487517766352486 4\zeta_5^2 - 247576878472893493248\zeta_5 - 261095350488487649280$$
$$p_4 = -4811775606726991872\zeta_5^3 - 22222473607315931136\zeta_5^2 + 10760403130155073536\zeta_5 - 25196314480061104128$$
$$p_3 = 276482894633799680\zeta_5^3 - 34165485402947584\zeta_5^2 + 191991256075173888\zeta_5 + 136710341577670656$$
$$p_2 = -3710317836828672\zeta_5^3 - 1489124112039936\zeta_5^2 - 1372773255708672\zeta_5 - 3782226124800000$$
$$p_1 = -15778289811456\zeta_5^3 - 37567936069632\zeta_5^2 + 13466842693632\zeta_5 - 47319383015424$$
$$p_0 = -308935655424\zeta_5^3 + 152202903552\zeta_5^2 - 284984082432\zeta_5 - 38730203136$$

$$q_{10} = -1392346457319198515564 48\zeta_5^3 + 50394937999765078507312\zeta_5^2 - 117197528018675407625680\zeta_5 - 35656805474111869647904$$
$$q_9 = -111330183157277384855 04\zeta_5^3 - 48432860478727686749 44\zeta_5^2 - 3887268321671333678528\zeta_5 - 1172386976436761458 8864$$
$$q_8 = -267620248511043235904\zeta_5^3 - 195920699157924560096\zeta_5^2 - 443127584782843957 44\zeta_5 - 361319108815442515904$$
$$q_7 = -7655477639433523840\zeta_5^3 - 10679039429870361280\zeta_5^2 + 1868663953570972608\zeta_5 - 15410384811160354304$$
$$q_6 = 197644809765196704\zeta_5^3 - 178968687167956944\zeta_5^2 + 232759941727336496\zeta_5 - 56817477037454624$$
$$q_5 = 2555466767339136\zeta_5^3 - 1423770951978304\zeta_5^2 + 2459304162104704\zeta_5 + 155594365551808$$
$$q_4 = 119805660949504\zeta_5^3 - 10777468449536\zeta_5^2 + 80704813041664\zeta_5 + 63266502106624$$
$$q_3 = -309233491968\zeta_5^3 + 1354458532864\zeta_5^2 - 1028218281472\zeta_5 + 1163342161408$$
$$q_2 = -9839472640\zeta_5^3 - 30416617472\zeta_5^2 + 12717301760\zeta_5 - 36497563648$$
$$q_1 = 247881728\zeta_5^3 + 309338112\zeta_5^2 - 37998592\zeta_5 + 462569472$$
$$q_0 = 2322432\zeta_5^3 - 1437696\zeta_5^2 + 2322432\zeta_5$$

**Sage code to generate $j$-invariants for Examples 4.2 and 4.3.**

```
##### TO CHANGE THE NUMBER FIELD:
# Choose a number field for K and name the generator.
# EXAMPLES:
#   K = CyclotomicField(7)
#   zeta7 = K.gen()
#
#   The complex numbers
#   K.<i> = NumberField(x^2 + 1)
#   i = K.gen()
#
#   K.<cuberoot2> = NumberField(x^3 - 2)
#   cuberoot2 = K.gen()

K = CyclotomicField(5)
zeta5 = K.gen()

#### TO CHANGE THE ROOTS OF THE DISCRIMINANT
# Choose six _distinct_ elements of K, whose squares are also distinct in K
# For example: rho1 = 1, rho2 = 2, ... rho6 = 6
rho1 = zeta5
rho2 = zeta5^2
rho3 = zeta5^3
rho4 = 1 + zeta5
rho5 = 1 + zeta5^2
rho6 = 1 + zeta5^3

#### DO NOT CHANGE STUFF BELOW
S.<T> = PolynomialRing(K)
U.<x> = PolynomialRing(S)

# DTred is short for 'D_T(x) reduced'
# In the paper this is D_T(x) / A, the discriminant divided by its
# leading coefficient
DTred = (x-rho1^2)*(x-rho2^2)*(x-rho3^2)*(x-rho4^2)*(x-rho5^2)*(x-rho6^2)
R = DTred.coefficients()

# A,B,C,D,a,b,c are constants that determine
# an elliptic curve with equation
#            y^2 = x^3
#                  + (2aT - B)x^2
#                  + (2bT - C)(T^2 + 2T - A + 1)x
#                  + (2cT - D)(T^2 + 2T - A + 1)^2
# For all but finitely many values of T in K, this curve will have rank 6.

A = 64*R[0]^3
c = 8*R[0]^2
b = 4*R[0]*R[1]
a = 4*R[0]*R[2] - R[1]^2
B = R[5]*A - a^2 - 2*b
C = R[4]*A - a^2- 2*b
D = R[3]*A - 2*a*b - 2*c

print "a = ",latex(a)
print "b = ",latex(b)
print "c = ",latex(c)
print "A = ",latex(A)
print "B = ",latex(B)
print "C = ",latex(C)
print "D = ",latex(D)

# Define the elliptic curve with the coefficients as above, and ask for its
# j-invariant. Please don't try to compute the rank of this curve; it will
# never finish.

E = EllipticCurve(S, [0,
                      (2*a*T - B),
                      0,
                      (2*b*T - C)*(T^2 + 2*T - A + 1),
                      (2*c*T - D)*(T^2 + 2*T - A + 1)^2])

#Print out the equation for the elliptic curve, if you would like.
print "E : ",latex(E)


#Compute the j-invariant of the curve
j = E.j_invariant()
#Since the j-invariant is so frequently so gross, a list of coefficients for
#numerator and denominator is nicer.
print "j numerator = ",latex(j.numerator().coefficients())
print "j denominator = ",latex(j.denominator().coefficients())
```