The Emergence of 4-Cycles in Polynomial Maps over the Extended Integers

Andrew Best[1], Patrick Dynes[2], Steven J. Miller[3],
Jasmine Powell[4], and Benjamin Weiss[5]

[1]Ohio State University
[2]Clemson University
[3]Williams College
[4]University of Michigan
[5]University of Maine, Orono

The Emergence of 4-Cycles in Polynomial Maps over the Extended Integers

Andrew Best [1], Patrick Dynes[2], Steven J. Miller[3],
Jasmine Powell[4], and Benjamin Weiss[5]

[1]Ohio State University
[2]Clemson University
[3]Williams College
[4]University of Michigan
[5]University of Maine, Orono

ABSTRACT. Let $f(x) \in \mathbb{Z}[x]$; for each integer α it is interesting to consider the number of iterates n_α, if possible, needed to satisfy $f^{n_\alpha}(\alpha) = \alpha$. The sets $\{\alpha, f(\alpha), \ldots, f^{n_\alpha-1}(\alpha)\}$ generated by the iterates of f are called cycles. For $\mathbb{Z}[x]$ it is known that cycles of length 1 and 2 occur, and no others. While much is known for extensions to number fields, we concentrate on extending $\mathbb{Z}$ by adjoining reciprocals of primes. Let $\mathbb{Z}[1/p_1, \ldots, 1/p_n]$ denote $\mathbb{Z}$ extended by adding in the reciprocals of the n primes $p_1, \ldots, p_n$ and all their products and powers with each other and the elements of $\mathbb{Z}$.

Interestingly, cycles of length 4, called 4-cycles, emerge under the appropriate conditions for polynomials in $\mathbb{Z}[1/p_1, \ldots, 1/p_n][x]$. The problem of finding criteria under which 4-cycles emerge is equivalent to determining how often a sum of four terms is zero, where the terms are ± 1 times a product of elements from the list of n primes. We investigate conditions on sets of primes under which 4-cycles emerge. We characterize when 4-cycles emerge if the set has one or two primes, and (assuming a generalization of the ABC conjecture) find conditions on sets of primes guaranteed not to cause 4-cycles to emerge.

1. INTRODUCTION

1.1. Background and Motivation.
Let R be a ring and $f(x) \in R[x]$ a polynomial over R. If we write

$$f^n(x) := \underbrace{(f \circ f \circ \cdots \circ f)}_{n \text{ times}}(x),$$

then for any fixed $\alpha \in R$ we define the cycle starting at α to be the sequence $(\alpha, f(\alpha), f^2(\alpha), \ldots)$. When this cycle consists of only finitely many distinct elements of R, then α is said to be pre-periodic. We study the cases where α is periodic; that is, where $f^n(\alpha) = \alpha$ for some positive integer n.

---

[1] *Corresponding author*

Definition 1.1. Given a ring R and a polynomial f in $R[x]$, an n-**cycle** (or a **cycle of length** n) is a sequence of n distinct elements of the ring, $(x_1, \ldots, x_n)$, such that

$$f(x_1) = x_2, \quad f(x_2) = x_3, \quad \ldots, \quad f(x_n) = x_1. \tag{1.1}$$

It is well known that when $R = \mathbb{Z}$ the only possible cycle lengths are 1 and 2, both of which occur. For one proof, see [8, Lemma 28]. In more generality, the possible cycle lengths for a polynomial in a number field has been related to the unit group of the ring of integers, see [5]. In his thesis Zieve [8] showed that if $R = \mathbb{Z}_{(2)}$, the localization[1] of $\mathbb{Z}$ at the ideal (2), then the only possible cycle lengths are 1, 2, and 4. It is thus natural to consider rings properly contained between $\mathbb{Z}$ and $\mathbb{Z}_{(2)}$. In particular, we are interested in the rings $\mathbb{Z}[1/p_1, \ldots, 1/p_n]$ which are formed by adjoining the reciprocals of n odd primes $\{p_1, \ldots, p_n\}$ along with all their products and powers with each other and the elements of $\mathbb{Z}$. We call $\{p_1, \ldots, p_n\}$ the ***inversion set*** associated to $\mathbb{Z}[1/p_1, \ldots, 1/p_n]$. Because $\mathbb{Z} \subset \mathbb{Z}[1/p_1, \ldots, 1/p_n]$, these intermediary rings of course have cycles of length 1 and 2.

While it is not known which rings $\mathbb{Z}[1/p_1, \ldots, 1/p_n]$ have polynomials that exhibit 4-cycles, there is an elegant connection between the existence of 4-cycles in a ring $\mathbb{Z}[1/p_1, \ldots, 1/p_n]$ and the solvability of special equations involving products of the primes in its inversion set.

Lemma 1.2. If there is a polynomial in $\mathbb{Z}[1/p_1, \ldots, 1/p_n]$ that exhibits a 4-cycle, then we can write

$$u_1 + u_2 + u_3 + u_4 = 0, \tag{1.2}$$

where $u_i = \pm p_1^{a_{i1}} \cdots p_n^{a_{in}}$ and each a_{ij} is a nonnegative integer. In order to discard pathological examples like $1 - 1 + 1 - 1 = 0$ or $p - p + 1 - 1 = 0$, we also insist that the u_i's have no proper subsum equal to 0.

