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Minimal Coverings of Surfaces

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Abstract. An elementary question of manifolds is calculation of the covering number – the least number of category-specific balls needed to cover the manifold. The following work calculates this fundamental invariant for 2-manifolds. A brief review of the classification of 2-manifolds is initially provided, followed by the details of calculation for all surfaces.

1. Introduction and Main Result

A manifold, M, is a second countable, Hausdorff space locally homeomorphic to Euclidean space ($\mathbb{R}^n$). The last property implies an important aspect of manifolds: they can be covered by some (at most countable) collection of open balls, $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$. This observation leads to the question: What is the minimal number of open balls needed to cover a given manifold?

The (smooth) **covering number** of a manifold M is the least number, denoted $B(M)$, such that M can be written as the union of $B(M)$ sets such that each set is diffeomorphic to $\mathbb{B}^n$. (Note: This definition can also apply to the topological or piecewise-linear category; replace *diffeomorphic* with *homeomorphic* or *PL homeomorphic*.) Define $\mathbb{B}^n_h = \mathbb{B}^n \cap \{x \in \mathbb{R}^n : x_n \geq 0\}$ to be the upper-half ball. Then the **covering number** of a manifold with boundary M is the least number, $B(M)$, such that M can be written as the union of $B(M)$ sets where each set is diffeomorphic to either $\mathbb{B}^n$ or $\mathbb{B}^n_h$. If no such finite number exists, let $B(M) = \infty$.

In all the following, assume all manifolds are connected. If M is one-dimensional, the question is decidedly trivial: there exist only two 1-manifolds, the line and the circle, which can be covered in one or two charts, respectively. Allowing boundary components does not make the question any less trivial; this allows for only two more manifolds, the one-ended line and two-ended line, which can be covered in a chart for each end.

In addition to the triviality of the one-dimensional case, higher dimensional calculations have been completed for various categories of manifolds. The most general result for closed manifolds, completed by Singhof [10], relates the Lusternik-Schnirelmann (LS) category of a manifold to the covering number. The LS category of a manifold, denoted $\operatorname{cat} M$, is the least number of open and contractible sets needed to cover M. This reduces the calculation to a homotopy invariant, which is easier to calculate (indeed, the LS category of a manifold can be estimated below by the cup-length). Singhof's theorem relates sufficient conditions for the dimension such that $\operatorname{cat} M$ and $B(M)$ are equal. Specifically, the following theorem is the main result [10].

Singhof's Theorem. Let M^n be a smooth (or piecewise-linear), closed, p-connected manifold, $n \geq 4$, $\operatorname{cat} M \geq 3$.

(1) If $\operatorname{cat} M \geq \dfrac{n+p+4}{2p+2}$, then $B(M) = \operatorname{cat} M$.

(2) If $\operatorname{cat} M < \dfrac{n+p+4}{2p+2}$, then $B(M) \leq \left\lceil \dfrac{n+p+4}{2p+2} \right\rceil$.

Rudyak and Schlenk [9] have calculated a similar result for symplectic manifolds. Since symplectic manifolds have additional structure, the theorem has greater complexity. The least number of Darboux charts needed to cover a symplectic manifold is constrained by the volume of the manifold, and the maximal volume of a ball symplectically embedded into M. As noted by Rudyak and Schlenk [9], we have $B(M) \leq n+1$ for M a closed smooth manifold. Their main result states that the volume consideration is the only aspect that increases the symplectomorphism invariant beyond $n+1$, see their paper for details.

Although this volume concern complicates the issue for symplectic manifolds, the case for closed contact manifolds never exceeds the bound of $n+1$. The main theorem on contact manifolds, as proven by Chekanov, Koert, and Schlenk [3], states that $B(M) \leq C(M,\xi) \leq n+1$, where $C(M,\xi)$ denotes the minimal number of contact balls needed to cover (M,ξ) (with ξ the contact structure on M).

Moving back to the category of smooth manifolds, there are a couple of obvious limitations to Singhof's theorem. First, the category must be sufficiently large. There are no closed manifolds such that $B(M) = 1$. If $B(M) = 2$, then M is a topological sphere and a PL sphere if $n \neq 4$; however, M could be an exotic sphere. If M is an exotic sphere ($n \neq 4$), note that the twisted sphere structure implies a 2-cover. To prove the existence of a 2-cover directly, first note that Montejano [6] proved Singhof's theorem in the PL category with $n \geq 5$ and $\operatorname{cat} M \geq 2$. We may then extend to the smooth category using the same strategy as Singhof [10] – there exists a smooth triangulation [14] where we may take a covering such that the interiors are diffeomorphic to $\mathbb{B}^n$ due to a theorem of Thom [11] and Munkres [7]. Therefore, the theorem is still true with the restriction $n \geq 5$ and no restriction on $\operatorname{cat} M$, as determined by Montejano [6] in the PL category. The case of $n = 4$ is true if any exotic 4-sphere may be covered in two balls, a question connected to the open part of the generalized Poincaré conjecture.

Another limitation is that the theorem only applies to closed manifolds. There has been some work in the covering number of manifolds with boundary, for example see [13]. Most of this work has been concerned with slightly nicer covers; Tsukui [13] concentrated on covers by PL balls that intersected only in the boundary, and such that the intersection was a manifold. For a quick review of these stronger ball covering numbers, see Todea [12].

These results do not consider noncompact manifolds. There is no immediate translation of Singhof's proof to noncompact manifolds. The proof is dependent upon (finite) handle decompositions. In order to avoid these finiteness concerns, the entire proof by Montejano [6] of Singhof's theorem can be translated into the *proper* category. This requires notions of the proper LS category, and proper connectedness. Cárdenas, Lasheras, and Quintero [2] have proven a statement similar to Singhof's theorem, applying to covers of one-ended manifolds by half spaces.

Finally, the dimensional constraint is also a limitation. Singhof [10] provides a basic result for three-dimensional manifolds. Gómez-Larrañaga and González-Acuña completely calculated the LS category of closed 3 manifolds [4], which contains Singhof's result. (It should be noted that the case of contact 3-manifolds is also completed, see [3].) The case of closed surfaces is mentioned in various sources (e.g., Todea [12] mentions the result). For surfaces, consideration of the LS category is not particularly helpful since every open, contractible set in $\mathbb{R}^2$ is diffeomorphic to $\mathbb{R}^2$.

Although the closed case is solved, consideration of general surfaces is easier than higher dimensions, since Ian Richards completed the classification of surfaces [8].

Classification of Surfaces. Two surfaces with a finite number of boundary components are homeomorphic if and only if they have an equivalent:

(1) Genus,

(2) Orientability class,

(3) Number of boundary components homeomorphic to a circle,

(4) Number and position of boundary components homeomorphic to a line,

(5) Triple of spaces $(B_0 \supseteq B_1 \supseteq B_2)$, where B_2 is the space of nonorientable ends, B_1 is the space of ends with genus, and B_0 is the space of ends.

Remark. The first, second, and fifth aspects are from Ian Richards [8]. The third and fourth are an obvious extension.

Remark. The "position" for boundary components homeomorphic to a line will be defined later and is necessary to distinguish the two following manifolds. Let B_x be the (open) ball of radius 1 centered at x.

$$M_1 = \mathbb{R}^2 - (B_{(\pm 2,0)} \cup (\pm 1,0) \cup (2,\pm 1)), \quad M_2 = \mathbb{R}^2 - (B_{(\pm 2,0)} \cup (\pm 1,0) \cup (\pm 2,1))$$

These manifolds are the plane, with two open disks and four boundary points removed. Both of these manifolds are orientable, have genus zero, have zero boundary components homeomorphic to a circle, and have five planar ends. Each also have four boundary components homeomorphic to a line.

For M_1, the left circle has one line, but the right circle has three lines. M_2 has two lines in each circle. These spaces fail to be homeomorphic.

None of the borderless surfaces on this list, except for the orientable genus 0 surface with one (planar) end, are open and contractible. If we allow boundary, then this only adds surfaces with any number of boundary components homeomorphic to a line (including

$\mathbb{B}^2_h$), or one boundary component homeomorphic to the circle (i.e., $\overline{\mathbb{B}^2}$). Therefore, $\operatorname{cat} M = B(M)$ for all surfaces without boundary. For surfaces with boundary, the bounds are less immediate.

Since the classifications is complete, the covering number can be calculated by simple consideration of a small handful of invariants. Recall that in dimensions less than four, the smooth, PL, and topological categories are equivalent. The main result for simple surfaces is the following theorem.