This fact is a consequence of Corollary 20 in [8], and we now provide a paraphrasing of an explanation from his thesis.

To show the necessity of the existence of such a linear relation suppose that $f \in R[x]$ has the 4-cycle (x_1, x_2, x_3, x_4). As the polynomial $x - y$ divides $f(x) - f(y)$ in $R[x, y]$, we find

$$x_i - x_{i-1} \mid f(x_i) - f(x_{i-1}) = x_{i+1} - x_i, \qquad 1 \le i \le 4. \tag{1.3}$$

From this, we obtain the chain of divisors

$$x_2 - x_1 \mid x_3 - x_2 \mid x_4 - x_3 \mid x_1 - x_4 \mid x_2 - x_1. \tag{1.4}$$

This shows that the pairwise ratios of $u_1 = x_2 - x_1$, $u_2 = x_3 - x_2$, $u_3 = x_4 - x_3$, and $u_4 = x_1 - x_4$ are units in R. Therefore, if an orbit exists, we are guaranteed a sum of units equaling zero.

Note that this condition is not sufficient for there to be a 4-cycle; see Lemma 2.1.

Definition 1.3. We say that a set of primes $\{p_1, \ldots, p_n\}$ **admits a 4-cycle** if we can write $\epsilon_1 u_1 + \epsilon_2 u_2 + \epsilon_3 u_3 + \epsilon_4 u_4 = 0$ with $\epsilon_i \in \{-1, 1\}$ and $u_i = p_1^{a_{i1}} \cdots p_n^{a_{in}}$ where each a_{ij} is a nonnegative integer; we further require that the u_i's have no zero proper subsum. If a set

---

[1]That is, $\mathbb{Z}_{(2)}$ consists of all fractions where the numerator is an integer and the denominator is not divisible by 2.

of primes does not admit a 4-cycle, we say it **avoids a 4-cycle**. Moreover, we say that this set **linearly admits (or avoids) a 4-cycle** if each $a_{ij} \in \{0,1\}$, and in general, we say that this set **admits (or avoids) a 4-cycle with n-powers** if each $a_{ij} \in \{0,1,\dots,n\}$.

We have justified in a natural way the requirements of Definition 1.3 in this section. With only a minor abuse of notation, we apply the same terminology for sets of primes to the ring R.

1.2. **Summary of Main Results.** We first attempt to classify inversion sets of low cardinality by whether they admit 4-cycles. Theorems 1.1 and 1.2 also appear in Narkiewicz [6][Theorem 1 & Theorem 2] with similar proofs (see also Appendix A for another proof and examples).

Theorem 1.1. $\mathbb{Z}[1/p]$ *admits a 4-cycle if and only if $p = 2$ or 3.*

Theorem 1.2. *Fix a positive integer n. An inversion set with two elements admits a 4-cycle if any of the following hold:*

(1) *it is of the form $\{p, p+2\}$, with p and $p+2$ both prime,*

(2) *it is of the form $\{p, p^n - 2\}$, with p and $p^n - 2$ both prime,*

(3) *it is of the form $\{p, 2p+1\}$, with p and $2p+1$ both prime.*

We prove related results for infinite sets, such as Corollary 2.2 (which states that any inversion set with positive upper density not only admits a 4-cycle, but does so linearly).

We then turn to the much harder problem of constructing inversion sets that are proven to avoid 4-cycles. Our main result assumes a generalized ABC conjecture. If we do not assume this conjecture we can prove that certain sets avoid 4-cycles with n powers (i.e., no prime occurs to a power greater than n); see §2.3 for detailed constructions.

Theorem 1.3. *If Conjecture 2.7 is true, then there exist infinitely many pairs of distinct primes p_1 and p_2 such that $\mathbb{Z}\left[\frac{1}{p_1}, \frac{1}{p_2}\right]$ does not have a 4-cycle.*

After proving some useful auxiliary results, we prove the above theorems in §2, and give conditions on the two primes in Theorem 1.3 that, under Conjecture 2.7 holding, ensure there is no 4-cycle. We conclude with a discussion of some future research problems in §3 and some examples in the appendices.

2. Proofs of Main Results

We begin with a result of Zieve that will be useful throughout the paper.

Lemma 2.1. *(Corollary 27, [8]) Let R be an integral domain. There exists a polynomial in $R[x]$ having a 4-cycle in R if and only if there exist units u and v in R for which $u + v$ and $u + 1$ are associates, and for which $1 + u + v$ is a unit.*

This leads us to a partial reformulation of the problem of characterizing sets of primes that admit a 4-cycle.