Main Theorem 1. For any closed surface S, let $S_{k,\ell}$ denote the surface with k open balls and ℓ closed balls, all disjoint, removed ($k, \ell < \infty$). Then:

$$B(S_{k,\ell}) = \max\{2k, 2\} \ : \ k + \ell \geq 2$$
$$B(S_{0,0}) = B(S_{1,0}) = 3, \ \ B(S_{0,1}) = 2 \ : \ S \neq \mathbb{S}^2$$
$$B(S_{0,0}) = B(S_{1,0}) = 2, \ \ B(S_{0,1}) = 1 \ : \ S = \mathbb{S}^2$$

Remark. Note that the covering number, $B(M)$, can be artificially increased, given the choice of definition. For example, take $\overline{\mathbb{B}^2}$, and remove $n > 0$ points from the boundary. Then the boundary requires n charts, and thus $B(M) = n$.

The final result, on more complicated surfaces, may be summarized similarly.

Main Theorem 2. Let S be a surface with k boundary components homeomorphic to a circle, and h boundary components homeomorphic to a line organized across p positions, such that $k + h$ is finite. Let $g = 0$ if S has genus 0, and otherwise let $g = 1$. Let ℓ be the number of planar ends outside of the p positions.

(1) $B(S) = \max\{2k + h, 2\} \ : \ k + p + \ell \geq 2$

(2) $B(S) = h \qquad : \ p = 1, \ h \geq 2$

(3) $B(S) = g + h \qquad : \ p = h = 1$

(4) $B(S) = g + 2 - \ell \qquad$ otherwise

If k or h is infinite, then $B(S) = \infty$.

Remark. Main Theorem 1 is a subset of this theorem: let $h = p = 0$.

The paper is organized in this format: Section 2 reviews the classification of surfaces. Section 3 provides the detailed geometric and topological arguments for surfaces with a finite number of planar ends and compact boundary. Section 4 extends the work to the remaining surfaces.

2. Review of the Classification of Surfaces

The classification of compact surfaces is well-known [1].

Theorem 2.1. Two compact surfaces with boundary are homeomorphic if and only if they have the same genus, the same orientability, and the same number of boundary components.

The extension to all surfaces is less well-known, and will thus be reviewed briefly. This requires a wealth of definitions.

Definition. The **Euler characteristic** of a manifold M, denoted $\chi(M)$, is the alternating sum of the number of cells in each dimension, i.e., if c_n is the number of cells of dimension n:

$$\chi(M) = \sum_{n=0}^{\infty}(-1)^n c_n.$$

Definition. The **genus** of an orientable surface, denoted g, is the maximum number of cuttings along closed, simple curves where the resultant manifold is still connected. For a nonorientable surface, $g = 2 - \chi$.

Definition. Let S be a compact surface with boundary, with Euler characteristic χ and q boundary components. Then the **reduced genus**, denoted r, is defined as $r = 1 - \frac{1}{2}(\chi + q)$.

The previous definitions are commonplace in the discussion of surfaces. The following definitions begin to cover noncompact surfaces.

Definition. An **end component** of a surface, S, is a nested sequence $U_1 \supseteq U_2 \supseteq \cdots$ of connected unbounded subsets of S with compact boundary such that for every compact K, there exists an n with $U_n \cap K = \varnothing$. Two end components, $U_1 \supseteq U_2 \supseteq \cdots$ and $U_1' \supseteq U_2' \supseteq \cdots$, are **equivalent** if for any n, m there exists n', m' such that $U_{m'} \subseteq U_m'$ and $U_{n'}' \subseteq U_n$.

Definition. An **end** of a surface, S, is an equivalence class of the end components of S. If p is an end component, let p^* denote the corresponding end. For any $U \subseteq S$ with compact boundary in S, let U^* be the set of ends such that there exists an end component $P_1 \supseteq P_2 \supseteq \cdots$ with $P_n \subseteq U$. The **space of ends** of S, denoted $B_0(S)$, is the set of ends with topology defined by the basis $\{U^*\}$ where $U \subseteq S$ has compact boundary.

Note that, based on the definition of an end component, every end component will either eventually lie within U or eventually not intersect U, where U has a compact boundary. Based on the definition of equivalence, if one end component is eventually a subset of U, then all representatives for that end will eventually be a subset of U. Also note that this agrees with the usual definition, where an end is a connected component of well-constructed exhaustions of M by compact sets – take the complement of any compact set, and we get such a U. Then the basis for $B_0(S)$ is the set of connected components of complements of compact sets.

Definition. A surface is called **planar** if every compact subsurface has genus zero.

Definition. An end, p^*, is called **planar** (resp. **orientable**) if there exists a representative $p = \{P_1 \supseteq P_2 \supseteq \cdots\}$ such that P_n is planar (resp. orientable) for all sufficiently large n.

Remark. It is trivial to check that this is well-defined.

An end is planar if the P_n are eventually annuli. An end is orientable if the P_n eventually do not contain a Möbius strip.

Definition. A surface S is of **infinite genus** (resp. is **infinitely nonorientable**) if there does not exist a compact subset K such that $S - K$ is planar (resp. orientable).

Definition. The **orientability class** of a surface S is one of four descriptions of its orientability: **orientable**, **infinitely nonorientable**, **odd**, and **even**. For a surface that has odd or even orientability class, there exists a sufficiently large compact subsurface K such that its complement is orientable. If K has half-integral reduced genus, it is called **odd**. If K has integral reduced genus, it is called **even**.

Definition. Let S be a surface. The **space of ends with genus**, denoted $B_1(S)$, is the subspace of $B_0(S)$ consisting of ends that are not planar. The **space of nonorientable ends**, denoted $B_2(S)$, is the subspace of $B_1(S)$ consisting of ends that are nonorientable. The **ideal boundary** of S, denoted $\mathrm{IB}(S)$, is a nested triple of space $(B_0(S) \supseteq B_1(S) \supseteq B_2(S))$.

The major theorem from Ian Richard's paper is the following.

Theorem 2.2. Let S and S' be two surfaces (without boundary) of the same genus and orientability class. Then S and S' are homeomorphic if and only if their ideal boundaries are homeomorphic, as a triple of spaces.

All that remains is the extension to surfaces with boundary. First, I restrict to surfaces with a finite number of boundary components for clarity. For circular components of the boundary, it is clear that the number of components is sufficient. The largest issue for lines is defining *position*. Let S be a surface, and let ℓ_1 and ℓ_2 be two boundary components homeomorphic to a line. Connect these lines by a compact simple line, k, whose interior does not intersect the boundary of S. This compact set is closed, and its complement U_k is an open submanifold. The lines ℓ_1 and ℓ_2 are part of the same position if there exists a U_k that is disconnected.

First note that being part of the same position defines an equivalence relation on lines. Reflexivity is a consequence of the Jordan Curve Theorem. Symmetry is trivial. Transitivity follows from keeping track of the two connected components from the Jordan Curve Theorem. Thus we define a **position** to be an equivalence class of boundary components homeomorphic to a line. We define the number of positions to be the number of equivalence classes.

For any position, p, let $|p|$ denote the number of different lines in p. Let $\{p_i\}$ denote the collection of positions. For a surface S, let $P_S = (p_1, p_2, \ldots)$ be an ordered set, such that $|p_i| \leq |p_{i+1}|$ (if S has no positions, let $P_S = \varnothing$). Then two surfaces, S and S', with a finite number of boundary components are homeomorphic if and only if they have the same genus, orientability class, triple of spaces of ends, number of boundary components homeomorphic to a circle, and $P_S = P_{S'}$.

Note that this can easily be extended to an infinite number of boundary components. This complication is avoided since $B(S)$ will always be infinite.

Ian Richards [8] proves the following two facts, which are helpful for consideration of noncompact surfaces.

Proposition 2.1. The ideal boundary of a surface is totally disconnected, separable, and compact (e.g., the ideal boundary is a triple of subsets of the Cantor set).

Theorem 2.3. Let (X, Y, Z) be any triple of compact, separable, totally disconnected spaces, $Z \subseteq Y \subseteq X$. There is a surface S such that $\mathrm{IB}(S) \cong (X \supseteq Y \supseteq Z)$.

3. Simple Surfaces

The classification of surfaces [8] was reviewed in Section 2. By this classification, a "simple surface" is a surface with finitely many planar ends, finite genus, and a compact boundary. These surfaces are among the most common, and consist of the set of compact surfaces with disjoint open and closed balls removed. The extension to the remaining surfaces is not difficult, and is completed in Section 4. In the following, g will denote the orientable genus, h the nonorientable genus, k the number of open balls removed, and ℓ the number of closed balls removed. For completeness, the definition of removed balls will be stated. If U is a topological space, let $\overline{U}$ denote its closure.

Definition. S' is a surface with k open balls and ℓ closed balls removed from S if $k, \ell < \infty$ and

$$S' \cong S - \bigcup_{i=1}^{k} \varphi_i^1(\mathbb{B}^2) - \bigcup_{j=1}^{\ell} \varphi_j^2(\overline{\mathbb{B}^2}) \ : \ \varphi_\alpha^a(\overline{\mathbb{B}^2}) \cap \varphi_\beta^b(\overline{\mathbb{B}^2}) = \varnothing \ \ \forall\, a, b \in \{1,2\}, \ \ \forall\, \alpha, \beta,$$

with the $\varphi_\alpha^a : \overline{\mathbb{B}^2} \to K \subseteq S$ diffeomorphisms for all α and $a \in \{1,2\}$. It is, furthermore, restricted that S' is a surface.

Remark. The infinite case is not needed, and is therefore excluded. There are some complications. For example, let $k = \infty$ and $S = \overline{\mathbb{B}^2}$. Define A as a surface with k open balls removed from S as follows:

$$A = S - \left\{ x \in S \ : \ |x - (2^{-n}, 0)| \geq 2^{-n-2} \ \ \forall\, n \geq 1 \right\}.$$

Then A is not even a manifold. It should be noted that the case of $k = \infty$ is not particularly interesting: the covering number of this surface would always be infinite, since there are an infinite number of boundary components. The case of $\ell = \infty$ is more well-behaved, and will be treated in Section 4.

Remark. Forcing S' to be a surface removes trivial cases. For example, if $S = \overline{\mathbb{B}^2}$, then we may not let $S' = S - \overline{\mathbb{B}^2} = \varnothing$. The restriction may be well-defined by forcing the balls to be defined relative to another ball, and thus properly contained within another ball. This adds additional, unneeded complexity.

Removing an open ball adds a boundary component diffeomorphic to a circle. Removing a closed ball adds a planar end. Therefore, the following proposition is immediate.

Proposition 3.1. Let S be a surface with a finite number of boundary components. If A and B are both surfaces diffeomorphic to S with k open balls and ℓ closed balls removed, then $A \cong B$.

Proof. Both A and B have the same fundamental invariants as determined by the classification reviewed in Section 2, and they are therefore diffeomorphic. □

Remark. Technically, it should be checked that A and B have the same genus and orientability class. It is clear they have the same genus from definition (see Section 2). It is also clear that A contains a Möbius strip if and only if B contains a Möbius strip, and thus A is orientable (resp. infinitely nonorientable) if and only if B is orientable (resp.

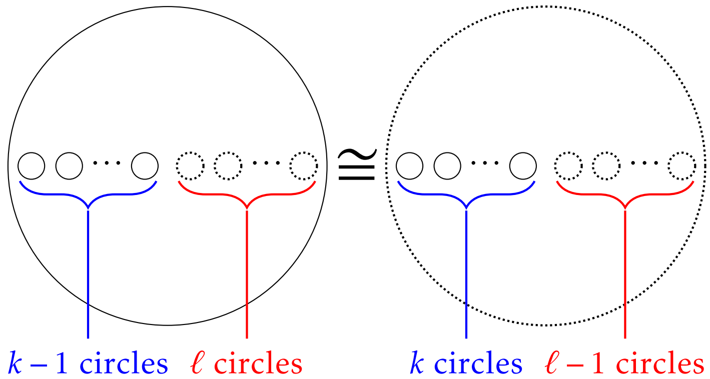

FIGURE 1. The surface $\Sigma_{0,k}^{\ell}$ such that $k,\ell \geq 1$. The left one is if an open ball is removed "first" and the right is if a closed ball is removed "first". Note that for all figures, a solid line will indicate a boundary, and a dotted line will indicate an open end.

infinitely nonorientable). Showing that A has odd orientability class if and only if B does is not as immediate. However, as will be seen in the proofs of Section 4, even and odd orientability class do not affect the covering number.

All of the following work uses this Proposition, so that the removed balls may be placed wherever is most convenient. Since orientable surfaces are embeddable in $\mathbb{R}^3$, the argument will first be provided for orientable surfaces and then extended to nonorientable surfaces.

3.1. **Orientable surfaces.** Let $\Sigma_{g,k}^{\ell}$ be the orientable surface of genus g with k open balls and ℓ closed ball removed. Note that $\Sigma_{g,k}$ is the standard compact 2-manifold with boundary. Recall the definition of $B(M)$ for manifolds with boundary, the minimal number such that M is covered by $B(M)$ charts from $\mathbb{B}^n$ or $\mathbb{B}_h^n$.

3.1.1. *Spheres.* By stereographic projection, we have $\Sigma_{0,0}^{0} \cong \mathbb{S}^2 \implies B(\Sigma_{0,0}^{0}) = 2$. Removing a closed ball about the South pole, stereographic projection about the North pole gives us

$$\Sigma_{0,0}^{1} \cong \mathbb{R}^2 \cong \mathbb{B}^2 \implies B(\Sigma_{0,0}^{1}) = 1.$$

If we remove an open ball about the South pole, the boundary is homeomorphic to $\mathbb{S}^1$, and thus

$$\Sigma_{0,1}^{0} \cong \overline{\mathbb{B}^2} \implies B(\Sigma_{0,1}^{0}) = 2.$$

These last two cases provide the basis for the remaining argument. It can now be immediately seen that $\Sigma_{0,k}^{\ell}$ is diffeomorphic to a closed ball with $k-1$ open balls and ℓ closed balls removed along the equator. Alternatively, it is an open ball with k open balls and $\ell-1$ closed balls removed. For a simple diagram, see Figure 1. From this figure, it is clear that we may define two charts that completely cover the interior. These two charts are the top half and bottom half of the balls. Therefore, if $k \in \{0,1\}$, exactly two charts are required (if $k+\ell \geq 2$).

The situation changes when $k > 1$. For each additional open ball removed, the boundary increases in size.

$$\delta\Sigma^{\ell}_{g,k} = \coprod_{i=1}^{k} \mathbb{S}^1_i$$

Therefore, the number of charts is at least $2k$. Since these charts more than cover the interior, this completes the result.

Lemma 3.1. $B(\Sigma^{\ell}_{0,k}) = \begin{cases} 1, & \ell = 1,\ k = 0 \\ \max\{2k, 2\}, & \text{otherwise} \end{cases}$

3.1.2. *Torus.* It is well known that the orientable surface of genus one requires three charts:

$$\Sigma^0_{1,0} \cong \mathbb{T}^2 \implies B(\Sigma^0_{1,0}) = 3.$$

The charts can be described as: the complement of two loops that form a basis for the fundamental group; a neighborhood about the intersection point of these two loops; and a final chart consisting of a tubular neighborhood of the two loops except a neighborhood about the intersection point, with a small band connecting the two components.

Proposition 3.2. $B(\Sigma^1_{1,0}) = 2,\ \ B(\Sigma^0_{1,1}) = 3$

Proof. By simple logic, the model looks like Figure 2. Taking a maximal boundary chart for $\Sigma^1_{1,0}$, there is a non-trivial loop, see the left part of Figure 2. Therefore, at least two more charts are required. Two more charts are adequate (in the left part of Figure 2, a neighborhood of the point of intersection of the loops that covers the remaining boundary, and a band covering the remaining parts). For $\Sigma^0_{1,1}$, two charts are more than adequate (see Figure 2 for an explicit cover). □

For a closed surface, it will always be the case that removing a closed ball decreases the cover by one. In essence, we are removing one of the charts from the closed surface. It will also be the case that removing an open ball does not change the covering number. This will always be due to a maximal boundary chart leaving a non-trivial loop, and the boundary requiring at least two charts.

Proposition 3.3. $B(\Sigma^1_{1,1}) = 2$.