Proposition 2.2. *If the set of primes $\{p_1, \ldots, p_n\}$ does not admit a 4-cycle, then the ring $\mathbb{Z}[1/p_1, \ldots, 1/p_n]$ has no polynomial with a 4-cycle.*

Proof. By means of contraposition, assume that $R = \mathbb{Z}[1/p_1, \ldots, 1/p_n]$ has a polynomial with a 4-cycle. Then by Lemma 2.1 we know that there exist units $u, v, w \in \mathbb{R}$ such that $1 + u + v = w$. Units in R are of the form $p_1^{a_1} \cdots p_n^{a_n}$ with $a_1, \ldots, a_n \in \mathbb{Z}$. We multiply through to eliminate negative exponents on the primes, which yields an equation of the form

$$\epsilon_1 t_1 + \epsilon_2 t_2 + \epsilon_3 t_3 + \epsilon_4 t_4 = 0, \tag{2.1}$$

where $\epsilon_i \in \{-1, 1\}$ and $t_i = p_1^{a_{i1}} \cdots p_n^{a_{in}}$ with each a_{ij} a positive integer. Therefore, we see that $\{p_1, \ldots, p_n\}$ admits a 4-cycle. $\qquad\square$

2.1. Singleton Inversion Sets.

We turn to the proof of Theorem 1.1, which states a singleton inversion set $\mathbb{Z}[1/p]$ admits a 4-cycle if and only if $p = 2$ or 3.

Proof of Theorem 1.1. First, consider the case when $p = 2$. Let $R = \mathbb{Z}[1/2], u = 2, v = 1$. Then, by Lemma 2.1, $\mathbb{Z}[1/2]$ admits a 4-cycle.

Next, consider the case when $p = 3$. Letting $R = \mathbb{Z}[1/3]$ and $u = v = 1$, by Lemma 2.1 we now have that $\mathbb{Z}[1/3]$ admits a 4-cycle.

Otherwise, let $p > 3$ be a prime. We know that $\mathbb{Z}[1/p]$ admits a 4-cycle if and only if there exist values of a_i such that

$$\pm p^{a_1} \pm p^{a_2} \pm p^{a_3} \pm p^{a_4} = 0 \tag{2.2}$$

and this equation admits no zero proper subsum.

By multiplying by the appropriate power of p, namely $p^{-\min a_i}$, we rewrite the equation as

$$1 \pm p^{b_1} \pm p^{b_2} \pm p^{b_3} = 0, \tag{2.3}$$

where $b_1, b_2, b_3 \geq 0$ and at least one sign is negative. Note that, disregarding solutions to this equation that admit a zero proper subsum, looking at this equation mod p we have either

$$1 \equiv 0, \; 2 \equiv 0, \; \text{or } 3 \equiv 0 \bmod p, \tag{2.4}$$

depending on the number of $b_i = 0$. However, since $p > 3$, this is a contradiction. Therefore $\mathbb{Z}[1/p]$ admits a 4-cycle if and only if $p = 2$ or 3. $\qquad\square$

Example 2.3. The polynomial $f(x) = -\frac{2}{3}x^3 + 4x^2 - \frac{19}{3}x + 5$ has the 4-cycle $(1, 2, 3, 4)$ in $\mathbb{Z}[1/3]$.

The proposition above answers for almost all rings $\mathbb{Z}[1/p]$ (except $p = 2$) the question of which cycle lengths are allowed. The case of $p = 2$ is handled completely by Narkiewicz [6][Lemmas 5 & 6]. In addition, Narkiewicz provides every example of a polynomial in $\mathbb{Z}[\frac{1}{2}]$ with a 4-cycle. For additional works and other methods of proofs for some of these results, see Appendix A of [1].

2.2. Other Inversion Sets Admitting 4-Cycles.

As soon as we consider slightly larger inversion sets, say of cardinality 2, the picture turns murky. Using our reformulation of the problem in the introduction, it is often a matter of algebra to find inversion sets with

special structure that admit 4-cycles. We give three examples in Theorem 1.2 (restated below for convenience), with full explanation for the first, and suggest others.

Theorem 1.2 (Partial Classification of Doubleton Inversion Sets) *Fix a positive integer n. An inversion set with two elements admits a 4-cycle if any of the following hold:*

(1) *it is of the form $\{p, p+2\}$, with p and $p+2$ both prime,*

(2) *it is of the form $\{p, p^n - 2\}$, with p and $p^n - 2$ both prime,*

(3) *it is of the form $\{p, 2p+1\}$, with p and $2p+1$ both prime.*

Proof of (1). Using our reformulation, $\{p, p+2\}$ admits a 4-cycle if we can write

$$u_1 + u_2 + u_3 + u_4 = 0 \tag{2.5}$$

with $u_i = \pm p^{a_{i1}}(p+2)^{a_{i2}}$, each a_{ij} a nonnegative integer, and the set of u_i's has no zero proper subsum.

Write

$$\begin{aligned}
u_1 &= p+2 \\
u_2 &= -1 \\
u_3 &= -1 \\
u_4 &= -p.
\end{aligned} \tag{2.6}$$

The result follows. $\square$

Similar proofs for the other cases are given in Appendix A of [1].

To actually construct a polynomial $f(x)$ in $\mathbb{Z}[1/p, 1/(p+2)][x]$ that admits a 4-cycle, simply choose a 4-cycle (x_1, x_2, x_3, x_4) with appropriate step sizes u_i. Using Lagrange interpolation with $f(x_1) = x_2$, and so on, one can construct f.

Example 2.4 (Example of (1)). Consider the polynomial

$$f(x) = -\frac{2}{35}x^3 - \frac{4}{35}x^2 + \frac{221}{35}x + \frac{101}{7} \in \mathbb{Z}\left[\frac{1}{5}, \frac{1}{7}\right][x]. \tag{2.7}$$

It is easy to verify that f has the 4-cycle $(-10, -3, -4, -9)$. This example also shows that the step sizes u_i may be reordered.