Proof. Remove an open ball from the non-compact version ($\Sigma^1_{1,0}$) of Figure 2, in the middle of the intersection neighborhood. Draw a line from the top-left corner to the bottom-right corner. This marks where the two charts would meet. See Figure 3 for explicit visualization. □

Lemma 3.2. $B(\Sigma^{\ell+1}_{1,k}) = B(\Sigma^{\ell}_{1,k})\ :\ \ell \geq 1$

Proof. Simply remove another ball along the diagonal line drawn previously. For completeness, an example is drawn as the right figure in Figure 3. □

Lemma 3.3. $B(\Sigma^{\ell}_{1,k+1}) = B(\Sigma^{\ell}_{1,k}) + 2\ :\ k \geq 1,\ k + \ell \geq 2$

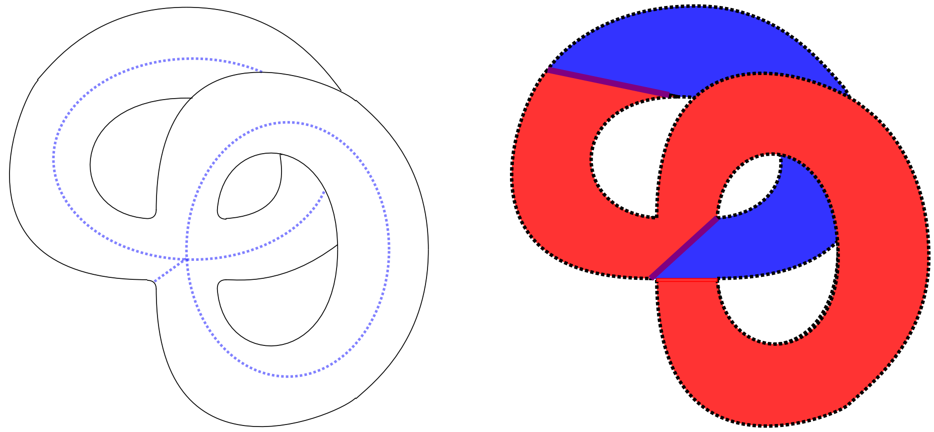

Figure 2. The left-hand figure shows a maximal boundary chart for $\Sigma^0_{1,1}$. Note that what remains (dotted in blue) has a nontrivial loop, and thus requires two more charts. The right-hand figure shows two charts covering $\Sigma^1_{1,0}$ (purple indicates overlap of the two charts).

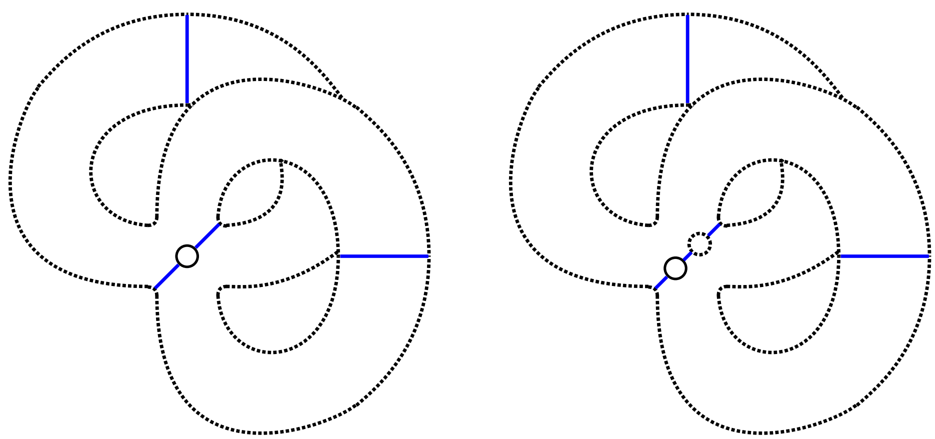

Figure 3. The left-hand figure is $\Sigma^1_{1,1}$, with blue lines showing the overlapping parts of two charts. The right-hand figure is $\Sigma^2_{1,1}$, representing the argument for $\Sigma^{\ell+1}_{1,k}$. Once again, blue lines show the overlap for charts.

Proof. Remove another ball along the diagonal line, except now adding another disjoint circle to the boundary requiring two more charts. See Figure 4 for the depiction, where this is reduced to the already proved spherical case. □

Thus the one case not yet covered is $\Sigma^0_{1,2}$. Note that $B(\Sigma^0_{1,1}) = 3$ (Proposition 3.2). To this model shape, remove an open ball from the central intersection area. Note that four charts is more than enough to cover all the interior (since it was previously enough), and

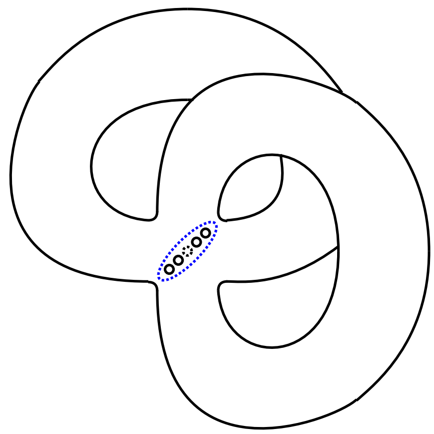

Figure 4. This is $\Sigma_{1,k}^{\ell}$ such that $k \geq 2$. Proposition 3.2 states that everything outside the blue ellipse can be covered in two charts. Lemma 3.1 states that everything inside the ellipse can be covered in $2k-2$ charts, with k the number of open balls removed (noting that one of the balls creates the boundary of the initial model shape). Since at least $2k$ charts are needed to cover the boundary, and $2k$ is sufficient, this determines the covering number.

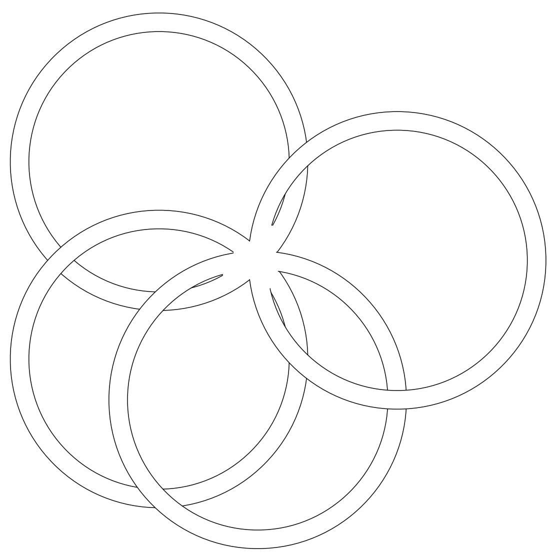

Figure 5. An example of the double torus with a ball removed ($\Sigma_{2,1}^{0}$). Note that the same arguments apply.

is enough to cover the boundary. As an explicit cover, we may take charts similar to the left figure of Figure 3. Therefore, $B(\Sigma_{1,2}^{0}) = 4$.

In summary we have the following lemma:

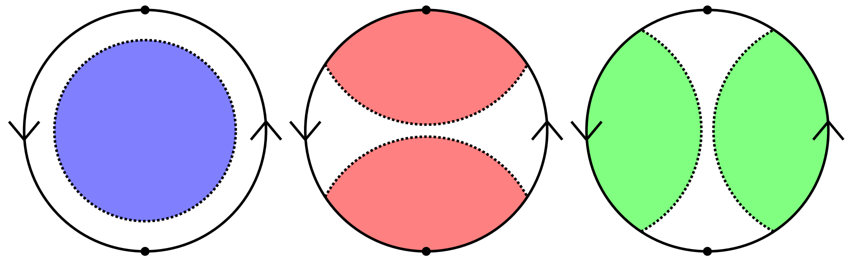

Figure 6. Cover of $\Xi^0_{1,0} = \mathbb{RP}^2$ in three charts. Two charts are not sufficient since any one of these (maximal) charts leaves a non-trivial loop.

Lemma 3.4. $B(\Sigma^\ell_{1,k}) = \begin{cases} 2k, & k \geq 1,\ k+\ell \geq 2 \\ 2, & k = 0,\ \ell \geq 1 \\ 3, & \text{otherwise} \end{cases}$

3.1.3. *Higher genus surfaces.* For higher genus surfaces, all that is needed is the following proposition.

Proposition 3.4. $B(\Sigma^\ell_{g+1,k}) = B(\Sigma^\ell_{g,k}) \ : \ 1 \leq g < \infty$

Proof. The model simply has more loops. For an example of $g = 2$, see Figure 5. All of the same arguments apply – the exact same charts can be used for $\Sigma^0_{g,0}$. Balls can still be aligned along the diagonal in the center. Therefore, the proposition is trivial. □

Remark. This argument, although correct, may appear to gloss over details. The argument in Section 3.2.3 using polygonal presentations could be applied, if a more technical argument is preferred.