Other interesting inversion sets of size 2 that admit 4-cycles exist. There are also larger inversion sets that admit 4-cycles. In fact, it is trivial to find inversion sets of arbitrary size that admit 4-cycles: Simply add in primes to an inversion set of size 2 that already admits 4-cycles. For example, to get an inversion set of size 3 that admits 4-cycles, you might consider the polynomial f in Example 2.4 viewed as an element of the ring $\mathbb{Z}[1/5, 1/7, 1/37][x]$. Thus, the bulk of our paper is dedicated to investigating inversion sets that **avoid** 4-cycles, as this problem is more interesting.

2.3. **Separations and Avoiding 4-Cycles.** The following theorem from Green is the starting point of our investigations of inversion sets avoiding 4-cycles.

Theorem 2.5 (Theorem 1.4, [4]). *Write $\mathcal{P}$ for the set of primes. Every subset of $\mathcal{P}$ of positive upper density contains a 3-term arithmetic progression.*

Green's result immediately implies the following useful characterization.

Proposition 2.1. If a set of primes does not linearly admit a 4-cycle, then it has density 0 in the primes.

Proof. Suppose we have a set of primes with a 3-term arithmetic progression; that is, we have distinct primes p_1, p_2, p_3 such that $p_2 = p_1 + a$ and $p_3 = p_2 + 2a$ where $a \in \mathbb{Z}$. Then

$$p_3 - p_2 - p_2 + p_1 \;=\; p_1 + 2a - (p_1 + a) - (p_1 + a) + p_1 \;=\; 0, \tag{2.8}$$

and this set of primes linearly admits a 4-cycle. Thus we have that if a set of primes does not linearly admit a 4-cycle, then it contains no arithmetic progressions. Then, by Theorem 2.5, we have that any set of primes that does not linearly admit a 4-cycle has density 0 in the primes. □

Note that this does not guarantee that $u + v$ and $u + 1$ will be associates, to satisfy the conditions of Lemma 2.1.

Corollary 2.2. Every subset of $\mathcal{P}$ of positive upper density must linearly admit a 4-cycle.

We now construct sets of primes that linearly avoid 4-cycles. We consider equations of the form

$$\epsilon_1 t_1 + \epsilon_2 t_2 + \epsilon_3 t_3 + \epsilon_4 t_4 \;=\; 0, \quad \epsilon_i \in \{-1, 1\}, \quad t_i \;=\; p_1^{a_{i1}} \cdots p_n^{a_{in}}, \quad a_{ij} \in \{0, 1\} \tag{2.9}$$

and discount trivial solutions, that is, instances where the set $\{\epsilon_i t_i\}$ contains a proper subsum equal to 0, as discussed in Definition 1.3.

Without loss of generality, let the term t_1 be maximal in the set of terms $\{t_1, t_2, t_3, t_4\}$. If $t_1 > t_2 + t_3 + t_4$ then it is easy to see that $\epsilon_1 t_1 + \epsilon_2 t_2 + \epsilon_3 t_3 + \epsilon_4 t_4 \neq 0$ for all choices of ϵ_i.

Lemma 2.6 (Separation Lemma). *Suppose we have an ordering of positive terms $\{x_1, x_2, \ldots, x_n\}$ such that $x_{i-1} < x_i$ holds for each $2 \leq i \leq n$. Also suppose that given an x_i, for any three distinct terms $x_{j_1}, x_{j_2}, x_{j_3} < x_i$ we have $x_i > x_{j_1} + x_{j_2} + x_{j_3}$. Then there is no set of t_i's that non-trivially satisfy*

$$\epsilon_1 t_1 + \epsilon_2 t_2 + \epsilon_3 t_3 + \epsilon_4 t_4 \;=\; 0, \tag{2.10}$$

where $t_i \in \{x_i\}$ and $\epsilon_i = \{-1, 1\}$.

Proof. It suffices to show the result holds for the ordering of terms $\{x_1, x_2, x_3, x_4\}$ such that $x_{i-1} < x_i$ for each $2 \leq i \leq 4$ and $x_4 > x_1 + x_2 + x_3$. In this case, let $t_i = x_i$ for $1 \leq i \leq 4$. Then $t_1 + t_2 + t_3 - t_4 < 0$. It is now easy to see that $\epsilon_i t_i + \epsilon_i t_i + \epsilon_i t_i + \epsilon_i t_i \neq 0$ for all choices of ϵ_i. □

With Lemma 2.6, we characterize a set of primes which linearly avoids a 4-cycle in the next result. In particular, we note that the conditions of the lemma are satisfied if $x_i > 3x_{i-1}$.

Proposition 2.3. Fix a positive integer k, and let $p_1 > 3$, $p_j > 3 \prod_{i=1}^{j-1} p_i$ for $2 \leq j \leq k$ be prime. Then the inversion set $\{p_1, p_2, \ldots, p_k\}$ linearly avoids a 4-cycle.