3.2. **Non-orientable surfaces.** To keep notation similar, let $\Xi^\ell_{h,k}$ denote the nonorientable surface of nonorientable genus h, with k open balls and ℓ closed balls removed. By Proposition 3.1, this manifold is unique. Therefore, $\Xi^0_{1,0}$ is the real projective space ($\mathbb{RP}^2$) and $\Xi^0_{2,0}$ is the Klein flask. By a proof in [5, pp. 174], $\Xi^0_{3,0} \cong \mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2 \cong \mathbb{RP}^2\#\mathbb{T}^2$ (where # denotes the connected sum).

To begin, $B(\mathbb{RP}^2) = 3$. It is at most three by the three standard charts $(x, y, z) \mapsto [x, y, 1]$. It cannot be two: any chart leaves a non-trivial loop (see the polygonal presentation in Figure 6 for the visualization, see [5, pp. 166] for a definition of polygonal presentations). The first two variants are also easily determined:

$$B(\Xi^1_{1,0}) = 2, \quad B(\Xi^0_{1,1}) = 3.$$

These shapes are a Möbius strip without and with boundary, respectively. Three charts are required when the boundary is included for the same reason as with the torus (Proposition 3.2). One could extend all of the orientable arguments by defining an equator on the Möbius strip and putting all new balls on this equator. Thus:

$$B(\Xi^\ell_{1,k}) = B(\Sigma^\ell_{1,k}).$$

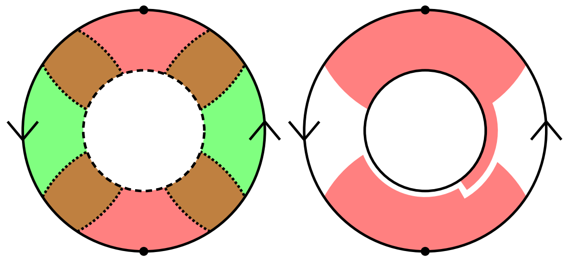

FIGURE 7. $\Xi^1_{1,0}$ can be covered in two charts, as shown on the left. Three charts are required for $\Xi^0_{1,0}$. A maximal boundary chart for $\Xi^0_{1,1}$ is shown on the right, and it can be seen a nontrivial loop will always exist.

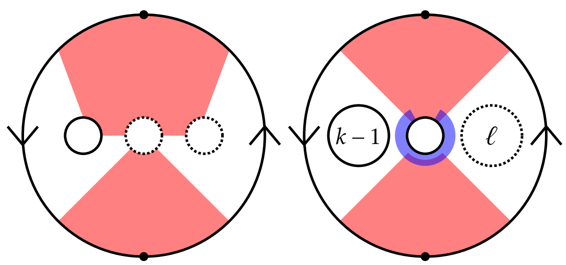

FIGURE 8. The left-hand figure is $\Xi^\ell_{1,1}$ ($\ell \geq 1$). Any more closed balls can be removed along the red chart boundary. The right-hand figure is $\Xi^\ell_{1,k}$ ($k \geq 2$). The remaining region is diffeomorphic to $\Sigma^{\ell+1}_{0,k-1}$, requiring $2k-2$ charts by Lemma 3.1 for a total of $2k$ charts. The notation inside the left and right circle indicates there are that many circles of each type.

Rather than relying on this haphazard treatment, easy visualization is possible if the surfaces are reduced to polygonal presentations. This more complete treatment will be preferred.

3.2.1. *Real-projective space.* The visualization for $\mathbb{RP}^2$ spans Figures 6, 7, and 8. All of the covers center around three chart "types:" a corner chart (covering the corners of the polygonal presentation), a "side" chart (covering some of the sides), and the center chart (the 2-cell of the polygonal presentation). Note that these same three charts (center, corner, and sides) will be used for all of the remaining presentations.

The argument that $B(\mathbb{RP}^2) = 3$ is provided as Figure 6. Next, $B(\Xi^0_{1,1}) = 3$ and $B(\Xi^1_{1,0}) = 2$ are shown in Figure 7. This shows the equivalent of Proposition 3.2.

Now, let $k = 1$ and $\ell \geq 1$. The claim is only two charts are required, which is evidenced in Figure 8.

The final $\mathbb{RP}^2$-type manifold left to explain is $k \geq 2$. In this case, at least $2k$ charts are needed, since the boundary is $\delta(S) = \coprod_{i=1}^k \mathbb{S}^1$. Since $2k$ is more than sufficient to cover the interior, then $2k$ is all that is required. Figure 8 gives a visualization of this argument.

The following lemma summarizes the above arguments.

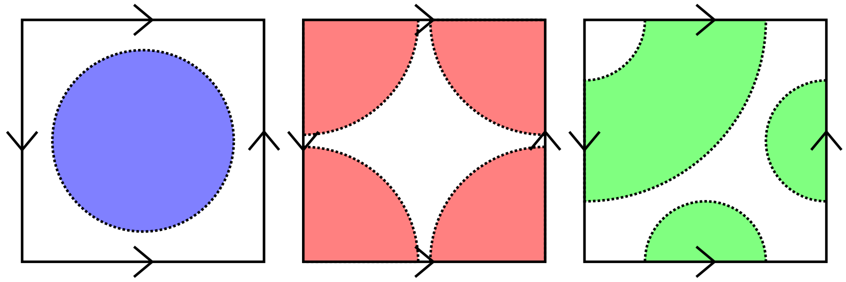

Figure 9. The Klein flask covered in the three standard charts. Note that the charts are similar to the cover of $\mathbb{RP}^2$ (see Figure 6).

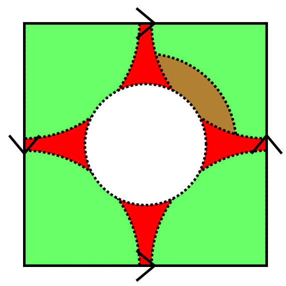

Figure 10. The two charted cover of $\Xi^1_{2,0}$. This figure exemplifies what has been meant by using a "band" to connect connected components.

Lemma 3.5. $B(\Xi^{\ell}_{1,k}) = \begin{cases} 3, & \ell = 0,\ k \leq 1 \\ \max\{2, 2k\}, & \text{otherwise} \end{cases}$

3.2.2. *Klein flask.* The Klein flask can be covered by a similar set of charts, see Figure 9. Thus $B(\Xi^0_{2,0}) = 3$. Once again, observe that all three maximal charts leave a non-trivial loop, and thus three is the minimum number. Consider $\Xi^1_{2,0}$. Removing a closed ball from a closed surface always decreases the covering number by one, as noted after Proposition 3.2. See Figure 10 for a visualization.

The same argument used for the torus applies to the Klein flask, thus forcing $B(\Xi^0_{2,1}) = 3$. Recall that the argument is that a maximal boundary chart leaves a non-trivial loop. Since this is similar to the case for $\mathbb{RP}^2$, see Figure 7 for a visualization. The same arguments for the projective space apply to showing that $B(\Xi^{\ell}_{2,k}) = B(\Xi^{\ell}_{1,k})$ (Figures 7 and 8, replacing a square as the perimeter as opposed to a circle). The argument will, thus, not be reproduced. The following lemma summarizes the result.

Lemma 3.6. $B(\Xi^{\ell}_{1,k}) = B(\Xi^{\ell}_{2,k})$.

3.2.3. *Higher genus surfaces.* The easiest (visualized) method to prove the general case is as follows. Consider $\Xi^0_{2h+1,0} \cong \mathbb{T}^2\#\cdots\#\mathbb{T}^2\#\mathbb{RP}^2$. The relation of the polygonal presentations is then:

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1} c_1 c_2.$$

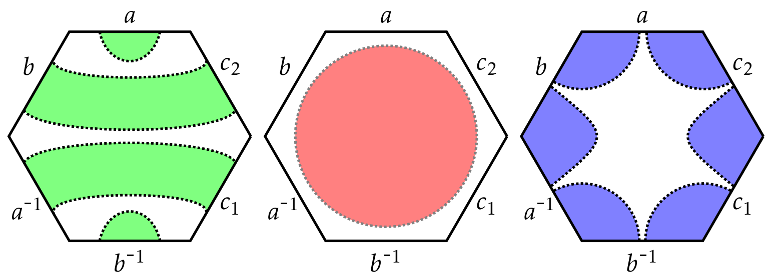

FIGURE 11. Charts showing the cover of $\mathbb{T}^2\#\mathbb{RP}^2$.