Proof. We establish that for any two terms s, t drawn from the set of products $\{p_1^{\alpha_1} \cdots p_n^{\alpha_n} : \alpha_i \in \{0, 1\}\}$, if $s < t$ then $3s < t$, giving enough separation to linearly avoid a 4-cycle.

We proceed by induction on the number of primes in the inversion set. If we only have one prime $p > 3$, then $\{p\}$ linearly (and in fact generally by Proposition 1.1) avoids a 4-cycle. Now suppose that a set of k primes $\{p_1, \ldots, p_k\}$ with the separation properties above linearly avoids a 4-cycle. Consider adding in another prime p_{k+1}, satisfying $p_{k+1} > 3p_1 \cdots p_k$. Let

$$S := \{p_1^{\alpha_1} \cdots p_k^{\alpha_k} : \alpha_i \in \{0, 1\}\} \tag{2.11}$$

denote the set of products for which the induction hypothesis applies, and let

$$S^* := \{p_1^{\alpha_1} \cdots p_{k+1}^{\alpha_{k+1}} : \alpha_i \in \{0, 1\}\} \tag{2.12}$$

denote the set of products extended with the possibility of a p_{k+1} factor.

Claim 2.4. For all $s, t \in S^*$, if $s < t$ then $3s < t$.

To see this, choose $s = p_1^{\alpha_1} \cdots p_{k+1}^{\alpha_{k+1}}$ and $t = p_1^{\beta_1} \cdots p_{k+1}^{\beta_{k+1}}$ in S^* such that $s < t$.

First, if $\alpha_{k+1} = 0$ and $\beta_{n+1} = 0$, then $s, t \in S$, so that $3s < t$ by hypothesis.

Next, if instead we have $\alpha_{k+1} = 1$ and $\beta_{k+1} = 1$, then $s/p_{k+1}, t/p_{k+1} \in S$ with $s/p_{k+1} < t/p_{k+1}$, so that by hypothesis we have $3s/p_{k+1} < t/p_{k+1}$, so that $3s < t$.

Further, suppose $\alpha_{k+1} = 1$ and $\beta_{k+1} = 0$. Then $s \geq p_{k+1} > 3p_1 \cdots p_k > t$, which contradicts $s < t$.

Finally, suppose that $\alpha_{k+1} = 0$ and $\beta_{k+1} = 1$. Then we have that $3s \leq 3p_1 \cdots p_k < p_{k+1} \leq t$, so that $3s < t$. The claim follows.

Therefore, following the discussion above, for any k, provided the separation conditions are met, the inversion set $\{p_1, p_2, \ldots, p_k\}$ linearly avoids a 4-cycle. $\square$

We have discussed the case in which the powers of primes are restricted to first powers (linear avoidance). We now consider the case where we allow powers of primes up to some integer n. The ideas behind the proof are very similar.

Proposition 2.5. Fix positive integers n and k, and let $p_1 > 3$, $p_j > 3 \prod_{i=1}^{j-1} p_i^n$ for $2 \leq j \leq k$ be prime. Then the inversion set $\{p_1, p_2, \ldots, p_k\}$ avoids a 4-cycle with n-powers.

Proof. We establish that for any two terms s, t drawn from the set of products $\{p_1^{\alpha_1} \cdots p_n^{\alpha_n} : \alpha_i \in \{0, 1, \ldots, n\}\}$, if $s < t$ then $3s < t$, giving enough separation to avoid a 4-cycle with n-powers.

We proceed by induction on the number of primes in the inversion set. If we only have one prime $p > 3$, then $\{p\}$ avoids a 4-cycle by Proposition 1.1. Now suppose that a set of k primes $\{p_1, \ldots, p_k\}$ with the separation properties above avoids a 4-cycle with n-powers. Consider adding in another prime p_{k+1}, satisfying $p_{k+1} > 3p_1^n \cdots p_k^n$. Let

$$S := \{p_1^{\alpha_1} \cdots p_k^{\alpha_k} : \alpha_i \in \{0, 1, \ldots, n\}\} \tag{2.13}$$

denote the set of products for which the induction hypothesis applies, and let

$$S^* := \{p_1^{\alpha_1} \cdots p_{k+1}^{\alpha_{k+1}} : \alpha_i \in \{0, 1, \ldots, n\}\} \tag{2.14}$$

denote the set of products extended with the possibility of a p_{k+1} factor.

Claim 2.6. For all $s, t \in S^*$, if $s < t$ then $3s < t$.

To see this, choose $s = p_1^{\alpha_1} \cdots p_{k+1}^{\alpha_{k+1}}$ and $t = p_1^{\beta_1} \cdots p_{k+1}^{\beta_{k_1}}$ in S^* such that $s < t$.

First, if $\alpha_{k+1} = 0$ and $\beta_{k+1} = 0$, then $s, t \in S$, so that $3s < t$ by hypothesis.

Next, if instead we have $\alpha_{k+1} = \beta_{n+1} \neq 0$, then $s/p_{k+1}^{\alpha_{k+1}}, t/p_{k+1}^{\alpha_{k+1}} \in S$ with $s/p_{k+1}^{\alpha_{k+1}} < t/p_{k+1}^{\alpha_{k+1}}$, so that by hypothesis we have $3s/p_{k+1}^{\alpha_{k+1}} < t/p_{k+1}^{\alpha_{k+1}}$, so that $3s < t$.