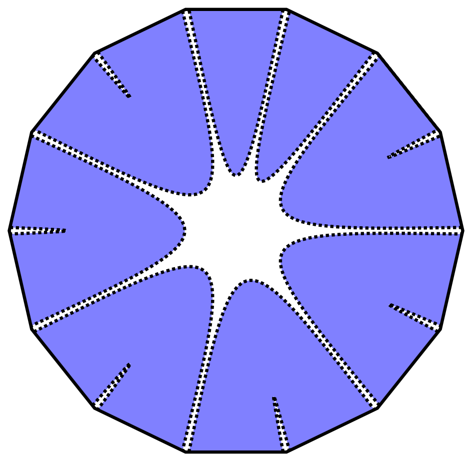

FIGURE 12. The side chart for $\mathbb{T}^2\#\mathbb{T}^2\#\mathbb{T}^2\#\mathbb{RP}^2$. It can be seen how a corner chart and center chart complete the cover.

The subnotation on the c is simply to distinguish between which side comes first and which side comes second. Define a **corner chart** of a polygonal presentation as a (maximal, symmetric) neighborhood of the corner point of a polygonal presentation (note that for all polygonal presentations that are not the sphere, the corners maps to the same point). Define a **center chart** as the chart about the center point (e.g., the 2-cell of the polygonal presentation). The other type of chart is a side chart. An example is:

$$(a, a^{-1} \to b, b^{-1} \to c, c^{-1} \to d^{-1}, d).$$

This indicates an open end on side a (excluding the corner), then a (tubular) path from a^{-1} to b (through the 2-cell), a path from b^{-1} to c, a path from c^{-1} to d^{-1}, and then an open end on side d. Using this notation, the set

$$\left\{(a, a^{-1} \to c_1, c_2 \to b, b^{-1}), \text{center}, \text{corner}\right\}$$

covers $\Xi^0_{3,0}$, see Figure 11 for the visualization, and thus an example of the side chart.

For the generic case of $\Xi^0_{2h+1,0}$ ($h \in \mathbb{N}$), the charts could be:

$$\left\{(a_1, a_1^{-1} \to b_1, b_1^{-1} \to a_2, a_2^{-1} \to b_2, \ldots, a_h^{-1} \to b_h, b_h^{-1} \to c_1, c_2), \text{center}, \text{corner}\right\}.$$

Figure 12 demonstrates the first chart for the case $h = 3$. (It should technically be shown that a 2-cover is insufficient. It is a common fact that a closed surface covered in two charts is a sphere, and this is not a sphere, so this will be ignored.)

Removing balls is the next step. For $\Xi^1_{2h+1,0}$, remove the center chart. As stated previously, the other two charts are preserved as a cover. Further closed balls could be removed along the corner chart. Therefore, $B(\Xi^\ell_{2h+1,0}) = 2$ if $\ell \geq 1$.

Suppose that at least two open balls were removed. Now at least four charts are required to cover the boundary. Four charts is more than sufficient to cover the interior. Therefore, the boundary determines the covering number.

Suppose one open ball and one closed ball were removed. Let the closed ball be the center chart. Remove an open ball along the corner chart. Then the corner chart and side chart will still cover the surface. Therefore, two charts are sufficient. Two charts are necessary since the fundamental group is non-trivial.

The final consideration is $\Xi^1_{2h+1,1}$. The argument is the exact same as in Proposition 3.2 and Figure 7: Suppose the first chart is a boundary chart. Then there exists a non-trivial loop not covered by the boundary chart. Therefore, two more charts are required. Suppose that the first chart is an interior chart. Then two more charts are required to cover the boundary. Therefore, three charts are required. The following lemma summarizes these results.

Lemma 3.7. $B(\Xi^\ell_{2h+1,k}) = B(\Xi^\ell_{1,k})$.

The remaining case is $\Xi_{2h} = \mathbb{T}^2 \# \cdots \# K$. The charts are precisely the same. The polygonal representation can be written as $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_h b_h a_h^{-1} b_h^{-1} c_1 d c_2 d^{-1}$. Then the set

$$\left\{(d, d^{-1} \to a_1, a_1^{-1} \to b_1, b_1^{-1} \to a_2, a_2^{-1} \to b_2, \ldots, a_h^{-1} \to b_h, b_h^{-1} \to c_1, c_2), \text{center}, \text{corner}\right\}$$

covers $\Xi^0_{2h,0}$. All cases for removed balls are precisely the same. Therefore, the last lemma may be stated.

Lemma 3.8. $B(\Xi^\ell_{2h,k}) = B(\Xi^\ell_{1,k})$.

Main Theorem 1. For any closed surface S, let $S_{k,\ell}$ denote the surface with k open balls and ℓ closed balls, all disjoint, removed ($k, \ell < \infty$). Then:

$$B(S_{k,\ell}) = \max\{2k, 2\} \ : \ k + \ell \geq 2$$
$$B(S_{0,0}) = B(S_{1,0}) = 3, \ \ B(S_{0,1}) = 2 \ : \ S \neq \mathbb{S}^2$$
$$B(S_{0,0}) = B(S_{1,0}) = 2, \ \ B(S_{0,1}) = 1 \ : \ S = \mathbb{S}^2$$

Proof. Lemmas 3.8, 3.7, 3.6, and 3.5 determine the result for nonorientable surfaces. Lemma 3.4 and Proposition 3.4 says that this result is the same for orientable surfaces of positive genus. Therefore, we have $B(S_{k,\ell}) = 3$ if $\ell = 0$ and $k \leq 1$ and $S \neq \mathbb{S}^2$, which says $B(S_{0,0}) = B(S_{1,0}) = 3$. Furthermore, $B(S_{k,\ell}) = \max\{2, 2k\}$ for all other values of k and ℓ, as determined by Lemma 3.5.

If $S = \mathbb{S}^2$, then the result is contained in Lemma 3.1. Thus $B(S_{0,1}) = 1$, and otherwise $B(S_{k,\ell}) = \max\{2, 2k\}$. Thus we are done. □

Remark. The entire proof can be done by polygonal presentations, as the general case for non-orientable surfaces. The charts are precisely the same.

Remark. The difference in the sphere results entirely because the corners fail to be identified in the polygonal presentation with two sides. This subtracts one needed chart. This becomes unimportant after enough closed balls are removed such that the first homotopy group is steadily nontrivial. This also becomes unimportant once enough open balls are removed such that the boundary components overwhelm the needed number of charts.

4. Consideration of other surfaces

The above considered *finitely* many *planar* ends, with *finite* genus, and a *compact* boundary. Therefore, these limitations will be addressed.

4.1. **Infinite planar ends.**

The result can be summarized with the following lemma:

Lemma 4.1. Let S be a surface with finite genus, compact boundary, and with at least two (planar) ends. Suppose S has k boundary components. Then $B(S) = \max\{2, 2k\}$.

Proof. First note that the case where S has a finite number of ends is complete by Main Theorem 1, since $k + \ell \geq \ell \geq 2$.

For an example, consider the plane $\mathbb{R}^2$, which is the sphere with one closed ball removed. Let C be the usual Cantor set, defined on the set $[0,1]$ (i.e., the set of all numbers in $[0,1]$ such that there exists a ternary expansion that does not contain a one). Embed C in $\mathbb{R}^2$, and let $S = \mathbb{R}^2 - C$. This set is open, and is thus a surface, and has uncountably many ends. Since the set is open, for every $(x,0) \in S$ such that $x \in [0,1]$, construct an open rectangle, $B_x \subset S$ centered about the line from $(x,-1)$ to $(x,1)$. Now let $U_x = B_x \cap \{(x,y) : y > -1/2\}$, and $L_x = B_x \cap \{(x,y) : y < 1/2\}$. Let:

$$U = \{(x,y) : x < 0 \lor x > 1 \lor y > 0\} \cup \bigcup_{x \in [0,1] \cap S} U_x$$

$$L = \{(x,y) : x < 0 \lor x > 1 \lor y < 0\} \cup \bigcup_{x \in [0,1] \cap S} L_x.$$

U and L are open, as a union of open sets, and contractible. Therefore (since the dimension is 2), $U \cong L \cong \mathbb{R}^2$, and $U \cup L = S$ gives $B(S) = 2$.

By Proposition 2.1, the set of ends must be a compact, separable, totally disconnected space, and thus must be some subset of the Cantor set. Clearly, if X is any closed subset of C, then the same construction above can be used. Since $X \subset C \subset [0,1]$, and X is compact, then X is closed by Heine-Borel. Therefore, the above construction gives a 2-cover for the genus 0 orientable borderless surface, with any number of (planar) ends.

Now begin with any other surface, S, satisfying the requirements of Lemma 4.1, and additionally such that S has a finite number of ends. Let $\mathscr{U}$ be a minimal cover of S. Choose two sets, $U, L \in \mathscr{U}$, such that $U \cap L \cap \operatorname{Int}(M) \neq \varnothing$. Let $R \cong [0,1] \times (-1,1)$ such that $R \subset U \cap L \cap \operatorname{Int}(M)$. Let X be a compact subset of the Cantor set. Let $S' = S - X \times 0$, under the homeomorphism for R and $[0,1] \times (-1,1)$. Use a similar construction as the above to construct U' and L'. Then these two charts, and all other members of $\mathscr{U}$, will be a cover for S'. Therefore, $B(S') \leq B(S)$.

Finally, let k be the number of boundary components. Clearly, S' still has k boundary components and has a nontrivial loop. Therefore, $B(S') \geq \max\{2k, 2\}$, and thus $B(S) = B(S')$.

By the classification of surfaces and Proposition 2.1, this is sufficient for a proof. □

4.2. **Adding an end with genus or a nonorientable end.** In order to construct the standard, infinite genus surface (resp. infinitely nonorientable surface), attach, via a connected sum, a torus (resp. projective plane) to a sphere, and then continue attaching tori (resp. projective planes) to the last attached torus (resp. projective plane). This surface will have one end with genus (resp. nonorientable end).

First, note that these surfaces can be covered in three charts. This can be done, for the infinite genus orientable surface, by taking the same three charts as for the torus: the complement of the loops defining the fundamental group, the loops defining the fundamental group excluding a neighborhood about the intersection point with trivial paths to connect the components, and a neighborhood of the intersection point. A similar construction works for the infinitely nonorientable surface. Second, apply the above constructions (removing open balls, removing closed balls, and removing some subset of the Cantor set) to arrive at the same conclusions. Therefore, the covering number for a surface with one end with genus (resp. nonorientable end) is equivalent to the covering number of the torus, with comparable planar ends and boundary components.

4.3. **Multiple ends with genus or nonorientable ends.** The results can be summarized by the following lemma.

Lemma 4.2. Let S be a surface with compact boundary. Let S' be a surface with: (1) the same number of boundary components, (2) the same number of planar ends, (3) finite positive genus if S has an end with genus, and (4) nonorientable if S has a nonorientable end. Then $B(S) = B(S')$.

To construct representative surfaces with any number of different ends, begin with a sphere. To each point in the Cantor set, considered as an infinite string possibly ending in infinitely many zeros, label it as one of: (not an end), (a planar end), (an end with genus), or (a nonorientable end). First, begin with the nonorientable ends. To explain by example, suppose 0.022 (followed by infinitely many zeros) is a nonorientable end. Then, remove the left third of the top of the sphere and attach $\mathbb{RP}^2$. Then remove the right third of the top of this $\mathbb{RP}^2$ and attach another copy. Then remove the right third, and then remove the left third of each attached copy afterwards attaching more real-projective planes. Therefore, remove left thirds for a 0, and right thirds for a 2, attaching projective planes along the way.

Follow a similar process for each nonorientable end. Then for each end with genus (that was not also nonorientable), attach tori in a similar fashion (note that the tori may be attached to previously attached real-projective planes – this is where the concept of *odd* and *even* nonorientability would develop). Finally, for each planar end, attach spheres similarly. This will create a surface, S, with the exact properties desired. Note that the exact definition of “left third”, “right third”, and “top” are not particularly important. As long as this is done consistently and logically, the result will be equivalent. The term “top” is only used to indicate that these ends are all being built “up” and are not intersecting or crossing each other. “Left third” and “right third” are used for the direct comparison to the Cantor set. Also note that, to properly visualize this surface, the attached surface should be considerably smaller.

Proof. A chart for this complicated surface will begin at the initial sphere. At any given branch point, the chart may split into two parts to continue moving up the ends. Therefore, the addition of these ends does not increase the number of needed charts, since the resulting chart will be contractible by construction. Since contractibility implies the set is $\mathbb{B}^2$, we are done. Explicitly, let K be a compact subsurface containing all of the boundary components and at least one layer of attached tori or real projective planes (if any exist). Then the number of charts needed to cover K is determined by Main Theorem 1. These charts could be constructed as described: starting at the initial sphere, and moving up through the attached surfaces. These same charts could be extended to the ends of S. Thus, $B(K) = B(S)$.

Note that this construction also contains Lemma 4.1. □

Note that odd and even nonorientability does not effect the covering number, as previously mentioned in the Remark following Proposition 3.1. This completes the statement of Main Theorem 2, except for the consideration of lines in the boundary.

4.4. **Allowing lines in the boundary.** “Adding” lines to the boundary is equivalent to “cutting” a circle into one ore more pieces (for one piece, remove a chunk from the circle). Thus, in addition to the $2k$ parameter, an additional sum for the number of lines must be added. In addition, the requirement of $k + \ell \geq 2$ must be modified to account for the positions of the lines. The reason for this restriction was that at least two disks, either open or closed, were removed. A position of lines is a disk that is neither open or closed. Therefore, these must also be considered.

Recall the definition of a position, given in Section 2. Let ℓ_1 and ℓ_2 be two boundary components homeomorphic to a line. Consider a topological embedding $f : [0,1] \to M$ such that $f(0) \in \ell_1$, $f(1) \in \ell_2$, and $f(0,1) \in \operatorname{Int} M$. Then ℓ_1 and ℓ_2 are part of the same position if there exists such an f such that $M - f[0,1]$ is disconnected. A position is an equivalence class of lines that are part of the same position.

Lemma 4.3. Let S be a surface with k boundary components homeomorphic to a circle and h boundary components homeomorphic to a line organized across p positions, such that $k+h$ is finite and $k > 0$. Then there exists a surface with $k-1$ boundary components homeomorphic to a circle, and $h+n$ boundary components homeomorphic to a line organized across $p+1$ positions, $n > 0$. In particular, one of these positions contains n lines.

Proof. Choose one of the circles in the boundary. From this circle, delete n points in the boundary (such that the result is still a manifold). Then we are done. □

Let $S[n]$ be the derived surface from S guaranteed by the above Lemma. Note that this becomes somewhat arbitrary with infinitely many chosen points, and we will thus required n to be finite for $S[n]$. Then the result will always be a manifold. Note that if n were countably infinite, we would create infinitely many boundary components homeomorphic to a line, and thus then the covering number would be infinite.

Lemma 4.4. Let k be the number of boundary components homeomorphic to a circle, and h the number of boundary components homeomorphic to a line. Suppose $B(S[n]) = 2k + h + n$. Then $B(S[n+1]) = 2k + h + n + 1$.

Proof. We must simply add one more boundary chart to cover the additional part of the boundary. □

Lemma 4.5. Suppose S has at least two boundary components homeomorphic to a circle. Then $B(S[1]) = B(S) - 1$.

Proof. Suppose S has two boundary components homeomorphic to a circle and no boundary components homeomorphic to a line. Then Main Theorem 1, Lemma 4.1, and Lemma 4.2 state that it takes four charts to cover S. Note that, in all of the arguments for Main Theorem 1, four was more than sufficient for the interior. Therefore, three charts work for $S[1]$ since we have removed a boundary chart. If S has even more boundary components, the argument is the same. □

Lemma 4.6. Suppose S has exactly one boundary component homeomorphic to a circle. Then if S has at least one planar end and no boundary component homeomorphic to a line, $B(S[1]) = B(S[2]) = 2$. Otherwise, $B(S[1]) = B(S) - 1$.

Proof. If S has any planar ends, then it will be the case that the fundamental group of S is non-trivial. Therefore, we need at least two charts. The first line of Main Theorem 1 states that this is sufficient, for simple surfaces. Lemmas 4.1 and 4.2 detail how this remains true with any number of ends of any type.

Suppose S does not have any planar ends. Suppose further that S does not have any ends and the fundamental group is trivial. Then S is a closed ball. Thus $S[1] \cong \mathbb{B}^n_h$. This clearly takes one chart.

Suppose the fundamental group is non-trivial, but S does not have any ends. Then we are in the case of the second line of Main Theorem 1; we have a closed surface with one open ball removed. Since the closed surface with one closed ball removed only takes two charts, it is clear that two charts would be sufficient.