Further, suppose $\alpha_{k+1} > \beta_{k+1}$. Then we have

$$s \;\geq\; p_{k+1}^{\alpha_{k+1}} \;>\; 3p_1^n \cdots p_k^n p_{k+1}^{\alpha_{k+1}-1} \;\geq\; 3p_1^n \cdots p_k^n p_{k+1}^{\beta_{k+1}} \;>\; p_1^n \cdots p_k^n p_{k+1}^{\beta_{k+1}} \;\geq\; t, \tag{2.15}$$

which contradicts $s < t$.

Finally, suppose that $\alpha_{k+1} < \beta_{k+1}$. Then we have

$$3s \;\leq\; 3p_1^n \cdots p_k^n p_{k+1}^{\alpha_{k+1}} \;<\; p_{k+1}^{\alpha_{k+1}+1} \;\leq\; p_{k+1}^{\beta_{k+1}} \;\leq\; t, \tag{2.16}$$

so that $3s < t$.

Therefore, following the discussion above, for any k and n, provided the separation conditions are met, the inversion set $\{p_1, p_2, \ldots, p_k\}$ avoids a 4-cycle with n-powers. □

Up until now, we have restricted the powers we allow on primes when constructing sets that avoid 4-cycles to some degree. We remove this restriction of possible powers after first proving the following helpful lemma.

Lemma 2.7. *Let $m > 7$ be an integer. Fix primes p_1 and p_2 such that for all non-negative integers k and all integers ℓ with $0 \leq \ell < m$ it holds that*

$$p_1^{\ell+k} \;<\; p_1^k p_2^\ell \;<\; \frac{1}{3} p_1^{\ell + \frac{\ell}{m} + k}. \tag{2.17}$$

Then for any non-negative integers $\ell, k, s,$ and t satisfying

(1) $0 \leq t < m$;

(2) $0 \leq \ell < m$;

(3) $t(1 + \frac{1}{m}) + s > \ell(1 + \frac{1}{m}) + k$,

it holds that $p_1^s p_2^t > p_1^k p_2^\ell$.

Proof. The lemma is clear whenever $t + s > \ell\left(1 + \frac{1}{m}\right) + k$, as $p_1^s p_2^t > p_1^{t+s} > p_1^{\ell\left(1+\frac{1}{m}\right)+k} > p_1^k p_2^\ell$.

Otherwise, we must have that $t(1 + \frac{1}{m}) + s > \ell(1 + \frac{1}{m}) + k \geq t + s$. Since both $\frac{t}{m}$ and $\frac{\ell}{m}$ are less than 1, it must be that $t + s = k + \ell$. In this case, since $t(1 + \frac{1}{m}) + s > \ell(1 + \frac{1}{m}) + k$, we have that $t/m > \ell/m$ and so $t > \ell$. Therefore, $k = s + t - \ell$. Thus,

$$p_1^s p_2^t \;=\; p_1^s p_2^{t-\ell} p_2^\ell > p_1^s p_1^{t-\ell} p_2^\ell \;=\; p_1^k p_2^\ell. \tag{2.18}$$

□

For our main theorems, we need to assume the following generalization of the ABC conjecture.

Conjecture 2.7. *[Browkin-Brzezinski] Given an integer $n > 2$ and an $\epsilon > 0$, there exists a constant $C_{n,\epsilon}$, such that for all integers $a_1,\ldots,a_n$ with $a_1 + \cdots + a_n = 0$ (and no proper subset having a zero sum), and $\gcd(a_1,\ldots,a_n) = 1$ we have*

$$\max(|a_1|,\ldots,|a_n|) \leq C_{n,\epsilon}(\mathrm{rad}(a_1 \times \cdots \times a_n))^{2n-5+\epsilon}, \tag{2.19}$$

where $\mathrm{rad}(n)$ is the product of the distinct prime factors of n.

Proof of Theorem 1.3. We may assume without loss of generality that $p_2 > p_1$, and that $a_1 + a_2 + a_3 + a_4 = 0$ is a sum of powers of p_1 and p_2 which vanish, and no sum with fewer terms is zero. Our proof deals with two cases: the first has at least one summand divisible by a large power of p_2, and the second assumes that all four summands are only divisible by a small number of multiples of p_2.

In applying the conjecture, we are concerned about the case when $n = 4$, and we fix $\epsilon = 1$ for simplicity. Note that each a_i is a product of powers of p_1 and p_2 and so $\mathrm{rad}(a_1 \times \cdots \times a_n) = p_1 p_2$. We then get

$$\max(|a_1|,|a_2|,|a_3|,|a_4|) \leq C_{4,1}(p_1 p_2)^4. \tag{2.20}$$

Choose $m \in \mathbb{Z}$ such that $m > \max\{8, \log_3(C_{4,1})\}$ and choose primes p_1 and p_2 such that the following conditions are satisfied:

(1) $p_1 \geq 18^m$;

(2) $p_2 \in \left(3p_1, \frac{1}{3}p_1^{1+\frac{1}{m}}\right)$.

We first show that we can always find a prime satisfying Condition 2. From Condition 1 we have that $p_1 \geq 18^m$, which we can obviously satisfy for any choice of m. We then get

$$p_1 \geq 18^m \quad \Rightarrow \quad \frac{1}{3}p_1^{1+\frac{1}{m}} \geq 2 \cdot 3p_1. \tag{2.21}$$

By Bertrand's postulate (see for example [3]) there is always a prime in $(x, 2x)$ for all $x > 1$. Thus we see that we can find a prime $p_2 \in (3p_1, 2 \cdot 3p_1) \subseteq \left(3p_1, \frac{1}{3}p_1^{1+\frac{1}{m}}\right)$.

We first deal with the case where at least one summand is divisible by a large power of p_2; in particular, we define large to be greater than m.

To show that any term divisible by p_2^k where $k \geq m$ cannot be a part of a 4-term sum, we only need to show that $p_2^m > C_{4,1}(p_1 p_2)^4$.

We required that $m > \max\{8, \log_3(C_{4,1})\}$. This gives us that $C_{4,1} < 3^m$, because $m > \log_3(C_{4,1})$; it also gives that $\frac{8m+4}{m^2} < 1$, because m is at least 9. Combining these inequalities, we see that

$$C_{4,1}^{\frac{1}{m}} p_1^{\frac{8m+4}{m^2}} < 3p_1. \tag{2.22}$$

Finally, since $3p_1 < p_2$, we get that

$$p_2 > C_{4,1}^{\frac{1}{m}} p_1^{\frac{8m+4}{m^2}} \tag{2.23}$$

and so

$$\begin{aligned} p_2^m &\geq C_{4,1} p_1^{\left(8+\frac{4}{m}\right)} \\ &> C_{4,1} p_1^4 p_1^{4\left(1+\frac{1}{m}\right)} \\ &> C_{4,1}(p_1 p_2)^4, \tag{2.24} \end{aligned}$$

the last substitution coming from the fact that $p_2 < p_1^{1+\frac{1}{m}}$. Thus if a 4-term sum exists, it cannot have any term divisible by p_2^m.

Next, we invoke Lemma 2.7 to show that when all the terms are divisible only by powers of p_2 that are less than m, we have enough separation between possible products of powers of primes to make the sum impossible.

To see this, we need to show, letting $S = \{p_1^k p_2^\ell | k \in \mathbb{Z}_{\geq 0}, 0 \leq \ell < m\}$, that for all $a, b \in S$ if $a < b$ then $3a < b$.

Our lemma gives us an ordering on the elements – now that we have this ordering, we only need to verify the two following cases.

(1) $3p_1^k p_2^\ell < p_1^{k-1} p_2^{\ell+1}$

(2) $3p_2^\ell < p_1^{\ell+1}$.

Case 1 corresponds to the case in which $a = p_1^k p_2^\ell$ and $b = a\frac{p_2}{p_1}$. By our original conditions, we have that $\frac{p_2}{p_1} > 3$ and therefore $3a < b$.

In Case 2 we have $a = p_2^\ell$ and $b = p_1^{\ell+1}$, and by our initial conditions, since $\ell \leq m$, we see that

$$p_2^\ell < \frac{1}{3} p_1^{\ell+\frac{\ell}{m}} \tag{2.25}$$

so

$$3p_2^\ell < p_1^{\ell+1}. \tag{2.26}$$

Therefore, for all $a, b \in S$, if $a < b$ then $3a < b$.

With all of these conditions in place, we then see that $\mathbb{Z}[1/p_1, 1/p_2]$ does not have a 4-cycle. $\qquad \square$

2.4. **Numerics.** We end by examining patterns that occur when counting the number of cycles for a given prime list. Based on our observations and the formulation of our results, we conjecture that the number of cycles that occur when considering a specific list of primes correlates with the spacing between the primes. Intuitively, if the primes are spaced far apart, the likelihood of them "interacting" in a way that gives a cycle – that is, finding some combination of four products of the primes that sums to zero – is small. We examine this conjecture through computation and find the pattern to hold.

Figure 1 gives a plot of the number of cycles based on the minimum gap between primes in the inversion set. The points are based on lists of five of the first 50 primes. For example, the inversion set $\{37, 73, 83, 127, 157\}$ admits two cycles. The minimum gap associated to this list is 10. In the plot, the size of the point at any given position represents the number of lists associated with that gap and number of cycles.
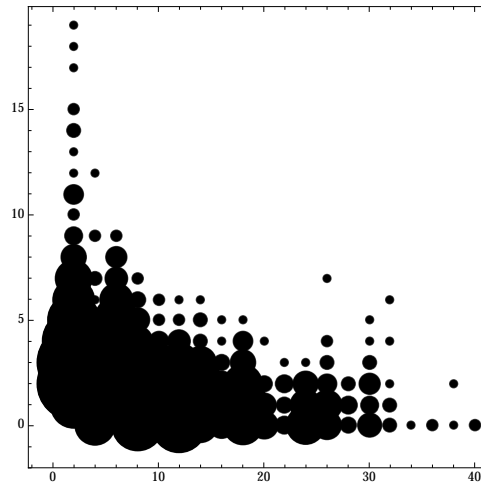

Figure 1. Minimum gap between inversion set vs. number of cycles, with size of points reflecting density of the data.