Finally, suppose S has an end with genus or a nonorientable end. Then by Lemma 4.2, it is once again true that this does not effect the calculation. □

Lemma 4.7. Suppose S does not have any planar ends, but has one boundary component homeomorphic to a circle. Then if $S = \overline{\mathbb{B}^n}$, $B(S[n]) = n$. Otherwise, $B(S[1]) = B(S[2]) = 2$.

Proof. The result is clear if $S = \overline{\mathbb{B}^n}$.

Suppose S has non-trivial fundamental group and no ends. By Lemma 4.6, $B(S[1]) = 2$. As argued above in Lemma 4.6, two charts is sufficient. This remains true for $S[2]$. In order to visualize this, look at the left figure of Figure 3. Remove two points from the boundary, one on each side of the circle, that touch the blue line. This is now a two cover of $S[2]$.

Suppose S has ends. Then by Lemma 4.1 and Lemma 4.2, this does not affect the covering number. □

4.5. **Proof of Main Theorem 2.** We first must well-define what it means for a planar end to be outside of a position. Let S be a surface. Let $S[n]$ be the derived surface as in Section 4.4. Then a planar end is outside of the newly created position if and only if it is a planar end of S. Thus, in particular, a planar end, k, of S is outside of all positions if there does not exist a surface S' such that $S'[n] = S$ and k is not a planar end of S'.

Main Theorem 2. Let S be a surface with k boundary components homeomorphic to a circle, and h boundary components homeomorphic to a line organized across p positions, such that $k+h$ is finite. Let $g = 0$ if S has genus 0, and otherwise let $g = 1$. Let ℓ be the number of planar ends outside of the p positions.

(1) $B(S) = \max\{2k+h, 2\}$: $k+p+\ell \geq 2$

(2) $B(S) = h$: $h \geq 2$

(3) $B(S) = g+h$: $h = 1$

(4) $B(S) = g+2-\ell$ otherwise

If k or h is infinite, then $B(S) = \infty$.

Proof. This result is identical to Main Theorem 1 if there are finitely many ends, the surface has finite genus, and the boundary is compact. Assume k and h are finite.

First, suppose $p = 0$ (and thus $h = 0$). Then the result follows from Lemmas 4.1 and 4.2 and Main Theorem 1.

Now suppose $p = 1$, and $k = \ell = 0$. This must mean that S is a surface created by starting with a surface S' with exactly one boundary component homeomorphic to a circle and no planar ends, i.e., $S = S'[n]$. Therefore, the result follows from Lemma 4.4, Lemma 4.6, Lemma 4.7, Main Theorem 1, Lemma 4.1, and Lemma 4.2. More explicitly, it follows because $B(S_{1,0}) = 2$ if $S \cong \mathbb{S}^2$, and $B(S_{1,0}) = 3$ otherwise, with S a simple surface. Then Lemma 4.1 and 4.2 says that we may attach as many ends with genus and nonorientable ends that we want. Then Lemma 4.6 says that if $h = 1$, the result is 2 (if S is not the sphere) or 1 (if S is the sphere). Lemma 4.7 says that if $h = 2$, then the result is 2. Finally, Lemma 4.4 says that we increase this by one for each additional boundary component homeomorphic to a line.

Suppose $p > 0$ and $k+p+\ell \geq 2$. Since there are finitely many boundary components homeomorphic to a line, there are finitely many positions. Thus there exists some surface S' such that $S = S'[n_1][n_2]\cdots[n_p]$ for positive integers $n_1, \ldots, n_p$. By Main Thereom 1, Lemma 4.1, and Lemma 4.2, we know that $B(S') = \max\{2(k+p), 2\}$. Suppose $k \geq 1$. Then $k+1 \geq 2$, and we may apply Lemma 4.5 at each step, along with Lemma 4.4. This will generate $B(S) = \max\{2(k+p) - 2p + h, 2\}$, which is the result. Suppose $k = 0$, $\ell \geq 1$. Then

Lemma 4.6 applies to show the minimum is 2. Otherwise, application of Lemma 4.4 and Lemma 4.5 will determine the result. Now suppose $k = \ell = 0$ and $p \geq 2$. Then it is the case that $h \geq 2$. To S', we may initially apply Lemma 4.4 or Lemma 4.5. To the final step, we may apply Lemma 4.6 followed by Lemma 4.4. This finishes the result.

Note that if there are infinitely many boundary components, some of the above arguments are more complicated. However, then the covering number must be infinite, so it does not affect the result. Therefore, this case was ignored. □

5. Summary

In summary, most of the time borderless surfaces require less than the maximum determined by Ostrand's theorem ($n+1 = 3$). The border does not significantly complicate the issue, modifying the result exactly as one would predict: for circles, increasing the number by two, and for lines, increasing the number by one. Finally, any number of ends greater than some minimal requirement (two for planar ends, one for ends with genus) does not effect the covering number. In short, most (borderless) surfaces can be covered in two charts.

A simple bound on $B(S)$ is that $B(S) \geq \max\{B(\operatorname{Int} S), B(\delta S)\}$. Main Theorem 2 says that this is almost always equality. For part (1), this is equality: the $2k+h$ bound is the boundary, and the 2 bound is the interior. For part (2), the cover is determined by the boundary. For part (3), the interior determines the cover. For part (4), if $\ell = 1$ or $k = \ell = 0$, then the manifold is borderless and thus this is trivially true. Otherwise, $k = 1$ and the boundary is a circle. Thus the only case where $B(S) > \max\{B(\operatorname{Int} S), B(\delta S)\}$ is when S has genus (possibly infinite), no planar ends, and one boundary component homeomorphic to a circle. This is an alternative, shorthand way of stating Main Theorem 2.

The exact reason for the simplicity in this dimension is that for any contractible set U, there exists a V such that $\mathbb{B}^2 \subseteq V \subseteq \overline{\mathbb{B}^2}$ and $V \cong U$. This property, along with Ian Richard's [8] classification of surfaces, eased calculation. For any higher dimension, the above arguments would fail. Instead, the arguments would calculate the Lusternik-Schnirelmann category. Therefore, the above work emphasizes the importance of this invariant in higher dimensions, as shown by other work (e.g., see [10, 9, 3]).

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References

[1] H. R. Brahana. Systems of circuits on two-dimensional manifolds. *Annals of Mathematics*, 23:144–168, 1921.

[2] M. Cárdenas, F. F. Lasheras, and A. Quintero. Minimal covers of open manifolds with half-spaces and the proper L-S category of product spaces. *Bulletin of the Belgian Mathematical Society*, 9:419–431, 2002.

[3] Yuri Chekanov, Otto van Koert, and Felix Schlenk. Minimal atlases of closed contact manifolds. In Miguel Abreu, François Lalonde, and Leonid Polterovich, editors, *New Perspectives and Challenges in Symplectic Field Theory*, volume 49 of *CRM Proceedings and Lecture Notes*, pages 73–112, Providence, RI, 2009. American Mathematical Society.

[4] J. C. Gómez-Larrañaga and F. González-Acuña. Lusternik-Schnirelmann category of 3-manifolds. *Topology*, 31:791–800, 1992.

[5] John M. Lee. *Introduction to Topological Manifolds*. Springer Publishing, 2 edition, 2011.

[6] Montejano. A quick proof of Singhof's cat$(M \times S^1) = cat(M) + 1$ theorem. *Manuscripta Mathematica*, 42:49–52, 1983.

[7] J. Munkres. Obstructions to the smoothing of piecewise-differentiable homeomorphisms. *Annals of Mathematics*, 72:521–554, 1960.

[8] Ian Richards. On the classification of noncompact surfaces. *Transactions of the American Mathematical Society*, 106:259–269, 1963.

[9] Yu B. Rudyak and Felix Schlenk. Minimal atlases of closed symplectic manifolds. *Communications in Contemporary Mathematics*, 9:811–855, 2007.

[10] Wilhelm Singhof. Minimal coverings of manifolds with balls. *Manuscripta Mathematica*, 29:385–415, 1979.

[11] R. Thom. Les structures différentiables des boules et des sphères. In *Colloque de Géométrie Différentielle Globale*, volume 31, pages 27–35, Brussels, Belgium, 1959. Organized by the Centre belge de recherches mathématiques.

[12] Mihaela Todea. Some remarks concerning the covering numbers of closed manifolds. *Acta Universitatis Apulensis. Mathematics - Informatics 3*.

[13] Yasuyuki Tsukui. On ball coverings for products of manifolds. *Yokohama Mathematical Journal*, 25:113–117, 1977.

[14] J. H. C. Whitehead. On C^1-complexes. *Annals of Mathematics*, 41:809–824, 1940.

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