3. Future Work

In terms of the main result, there are at least two directions in which to proceed. First, we would like to extend Theorem 1.3 to sets of n primes. The main difficulty with extending our method of proof is constructing a set of primes so that both methods in the proof still apply. In particular, a generalization of Lemma 2.7 would be needed. Second, we would like to eliminate the dependence of Theorem 1.3 on the generalized ABC conjecture given as Conjecture 2.7.

In Section 2.2 we considered particular shapes of doubleton inversion sets that admit 4-cycles. Related questions of the following flavor suggest themselves: Given an inversion set consisting of a particular odd prime p, what is the minimal number of primes we need to add to the set to ensure that it admits a 4-cycle? Per the results in that subsection, this answer might usually be one (if p happens to be a twin prime, for example), but occasionally it might be two. Or, if we have an inversion set with two primes that is known to avoid 4-cycles, how "easy" is it to introduce 4-cycles by adding primes to it? (Maybe such sets happen to be difficult to disrupt in this way, and require essentially grafting an entire inversion set that is known to work, such as a pair of twin primes, or maybe not.) These and other questions would be interesting to consider further.

Appendix A. Proofs and Examples of Theorem 1.2

Proof of Theorem 1.2 (2). Using our reformulation, we write

$$u_1 = p^n - 2$$
$$u_2 = 1$$
$$u_3 = 1$$
$$u_4 = -p^n. \tag{A.1}$$

It is clear that $u_1 + u_2 + u_3 + u_4 = 0$ and there are no zero proper subsums of u_i's, so that $\{p, p^n - 2\}$ admits a 4-cycle. □

We give an example of this case when $n = 2$ and $p = 5$.

Example A.1 (Example of (2)). Consider the polynomial

$$g(x) = -\frac{2}{575}x^3 + \frac{112}{115}x^2 + \frac{3127}{575}x - \frac{16019}{115} \in \mathbb{Z}\left[\frac{1}{5}, \frac{1}{23}\right][x]. \tag{A.2}$$

It is easy to verify that g has the 4-cycle $(-14, -15, 10, 9)$. This example also shows that the step sizes u_i can appear with all polarities reversed, too.

Proof of Theorem 1.2 (3). Using our reformulation, we write

$$u_1 = 2p + 1$$
$$u_2 = -p$$
$$u_3 = -p$$
$$u_4 = -1. \tag{A.3}$$

It is clear that $u_1 + u_2 + u_3 + u_4 = 0$ and the set of u_i's has no zero proper subsum, so that $\{p, 2p + 1\}$ admits a 4-cycle. □

Example A.2 (Example of (3)). Consider the polynomial

$$h(x) = -\frac{2}{11}x^3 - \frac{146}{55}x^2 - \frac{39}{5}x + 7/11 \in \mathbb{Z}\left[\frac{1}{5}, \frac{1}{11}\right][x]. \tag{A.4}$$

It is easy to verify that h has the 4-cycle $(-10, -5, -4, 1)$.

Appendix B. Acknowledgments

References

[1] A. Best, P. Dynes, S. J. Miller, J. Powell and B. Weiss, *The Emergence of 4-Cycles in Polynomial Maps over the Extended Integers* (expanded arXiv version), http://arxiv.org/pdf/1507.03597.

[2] J. Browkin and J. Brzezinski, *Some Remarks on the abc-Conjecture*, Mathematics of Computation **62** (1994), 931–939. http://cr.yp.to/bib/1994/browkin.pdf.

[3] H. Davenport, *Multiplicative Number Theory*, 2nd edition, revised by H. Montgomery, Graduate Texts in Mathematics, Vol. 74, Springer-Verlag, New York, 1980.

[4] B. Green, *Roth's theorem in the primes*, Annals of Mathematics **161** (2005), 1609–1635. http://annals.math.princeton.edu/wp-content/uploads/annals-v161-n3-p11.pdf.

[5] H. W. Lenstra, Jr. *Euclidean number fields of large degree*, Inventiones Mathematicae **38** (1977), 237–254. http://www.math.leidenuniv.nl/~hwl/PUBLICATIONS/1977b/art.pdf.

[6] W. Narkiewicz, *Polynomial cycles in certain rings of rationals*, Journal de Théorie des Nombres de Bordeaux **14** (2002), 529–552.

[7] J. H. Silverman, *The Arithmetic of Dynamical Systems*, Springer-Verlag, 2007.

[8] M. Zieve, *Cycles of Polynomial Mappings*, Ph.D. thesis, Berkeley, 1996. http://www.math.lsa.umich.edu/~zieve/papers/ucthesis.pdf.

Student biographies

Andrew Best: (*Corresponding author:* andrewbest312@gmail.com) Andrew graduated from Williams College with a mathematics and Chinese double major. A second year mathematics graduate student at the Ohio State University, he is working on problems in ergodic theory.

Patrick Dynes: Patrick will graduate from Clemson University in May 2017 with a mathematical sciences and computer engineering double major. He will be entering a graduate program in mathematics in the Fall 2017 academic semester.

Jasmine Powell: Jasmine graduated from Northwestern University with a mathematics major and a computer science minor. She is currently a second year mathematics graduate student at the University of Michigan, working in complex dynamics.